CHAPTER- 1

A STUDY OF BASIC HYPERGEOMETRIC FUNCTIONS AND THEIR APPLICATIONS IN PARTITION THEORY AND CONTINUED FRACTIONS

(A Historical Survey)

1.1 In this chapter we give a brief account of some of the researches carried out in the field of generalized hypergeometric series. It shall not be our endeavour to give a complete chronological survey of all developments in this field but shall mention only those relevant to the present work. The following notations and definitions shall be used throughout this and subsequent chapters.

Let

$$[a]_n = a[a+1]\ldots[a+n-1]. \quad [a]_0 = 1,$$

$$[a]_{-n} = \frac{(-1)^n}{(1-a)_n}$$

then the generalized ordinary hypergeometric series is defined as

$$(1) \qquad {}_rF_s\left[\begin{matrix} a_1, & a_2, & \ldots, & a_r; & z \\ b_1, & b_1, & \ldots, & b_s & \end{matrix}\right] = {}_rF_s\left[\begin{matrix} (a)_r; z \\ (b)_s \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{[a_1]_n [a_2]_n \cdots [a_r]_n}{[1]_n [b_1]_n [b_2]_n \cdots [b_s]_n} z^n \equiv \sum_{n=0}^{\infty} \frac{[(a_r)]_n z^n}{[1]_n [(b_s)]_n}.$$

The ${}_rF_s$ series converses for $|z| < 1$ when $r = s+1$ and for all values of z, when $r \leq s$. When r > s+1, the series diverges for all $z \neq 0$. The later type of series $(r > s+1)$ have been the subject of detailed study by Mac Robert [1-10], Mejjer [1-11], Agarwal [1] and others. They have attempted to give a meaning to the F-symbol in this case also.

In 1.1(1) (a_r) stands for the sequence of parameters $a_1, a_2, \ldots, a_r$. We may also have occasion to use symbol (ap,q) to mean the q-p+1 parameters $a_p, a_{p+1}, \ldots, a_q$. If $p = 1$ in it, we shall write (a_q) instead of (a_1, q) but when $q = A$, we shall simply denote it by (a).

In the case when $z = 1$, ${}_{s+1}F_s(1)$ converges for $Rl[\Sigma b_s - \Sigma a_{s+1}] > 0$ and when $z = -1$ it converges for $Rl[\Sigma b_s - \Sigma a_{s+1} + 1] > 0$.

As usual, the argument z shall be omitted in the F-symbol when it is equal to unity.

When r = s+1, the series 1.1(1) is called 'Saalschutzian' if

$$\Sigma b_s - \Sigma a_{s+1} = 1$$

'Well-poised' if

$$1 + a_1 = b_1 + a_1 = \ldots = b_s + a_{s+1}$$

and nearly-poised if all but one of the pairs of parameters have the same sum, when unity is regarded as the first denominator parameter.

if $b_1 + a_2 = b_2 + a_3 = ... = b_s + a_{s+1}$, so that the break-down in the equality of sums of pairs occurs with the first pair, regarding unity as the first denominator parameter, the series is called a 'nearly-poised series of first kind'. If, however, the break-down occurs with the fast pair so that

$$1 + a_1 = b_1 + a_2 = ... = b_{s-1} + a_s$$

the series is called a 'nearly-poised' series of second kind.

A further generalization of the ordinary hypergeometric series is provided in the form of bilateral hypergeometric series. Though a few scattered results for what we now call the ordinary bilateral hypergeometric series were given by Dougall [1] as long as 1907, Yeta systematic study of such series was made first by Bailey [1] in 1936.

Dougall [1] obtained a formula which can be written as

(2) $$\sum_{n=-\infty}^{\infty} \frac{[a]_n [b]_n}{[c]_n [d]_n} = \Gamma\left[\begin{matrix} c, & d, & 1-a, & 1-b, & c+d-a-b-1; \\ c-a, & d-a, & c-b, & d-b & \end{matrix}\right],$$

where

$Rl\,[c+d-a-b-1] > 0$.

If we take d = 1 in 1.1(2), it reduces to well known Gauss's sum of ${}_2F_1[a,b,;c;1]$ (cf, slater [1;1.1.5]).

Bailey regarded series of the type 1.1(2) as hypergeometric series with unit argument which are infinite in both directions and called them 'bilateral' hypergeometric series of the ordinary bilateral hypergeometric series as

$$\textbf{(3)} \quad {}_pH_p\begin{bmatrix} a_1, & a_2, & \ldots, & a_p; & z \\ b_1, & b_2, & \ldots, & b_p & \end{bmatrix} = {}_pH_p\begin{bmatrix} (a_p); z \\ (b_p); \end{bmatrix}$$

$$= \sum_{n=-\infty}^{\infty} \frac{[(a_p)]_n z^n}{[(b_p)]_n}$$

where $|z| = 1$, for convergence.

Obviously, the bilateral series ${}_pH_p$ can be expressed as the sum of two hypergeometric series of the type ${}_{P+1}F_P$. The series ${}_pH_p(z)$ converges, provided $Rl[(\Sigma b_p - \Sigma a_{p-1})] > 0$ for $z = 1$ and when $z = -1$, provided $Rl[(\Sigma b_p - \Sigma a_p)] > 0$. When $b_p = 1$, it evidently reduces to a series of the type ${}_pF_{P-1}(z)$.

We call a bilateral series ${}_pH_p$ 'well-poised' if

$$a_1 + b_1 = a_2 + b_2 = \ldots = a_p + b_p.$$

and 'nearly-poised' if

$$a_1 + b_1 = a_2 + b_2 = \ldots = a_{p-1} + b_{p-1}.$$

i.e. the equality breaks for any one pair.

It is sais to be "Saalschutzian" if

$$b_1 + b_2 \ldots b_p = 1 + a_1 + a_2 + \ldots + a_p.$$

A further generalization of ordinary hypergeometric series has been provided with the help of following functions :

(4) $$F_1[a;b,b';c;x,y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a]_{m+n}[b]_m[b']_n x^m y^n}{[c]_{m+n}[1]_m[1]_n},$$

which exists for all real or complex values of a, b, b', c, x and y, except c a non-negative integer.

(5) $$F_2[a;b,b';c,c';x,y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a]_{m+n}[b]_m[b']_n x^m y^n}{[c]_m[c']_n[1]_m[1]_n},$$

which exists for all real or complex values of a,b, b',c, c',x and y, except c, c' negative integers ;

(6) $$F_3[a,a';b,b';c;x,y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a]_m[a']_n[b]_m[b']_n x^m y^n}{[c]_{m+n}[1]_m[1]_n},$$

which exists for all values of parameters except c a negative integer ;

(7) $$F_4[a;b;c,c;x,y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a]_{m+n}[b]_{m+n} x^m y^n}{[c]_m[c']_n[1]_m[1]_n},$$

which exists for all values of parameters except c, c' negative integers.

These series are called Appell series (cf. Slater [1;(8.1.3)-(8.1.6)]). All four Appell functions reduce to ordinary Gaus series ${}_2F_1[x]$ when $y = 0$. The first three function also reduces to ordinary ${}_2F_1[x]$ series when b' is zero.

Horn [1,2 and 3] systematically investigated all the, double hypergeometric functions of second order. Horn's final list consisted of fourteen complete (non-confluent) series and twenty distinct limiting form of them. This includes the four

Appell functions and the seven Humbert functions. The following list gives the complete functions exluding the Appell functions,

$$G_1(a,b,b';x,y)=\sum\frac{[a]_{m+n}[b]_{n-m}[b']_{m-n}x^m y^n}{[1]_m[1]_n},$$

$$G_2(a,a';b,b';x,y)=\sum\frac{[a]_m[a']_n[b]_{n-m}[b']_{m-n}x^m y^n}{[1]_m[1]_n},$$

$$G_3(a,a';x,y)=\sum\frac{[a]_{2n-m}[a']_{2m-n}x^m y^n}{[1]_m[1]_n},$$

$$H_1(a,b,c;d;x,y)=\sum\frac{[a]_{m-n}[b]_{m+n}[c]_n x^m y^n}{[d]_m[1]_m[1]_n},$$

$$H_2(a,b,c,d;e;x,y)=\sum\frac{[a]_{m-n}[b]_m[c]_n[d]_n x^m y^n}{[e]_m[1]_m[1]_n},$$

$$H_3(a,b;c;x,y)=\sum\frac{[a]_{2m+n}[b]_n x^m y^n}{[c]_{m+n}[1]_m[1]_n},$$

$$H_4(a,b;c,d;x,y)=\sum\frac{[a]_{2m+n}[b]_n x^m y^n}{[c]_m[d]_n[1]_m[1]_n},$$

$$H_5(a,b;c;x,y)=\sum\frac{[a]_{2m+n}[b]_{n-m}x^m y^n}{[c]_m[1]_m[1]_n},$$

$$H_6(a,b,c;x,y)=\sum\frac{[a]_{2m-n}[b]_{n-m}[c]_n x^m y^n}{[1]_m[1]_m}$$

and

$$H_7(a,b,c;d;x,y)=\sum\frac{[a]_{2m-n}[b]_n[c]_n x^m y^n}{[d]_m[1]_m[1]_n}$$

A further extension of the Appell series was provided with the help of double hypergeometric function of higher order defined by (cf. Ragab [1])

(8)
$$F\left[\begin{array}{ccc|ccc|c} & P & & & (\xi_p) & & \\ t & : & t' & (\gamma_p) & ; & \gamma'_{t'} & x \\ & s & & & (\delta_s) & & y \\ q & ; & q' & (\beta_q) & ; & (\beta'_{q'}) & \end{array}\right]=F\left[\begin{array}{ccc} (\xi_p); & (\gamma_t); & (\gamma'_{t'}); \\ & & & x,y \\ (\delta_s); & (\beta_q); & (\beta'_{q'}); \end{array}\right]$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(\xi_p)]_{m+n}[(\gamma_t)]_m[(\gamma'_{t'})]_n x^m y^m}{[(\delta_s)]_{m+n}[(\beta_q)]_m[(\beta'_{q'})][1]_m[1]_n}.$$

The concept of a double hypergeometric series can be extended to hypergeometric series of several variables, such series were first studied by Lauricella [1] in 1893. Lauricella defined the following four functions of n-variables :

(9) $F_A[a;b_1,b_2\dots;b_n;c_1,c_2\dots c_n;x_1,x_2\dots x_n]$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{[a]_{m_1+m_2 \dots m_n} [b_1]_{m_1} [b_2]_{m_2} \cdots [b_n]_{m_n} x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}}{[c_1]_{m_1} [c_2]_{m_2} \cdots [c_n]_{m_n} [1]_{m_1} [1]_{m_2} \cdots [1]_{m_n}},$$

where for convergence

$$|x_1| + |x_2| + \dots + |x_n| < 1,$$

(10) $F_B[a_1, a_2, \dots a_n; b_1, b_2 \dots b_n; c; x_1, x_2 \dots, x_n]$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{[a_1]_{m_1} \cdots [a_n]_{m_n} [b_1]_{m_1} \cdots [b_n]_{m_n}}{[c]_{m_1+m_2+\dots+m_n} [1]_{m_1} \cdots [1]_{m_n}} x_1^{m_1} \dots x_n^{m_n},$$

where for convergence,

$$|x_1| < 1; |x_2| < 1, \dots, |x_n| < 1,$$

(11) $F_C[a; b; c_1, c_2 \dots c_n; x_1, x_2 \dots x_n]$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{[a]_{m_1+m_2 \dots m_n} [b]_{m_1+m_2 \dots m_n} x_1^{m_1} \dots x_n^{m_n}}{[c]_{m_1} \cdots [c]_{m_n} [1]_{m_1} \cdots [1]_{m_n}},$$

where for convergence,

$$|x_1^{1/2}| + |x_2^{1/2}| + \dots |x_n^{1/2}| < 1,$$

(12) $F_D[a; b_1, b_2, \dots, b_n; c; x_1, x_2, \dots, x_n]$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{[a]_{m_1+m_2 \dots m_n} [b_1]_{m_1+\dots} [b_n]_{m_n} x_1^{m_1} \dots x_n^{m_n}}{[c]_{m_1+m_2+\dots+m_n} [1]_{m_1} [1]_{m_2} \cdots [1]_{m_n}},$$

where for convergence,

$$|x_1|, |x_2|, \dots |x_n| < 1.$$

If n, the number of variables, is 2, these four functions reduce to the Appell functions F_2, F_3, F_4 and F_1 respectively ; and if n = 1, all four functions become the Gauss's function ${}_2F_1$.

1.2 Heine E. [1; 1878] generalized the hypergeometric series in the form of basic hypergeometric series by introducing the basic number as :

$$a_q = (1-q^a)/(1-q)$$

where q and a are real or complex numbers, so that $q \to 1$, $(1-q^a)/(1-q) \to a$.

A basic hypergeometric series is a series $\sum C_n$ with C_{n+1}/C_n, a rational function of q^n for a fixed parameter q.

Further assuming $|q| < 1$, let

(1) $$[a]_n = [a;q]_n = (1-a)(1-aq)\dots(1-aq^{n-1}) \quad (a)_0 = 1,$$

(2) $$[a]_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{a^n [q/a]_n},$$

(3) $$[a;q]_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$

and

(4) $$[a;q]_\infty = [a;q]_\infty / [aq^n;q]_\infty$$

then the generalized basic hypergeometric series is defined as :

$${}_r\Phi_s\begin{bmatrix} a_1, & a_2, & \dots, & a_r; & q, z \\ b_1, & b_2, & \dots, & b_s & \end{bmatrix} = {}_r\Phi_s\begin{bmatrix} (a_r); & q, & z \\ (b_s) & & \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{[a_1;q]_n [a_2;q]_n \ldots [a_r;q]_n z^n}{[q;q]_n [b_1;q]_n \ldots [b_s;q]_n}$$

$$= \sum_{n=0}^{\infty} \frac{[(a_r);q]_n z^n}{[q;q]_n [(b_s);q]_n}.$$

The series ${}_r\Phi_s$ converges for max. $\{|z|,|q|\} < 1$.

For $r = s+1$, the series 1.2(5) is called 'Saalschutzian'

when $b_1, b_2 \ldots b_s = qa_1 a_2 \ldots a_{s+1}$,

'Well-poised' when

$$qa_1 = b_1 a_1 = \ldots = b_s a_{s+1}$$

and 'nearly-poised' if all but one of the pairs of parameters have the same product. The series is called a nearly-poised series of the first or of the second kind according as the break-down in equality of products of pair of parameters occur in the first or last pair, when q is regarded as the first denominator parameter.

A basic hypergeometric series reduces to an ordinary hypergeometric series when $q \to 1$.

Also for brevity, we shall write ,

$$\Gamma\begin{bmatrix} a_1, & a_2, & \ldots a_r; \\ b_1, & b_2, & \ldots b_s \end{bmatrix} \quad \text{or} \quad \Gamma\begin{bmatrix} (a_r); \\ (b_s) \end{bmatrix}$$

for $\dfrac{\Gamma[a_1]\Gamma[a_2]\ldots\Gamma[a_r]}{\Gamma[b_1]\Gamma[b_2]\ldots\Gamma[b_s]}$

and

$$\prod\begin{bmatrix} a_1, & a_2, & \dots & a_r; \\ b_1, & b_2, & \dots & b_s \end{bmatrix} \text{ or } \prod\begin{bmatrix} (a_r); \\ (b_s) \end{bmatrix} \text{or} \qquad \frac{[(a_r);q]_\infty}{[(b_s);q]_\infty}$$

for

$$\prod_{n=0}^{\infty}\frac{[1-a_1q^n][1-a_2q^n]\dots[1-a_rq^n]}{[1-b_1q^n][1-b_2q^n]\dots[1-b_sq^n]}.$$

An abnormal basic hypergeometric series is defined as

(6) $$ {}_A\Phi_B\begin{bmatrix} a_1, & a_2, & \dots, & a_A; & q, & x \\ b_1, & b_2, & \dots & b_B; & \lambda & \end{bmatrix} = {}_A\Phi_B\begin{bmatrix} (a_A); & x \\ (b_B); & \lambda \end{bmatrix}$$

$$= \frac{[a_1;q]_n[a_2;q]_n\dots[a_A;q]_n x^n q^{\lambda n(n-1)/2}}{[b_1;q]_n[b_2;q]_n\dots[b_B;q]_n[q;q]_n},$$

where for convergence

$|q| < 1$ and $|z| < \infty$ when $\lambda \in N$,

or max. $\{|q|, |x|\} < 1$ when $\lambda = 0$,

provided that no zero appears in the denominator.

Very Well-poised basic hypergeometric series is written as :

$$ {}_{r+1}\Phi_r\begin{bmatrix} a_1, & q\sqrt{a_1}, & -q\sqrt{a_1}, & -a_4, & \dots & a_{r+1}; & q, & z \\ & \sqrt{a_1}, & -\sqrt{a_1} & a_1q/a_4 & \dots, & a_1q/a_{r+1}; & & \end{bmatrix}$$

and in a compact notation

$$ {}_{r+1}W_r[a_1;a_4,a_5,\dots,a_{r+1};q;z].$$

Bailey [1] defined a generalized basic bilateral hypergeometric series as

(7) $${}_P\Psi_P\begin{bmatrix} a_1, & a_2, & \ldots, & a_P; & z \\ b_1, & b_2, & \ldots, & b_p; & \end{bmatrix} = {}_P\Psi_P\begin{bmatrix} (a_P);z \\ (b_P) \end{bmatrix}$$

$$= \sum_{n=-\infty}^{\infty} \frac{[(a_P);q]_n}{[(b_P);q]_n} z^n .$$

where for convergence,

$$|b_1 b_2 \ldots b_P / a_1 a_2 \ldots a_P| < |z| < 1.$$

The series of the type 1.2(7) can easily be written as the sum of two ${}_{P+1}\Phi_P$ - series and $b_P = q$, it reduces to a series of type ${}_P\Phi_{P-1}(z)$.

A bilateral ${}_P\psi_P$ series is called 'Well-poised' if

$$a_1 b_1 = a_2 b_2 = \ldots = a_P b_P .$$

and if the equality breaks-down for any one-pair, it will be called 'nearly-poised'.

It is said to be 'Saalschutzian' if

$$b_1 b_2 \ldots . b_P = q a_1 a_2 \ldots a_q .$$

The bilateral basic hypergeometric series shown by Bailey to have valuable application in proving numerous identities in combinatory analysis. He obtained the following sum of a well poised ${}_6\psi_6$ namely.

(8) $${}_6\Psi_6\begin{bmatrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e; & a^2q/bcde \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e; & \end{bmatrix}$$

$$= \prod \begin{bmatrix} aq, & q/a, & \frac{aq}{bc}, & \frac{aq}{bd}, & \frac{aq}{be}, & \frac{aq}{cd}, & \frac{aq}{ce}, & \frac{aq}{de}, & q; & q \\ \frac{aq}{b}, & \frac{aq}{c}, & \frac{aq}{d}, & \frac{aq}{e}, & \frac{q}{b}, & \frac{q}{c}, & \frac{q}{d}, & \frac{q}{e}, & \frac{a^2q}{bcde} & \end{bmatrix}.$$

The formula (1.2) (8) has been proved to have extensive application in the theory of elliptic functions, combinatory analysis and deduction of Rogers-Ramanujan type identities.

Ramanujan obtained the following bilateral summation formula

(9) $${}_1\Psi_1\begin{bmatrix} a; & z \\ b & \end{bmatrix} = \frac{[q;q]_\infty [b/a;q]_\infty [az;q]_\infty [q/az;q]_\infty}{[b;q]_\infty [q/a;q]_\infty [z;q]_\infty [b/az;q]_\infty},$$

$|b/a| < |z| < 1.$

Euler [1] showed that

(10) $$\sum_{n=-\infty}^{\infty} (-)^n q^{(3n^2+n)/2} = [q;q]_\infty$$

In an unpublished work gauss [1] pushed this work along. Among other results he proved that

(11) $$\sum_{n=-\infty}^{\infty} (-)^n q^{\binom{n}{2}} x^n = (q,x,q/x;q)_\infty .$$

This was found later by Jacobi [1] and is called triple product identity.

The basic analogues of four Appell series [1.1(4)-1.1(7)] are defined as (cf. Slater [1;9.1])

(12) $$\Phi^1[a,b,b';c;x,y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a;q]_{m+n}[b;q]_m[b';q]_n}{[c;q]_{m+n}[q;q]_m[q;q]_n}x^m y^n,$$

(13) $$\Phi^2[a;b,b';c,c';x,y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a;q]_{m+n}[b;q]_m[b';q]_n}{[c;q]_m[c';q]_n[q;q]_m[q;q]_n}x^m y^n,$$

(14) $$\Phi^3[a,a';b,b';c;x,y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a;q]_m[a';q]_n[b;q]_m[b';q]_n}{[c;q]_{m+n}[q;q]_m[q;q]_n}x^m y^n,$$

(15) $$\Phi^4[a,b;c,c';x,y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[a;q]_{m+n}[b;q]_{m+n}}{[c;q]_m[c';q]_n[q;q]_m[q;q]_n}x^m y^n.$$

The generalized basic hypergeometric series of two variables is defined as,

(16) $$\Phi\begin{bmatrix}(a); & (b); & (b); & x, & y\\ (c); & (d); & (d'); & & \end{bmatrix}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(a);q]_{m+n}[(b);q]_m[(b');q]_n x^m y^n}{[(c);q]_{m+n}[(d);q]_m[(d');q]_n[q;q]_m[q;q]_n}$$

One can easily define the abnormal basic hypergeometric series of two variables as :

(17) $$\Phi\begin{bmatrix}(a); & (b); & (b); & x, & y\\ (c); & (d); & (d'); & & \end{bmatrix}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(a);q]_{m+n}[(b);q]_m[(b');q]_n x^m y^n}{[(c);q]_{m+n}[(d);q]_m[(d');q]_n[q;q]_m[q;q]_n}q^{1/2\{im(m-1)+jn(n-1)+kmn\}},$$

where for convergence,

$|q| < 1, |x| < \infty$ and $|y| < \infty$ when $i, j, k \in N$

or

max. $(|q|, |x|, |y|) < 1$ when $i = j = k = 0$

In the special case when $i = j = k = 0$ 1.2(17) reduces into 1.2(16).

Analogously we can define the basic analogue of Lauricella's four functions [1.1(9)-1.1(12)].

These multiple basic hypergeometric series are special cases of the following q-extension of the generalized basic hypergeometric series in n-variables given by (cf. Math. Review 85 g: 33001, P.2900, July 1985)

$$\sum_{m_1,\ldots,m_n=0}^{\infty} A_{(m_1,\ldots,m_n)} x_1^{m_1} \ldots x_n^{m_n} q^{\lambda_1 m_1 (m_1+1)+\ldots+\lambda_n m_n (m_n+1)},$$

where $A_{(m_1,\ldots,m_n)} / A_{(m_1+1,\ldots,m_n+1)}$ is a rational function of $q^{m_1}, q^{m_2}, \ldots, q^{m_n}$, is a multiple basic hypergeometric series, which is convergent for $\lambda_i > 0$ or $|x_i| < 1$ when $\lambda_i = 0$ $(i = 1, 2, \ldots n)$.

1.3 Basic hypergeometric identities and partition theory:

Let us start with the following well-known generating functions which are very useful in the study of the theory of partitions.

$$\sum_{n=0}^{\infty} P_m(n) q^n = \frac{1}{(q;q)_m} \tag{1}$$

where $P_m(n)$ stands for the number of partitions of n in which no part is greater than m.

$$\sum_{n=0}^{\infty} P(n)q^n = \frac{1}{(q;q)_m} \tag{2}$$

where $P(n)$ denotes the number of partitons of n.

$$\sum_{n=0}^{\infty} d_m(n)q^n = (-q;q)_m \tag{3}$$

where $d_m(n)$ is the number of partitions of n into distinct parts no greater than m.

$$\sum_{n=0}^{\infty} d(n)q^n = (-q;q)_m \tag{4}$$

where d(n) denotes the number of partitions of n into distinct parts.

L.J. Rogers in 1894 established identities which were later discovered by Ramanujan in 1913, namely,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{5}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}, \tag{6}$$

These identities have the following very elegant combinatorial interpretations:

"The number of partitions of n into parts with least difference 2 is equal to the number of partitions of n into parts $\equiv 1$ or 4(mod 5)" and

The number of partitions of n into parts not less than 2 and with minimal difference 2 in equal to the number of partitions of n into parts $\equiv 2$ or 3 (mod 5)", respectively.

The simplest of the serveral proofs of [1.3.5] and [1.3.6] given by Rogers and they by Ramanujan depends on general formula-

$$1+\sum_{n=0}^{\infty}(-1)^n a^{2n} q^{n(5n-1)/2}\left[\frac{1-aq^{2n}}{1-a}\right]\frac{(a;q)_n}{(q;q)_n} = (a;q)_\infty \sum_{n=0}^{\infty}\frac{a^n q^{n^2}}{(q;q)_n} \qquad (7)$$

and Jacobi's triple product identity

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} z^n = \left(q^2, zq, q/z; q^2\right)_\infty, \ |q|<1$$

Identities [1.3.5] and [1.3.6] can be obtained from [1.3.7] by taking a = 1 and a=q, respectively and using [1.3.8].

In 1929, an interesting proof depending on the transformation [1.2(13).26] was given by G.N. Watson [1]. Letting b,c,d, e $\rightarrow \infty$ in [1.2(B).26] we get :

$$1+\sum_{r=1}^{\infty}\frac{(aq;q)_{r-1}\left(1-aq^{2r}\right)\left(q^{-n};q\right)_r a^{2r} q^{2r^2+nr}}{(q;q)_r\left(aq^{n+1};q\right)_r}$$

$$=(aq;q)_n\left\{1+\sum_{r=1}^{\infty}\frac{(-1)^r q^{r(r+1)/2}\left(q^{-n};q\right)_r a^r q^{nr}}{(q;q)_r}\right\} \qquad (9)$$

For $n \to \infty$, [1.3.9] reduces to [1.3.7].

Further, if we take $bc = aq$ and $d, e, n \to \infty$ in [1.2(B).26] we find

$$1+\sum_{n=0}^{\infty}(-1)^n a^n q^{n(3n-1)/2}\left(1-aq^{2n}\right)\frac{(aq;q)_n}{(q;q)_n}=(aq;q)_\infty \tag{10}$$

which for $a = 1$, yields Euler's identity viz.,

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}=(q;q)_\infty, \tag{11}$$

again, taking $bc = aq$, $d = q\sqrt{a}$, $e, n \to \infty$ and $a = 1$ in [1.2(B)26],

we have;

$$1+\sum_{r=1}^{\infty}q^{r(r+1)/2}=\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}, \tag{12}$$

which is a classical identity due to Gauss.

W.N. Bailey [4,5] combined Rogers method freely with the known summation formulae of basic hypergeometric series to obtain some general transformations leading to Rogers-Ramanujan identities as limiting cases. The fundamental theorem used by him was

"if $\quad \beta_n=\sum_{r=0}^{n}u_{n-r}v_{n+r}\alpha_r$

and $\quad \gamma_n=\sum_{r=n}^{\infty}u_{n+r}v_{n-r}\delta_r$

where $\alpha_r, \delta_r, \gamma_r$ and u_r are functions of r only then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n,$$

provided that all the series, involved are either convergent or terminating.

Similarly γ_n, by basic analogue of Gauss theorem one obtains the transformations :

$$\sum_{n=0}^{\infty} (y,z;q)_n \beta_n (x/yz)^n = \frac{(x/y,x/z;q)_\infty}{(x,x/yz;q)_\infty} \sum_{n=0}^{\infty} \frac{(y,z;q)_n \alpha_n}{(x/y,x/z;q)_n} (x/yz)^n, \qquad (13)$$

By giving different values to the sequence α_n, Bailey obtained different identities from it. However, a systematic attempt was made not until 1951, to use the above theorem to obtain various identities, when L.J. Slater [1] obtained a number of general transformation by different values to α_r, δ_r, U_r and V_r with the help of known basic bilateral q-series summation theorems. The most widely and fruitfully used summation theorem in this direction is

$${}_6\Psi_6\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e', & q; & \dfrac{a^2q}{bcde} \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e & & \end{matrix}\right]$$

$$= \frac{(aq,aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de,q,q,a;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,q/b,q/c,q/d,q/e,qa^2/bcde;q)_\infty}, \qquad (14)$$

Putting $b = q^{-n/3}$, $c = q^{(1-n)/3}$, $d = q^{(2-n)/3}$ and replacing 2 by q^3 and then taking $a = q$ in [1.3.14] we get :

$$\sum_{r=[-n/3]}^{[n/3]} \frac{(1-q^{6r+1})(-)^r (e;q^3)_r q^{r(9r+1)/2}}{(q;q)_{n+3r+1}(q;q)_{n-3r}(q^4/e;q^3)_r e^r} = \frac{(q^2/e;q^3)_n}{(q;q)_{2n}(q^2/e;q)_n}, \tag{15}$$

Letting $e \to \infty$, we find

$$\sum_{r=[-n/3]}^{[n/3]} \frac{(1-q^{6r+1})q^{r(6r-1)}}{(q;q)_{n+3r+1}(q;q)_{n-3r}} = \frac{1}{(q;q)_{2n}} \tag{16}$$

Since

$$q^{r(6r-1)}(1-q^{6r+1}) = q^{r(6r-1)}(1-q^{n+3r+1}) - q^{(2r+1)(3r+1)}(1-q^{n-3r})$$

So we get from [1.3.16] on simplification

$$\frac{(1-q)}{(q;q)_n(q;q)_{n+1}} = \sum_{r=1}^{[n/3]} \frac{q^{r(6r-1)}+q^{r(6r+1)}}{(q;q)_{n-3r}(q;q)_{n+3r}} - \sum_{r=1}^{[n+1/3]} \frac{q^{(2r+1)(3r+1)}}{(q;q)_{n+3r+1}(q;q)_{n-3r-1}}$$

$$-\sum_{r=1}^{[n+1/3]} \frac{q^{(2r-1)(3r-1)}}{(q;q)_{n-3r+1}(q;q)_{n+3r-1}}$$

$$= \frac{1}{(q;q)_{2n}}, \tag{17}$$

Again taking

$$\beta_n = \frac{1}{(q;q)_{2n}}, \qquad \alpha_{3n-1} = -q^{(2n-1)(3n-1)}$$

$$\alpha_{3n} = q^{n(6n-1)} + q^{n(6n+1)} \qquad \alpha_{3n+1} = -q^{(3n+1)(2n+1)}$$

and $x = q,\ y = q^{1/2}/u,\ z = q^{1/2}/v$ in [1.3.13] we obtain :

$$\sum_{n=0}^{\infty}\frac{\left(q^{1/2}/u,q^{1/2}/v;q\right)_n}{(q;q)_{2n}}u^n v^n=\frac{\left(uq^{1/2},vq^{1/2};q\right)_\infty}{(q,uv;q)_\infty}\times$$

$$\sum_{n=0}^{\infty}\frac{\left(q^{1/2}/u,q^{1/2}/v;q\right)_{3n}}{\left(uq^{1/2},vq^{1/2};q\right)_{3n}}u^{3n}v^{3n}\left\{q^{n(6n-1)}-\frac{\left(1-uq^{3n-1/2}\right)\left(1-vq^{3n-1/2}\right)q^{(2n-1)(3n-1)}}{\left(u-q^{2n-1/2}\right)\left(v-q^{3n-1/2}\right)}\times\right.$$

$$\left.\times q^{n(6n+1)}-\frac{\left(u-q^{3n+1/2}\right)\left(v-q^{3n+1/2}\right)q^{(2n+1)(3n+1)}}{\left(1-uq^{3n+1/2}\right)\left(1-vq^{3n+1/2}\right)}\right\} \qquad (18)$$

This is classical result proved by Rogers. He expressed it in the form :

$$a_0+a_2+a_4+\ldots=b_0-b_2q-b_4q^2+b_6\left(q^5+q^7\right)-b^3q^{12}\ldots \qquad (19)$$

where

$$a_{2n}=\frac{\left(q^{1/2}/u,q^{1/2}/v;q\right)_n u^n v^n}{(q;q)_{2n}}$$

and

$$b_{2n}=\frac{\left(uq^{1/2},vq^{1/2},q^{1/2}/u,q^{1/2}/v;q\right)_n}{\left(q,uv,uq^{1/2},vq^{1/2};q\right)_n}$$

Giving different values to u and v in [1.3.18], different identities can be obtained. For example, for $u=v=0$, we find

$$(q;q)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_{2n}}=\left(q^{30},q^{40},q^{16};q^{30}\right)_\infty-q^2\left(q^{30},q^4,q^{26};q^{30}\right)_\infty. \qquad (20)$$

Again taking $u=1$, $v=1$ and then replacing q by q^2, [13.18] yields.

$$\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\sum_{n=0}^{\infty}\frac{(-)^n q^{n^2}}{\left(-q;q^2\right)_n\left(q^4;q^4\right)_n}=\left(q^{42},q^{19},q^{23};q^{42}\right)_\infty+q^3\left(q^{42},q^5,q^{37}:q^{42}\right)_\infty.$$

........(21)

using similar technique, Slater [1] obtained a list of 130 identities with single and double products. Still there were gaps left out in her list of modular identities.

In 1951, Bailey showed that the identities with double products, deduced in the works of Slater and Rogers, can be reduced to single product identities. He gave two such identities but was not able to give any general identity.

He showed that

(i) When the power 0/q in the product advances by n and n is multiple of 3, then two products can be reduced to a single one by using following result :

$$(q;q)_\infty(-z,-q/z;q)_\infty(q/z^2,qz^2;q^2)_\infty$$
$$=(q^3,z^3q,q^2/z^3;q^3)_\infty+z(q^3,z^3q^2,q/z^3;q^3)_\infty \tag{22}$$

Replacing q by q^{14} and $z=q^3$ we get :

$$(-q^3,-q^{13},q^{14};q^{14})_\infty(q^8,q^{20};q^{28})=(q^{42},q^{19},q^{23};q^{42})_\infty+q^3(q^5,q^{37},q^{42};q^{42})_\infty.$$

.....(23)

Now, comparing [1.3.21] and (1.3.23), we have

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-q;q^2)_n(q^4;q^4)_n}=(-q^3,-q^{11},q^{14};q^{14})_\infty+(q^8,q^{20},q^{28};q^{28})_\infty$$

......(24)

(ii) When the power of q in product advance by n and n is not a multiple of 3 then we have :

$$(-qz^2,-q^3/z^2,q^4;q^4)_\infty + z(-q^3z^2,-q/z^2,q^4;q^4)_\infty$$

$$= (-z,-q/z,q;q)_\infty \tag{25}$$

Recently, Jain and Verma [1] in 1982, used a quadratic transformation :

$$ {}_4\Phi_3\left[\begin{matrix} a^2, & b^2, & c, & d; & q; & q \\ ab\sqrt{q}, & -ab\sqrt{q}, & -cd & & & \end{matrix}\right]$$

$$= {}_4\Phi_3\left[\begin{matrix} a^2, & b^2, & c^2, & d^2; & q^2; & q^2 \\ a^2, & b^2, & -cd, & -cdq & & \end{matrix}\right] \tag{26}$$

where a, b, c or d is of the form q^{-N}, N being a non-negative integer to obtain a new form of q-analogue of Whipple's transformation viz.

$$ {}_8\Phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & c, & e, & -e, & -q^n, & q^{-n}; & q; & \dfrac{a^2q^{2n+2}}{e^2c} \\ \sqrt{a}, & -\sqrt{a}, & aq/c, & aq/e, & -aq/e, & -aq^{n+1}; & aq^{n+1} & & & \end{matrix}\right]$$

$$= \frac{(a^2q^2;q^2)_n(-aq/e^2;q)_{2n}(a^2q^2/e^2c^2;q^2)_n e^{2n}}{(a^2q^2/c^2,a^2q^2/e^2;q^2)_n(-aq;q)_{2n}} \times$$

$$\times {}_4\Phi_3\left[\begin{matrix} ce^2q^{-2n-1}/a^2, & ce^2q^{-2n}/a^2, & e^2, & q^{-2n}; & q^2; & q^2 \\ e^2c^2q^{-2n}/a^2, & -e^2q^{-2n}/a, & -e^2q^{-2n+1}/a & & & \end{matrix}\right], \tag{27}$$

which yields an identity of method 13 for $c \to \infty$.

Verma and Jain [2] observing that all the transformations used by them or by others are either between two series with the same base or between two series one with base q and other different from q, developed transformations for terminating basic hypergeometric series, using the general theory of bibasic hypergeometric series given by Agarwal and Verma [1,2] in 1967-68. From these transformations they obtained a number of new Rogers-Ramanujan type of identities related to the moduli 11, 13, 17, 19, 22, 23, 26 and 38 etc. Later, verma and Jain [2] extended their own transformations to obtain identities for moduli 33, 39, 51 and 57.

In 1984, Prabha Rastogi [1] in her thesis approved for Ph.D. Degree of Lucknow University, Lucknow established some bibasic hypergeometric transformations, with the help of some known summation theorems. She made an attempt to fill up the gaps in the Slater's list [1951-52].

Recently, M.D. Hirschhorn [1], M.V. Subbarao [1] and other have made attempts to give partitions theoretic interpretations of certain identities due to Slater [2] One of the Hirschhorn's theorem is

"The number of partitions of K

$K = a_1 + a_2 + a_3 + \ldots$

with $a_1 \geq a_2 \geq a_3 \geq \ldots$

is equal to the number of partitions of K into parts congruent to 1,3,4,5,6,7,9,11,13,15,16,17 or 19 (mod 20)"

He used the folowing identity due to Slater [2] in order to establish the theorem [1.3.28].

$$\sum_{r=0}^{\infty}\frac{q^{r^2}}{(q;q)_{2r}}=\frac{1}{\left(q,q^3,q^4,q^5,q^7,q^9,q^{11},q^{13},q^{15},q^{16},q^{17},q^{19},q^{20}\right)_{\infty}} \qquad (29)$$

M.V. Subbarao [1] making use of an identity [Slater [2];(94)] established an other theorem, viz.,

"The number of partitions of n such that the parts in the first half of each partition have minimal difference 2, is equal to the number of partitions of n into parts $\equiv \pm1,\pm2,\pm5,\pm6,\pm8,\pm9 \pmod{20}$."

1.4 BASIC HYPERGEOMETRIC SERIES AND CONTINUED FRACTIONS :

In this first letter to Hardy (dated January 16,1913), Ramanujan stated several marvelous theorems on continued fractions, Two are of special interest to us:

$$\frac{1}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+}\,\frac{e^{-6\pi}}{1+\ldots}=\left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+1}{2}\right)e^{\frac{2\pi}{5}}, \qquad (1)$$

$$1-\frac{e^{\pi}}{1+}\,\frac{e^{-2\pi}}{1-}\,\frac{e^{-3\pi}}{1+\ldots}=\left(\sqrt{\frac{5-\sqrt{5}}{2}}-\frac{\sqrt{5}-1}{2}\right)e^{\frac{\pi}{5}}, \qquad (2)$$

of these (and related formula), Hardy says, in the article. "The Indian Mathematician Ramanujan" (Amer. Math. Monthly 44 (1937), P.144), [These formulas] defeated me completely. I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true, if they were not true, no would have had the imagination to invent them."

In the 'Lost' Note book of Ramanujan, there are a number of continued fractions and their basic hypergeometric equivalents. One day unusual continued fraction of the 'Lost' Note book is:

$$F(a,b,\lambda;q) = 1 + \frac{aq+\lambda q}{1+}\frac{bq+\lambda q^2}{1+}\frac{aq^2+\lambda q^3}{1+}\ldots, \tag{3}$$

where

$$f(a,b,\lambda;q) = \frac{\sum_{n=0}^{\infty}\frac{(-\lambda/a;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{(-\lambda/a;q)_n (aq)^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}},$$

Ramanujan deduced the following five corollaries of this viz.;

$$\frac{1}{1+}\frac{q}{1+}\frac{q^2+q}{1+}\frac{q^3}{1+}\frac{q^4+q^2}{1+}\ldots = \prod_{n=0}^{\infty}\frac{(1-q^{2n+1})}{(1-q^{4n+2})^2}, \tag{4}$$

$$\frac{1}{1+}\frac{q}{1+}\frac{q^2-q}{1+}\frac{q^3}{1+}\frac{q^4-q^2}{1+}\ldots = \prod_{n=0}^{\infty}(-)^n q^{n(n+1)/2} \tag{5}$$

$$\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+}\ldots = \prod_{n=0}^{\infty}\frac{(1-q^{6n+1})(1-q^{6n+5})}{(1-q^{6n+5})^2}, \tag{6}$$

$$\frac{1}{1+}\frac{q^2+q}{1+}\frac{q^4}{1+}\frac{q^6+q^3}{1+}\ldots = \prod_{n=0}^{\infty}\frac{(1-q^{8n+1})(1-q^{8n+7})}{(1-q^{8n+3})(1-q^{8n+5})}, \tag{7}$$

$$\frac{1}{1+}\frac{q^2-q}{1+}\frac{q^4-q^2}{1+}\frac{q^6-q^3}{1+}\ldots=\frac{1}{\prod_{n=0}^{\infty}(-)^n q^{3n^2+2n}\left(1+q^{2n+1}\right)}, \tag{8}$$

The oldest and the most famous theorem associated with Ramanujan's career is the Roger's-Ramanijan Continued fraction :

$$C(q)=1+\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\ldots=\frac{\left(q^2,q^3;q^5\right)_\infty}{\left(q,q^4;q^5\right)_\infty}, \tag{9}$$

which is also a special case of [1.4.3] for $a=b=0$ and $\lambda=1$.

Adiga [1] established a new simple and self continued approach to prove the following identities.

$$\frac{\sum_{n=0}^{\infty}\frac{(-\lambda/a;q)_n(aq)^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{(-\lambda/a;q)_n(a)^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}},$$

$$=\frac{1}{1+}\frac{aq+\lambda q}{1+}\frac{bq+\lambda q^2}{1+\ldots+}\frac{aq^{n+1}+\lambda q^{2n+1}}{1+}\frac{bq^{n+1}+\lambda q^{2n+2}}{1+\ldots}, \tag{10}$$

$$=\frac{1}{1+}\frac{aq+\lambda q}{1-aq+bq+\ldots}\frac{aq+\lambda q^n}{+1-aq+bq^n+\ldots}, \tag{11}$$

$$=\frac{1}{1-b+aq+}\frac{b+\lambda q}{1-b+aq^2+\ldots}\frac{b+\lambda q^n}{+1-b+aq^{n+1}+\ldots}; \tag{12}$$

$$=\frac{1}{1+aq+}\frac{\lambda q-abq^2}{1+q(aq+b)+}\cdots\frac{\lambda q^n-abq^{2n}}{1+q^n(aq+b)+}\cdots, \tag{13}$$

and

$$\frac{1}{a+c-}\frac{ab}{a+b+cq-}\cdots\frac{ab}{-a+b+cq^n-}\cdots$$

$$=\frac{1}{c-b+a+}\frac{bc}{c-b+a/q+}\cdots\frac{bc}{+c-b+a/q^n}\cdots \tag{14}$$

[1.4.12] is due to Hirschhorn [1] who pointed out that it contains several identities of Carlitz [1] and Gordon [1] as special cases.

B. Srivastava [1] has provided the generalization of continued fraction [1.4.3] of Ramanujan in the form :

$$\frac{\sum_{n=0}^{\infty}\frac{(-\lambda/a;q)_n(c;q)_n(-aq^2/c)^n}{(q;q)_n(-bq;q)_n}}{\sum_{n=0}^{\infty}\frac{(-\lambda/a;q)_n(c;q)_n(-aq/c)^n}{(q;q)_n(-bq;q)_n}}=\frac{1}{1+}\frac{(aq+\lambda q)(1-1/c)}{1+aq/c+}\frac{bq+\lambda q^2}{1+}\cdots$$

$$+\frac{(aq^{n+1}+\lambda q^{2n+1})(1-1/cq^n)}{1+aq/c+}\frac{bq^{n+1}+\lambda q^{2n+2}}{1+}\cdots \tag{15}$$

B. Srivastava [1] transformed this identity in various forms, so that, in the limit many of the classical results involving continued fractions due to Ramanujan [1], Gordon [1] and Carltiz [1] are obtained. In particular, for $c\to\infty$, this identity gives Ramanujan's continued fraction. (1.4.10).

Denis [1] established the following result :

$$\frac{{}_2\Phi_1\left[\begin{matrix}\alpha, & \beta; & q; & Kq\\ & \gamma & & \end{matrix}\right]}{{}_2\Phi_1\left[\begin{matrix}\alpha, & \beta; & q; & k\\ & \gamma & & \end{matrix}\right]}$$

$$=\frac{1}{1+}\,\frac{(1-\alpha)(1-\beta)k}{(1-k)+}\,\frac{\alpha\beta kq-\gamma}{1+}\,\frac{(1-\alpha q)(1-\beta q)k}{(1-k)+}\,\frac{\alpha\beta kq^3-\gamma q}{1+\ldots}, \qquad (16)$$

where ${}_2\Phi_1$ is the q-Gaussian hypergeometric series which includes the results due to B. Srivastava [1.2.1(3)] and Ramanujan (cf. Andrews [1.(14)]).

Denis [2], Verma, Denis and Rao [1] established the following results, respectively

$$\frac{{}_3\Phi_2\left[\begin{matrix}a, & b, & c;q; & ef/abc\\ & e, & f & \end{matrix}\right]}{{}_3\Phi_2\left[\begin{matrix}aq, & b, & c;q; & ef/abcq\\ & e, & f & \end{matrix}\right]}$$

$$=\frac{1}{1+}\,\frac{\eta(1-b)(1-c)}{\mu+}\,\frac{a(1-e/a)(1-f/a)}{1+}\,\frac{\eta(1-bq)(1-cq)}{\mu+}$$

$$\frac{a(1-eq/a)(1-fq/a)}{1+\ldots}, \qquad (17)$$

where $\eta = ef/abc$ and $\mu = (1-a)(1-\eta)$,

$$\frac{{}_3\Phi_2\left[\begin{matrix} a, & b, & c;q; & ef/abc \\ & e, & f & \end{matrix}\right]}{{}_3\Phi_2\left[\begin{matrix} aq, & b, & c;q; & ef/abcq \\ & e, & f & \end{matrix}\right]} = 1 + \frac{\alpha_0}{\beta_0 +}\frac{\gamma_0}{1+}\frac{\alpha_1}{\beta_1 +}\frac{\gamma_1}{1+\ldots}$$

where

$$\alpha_i = \frac{(1-aq^i)(1-bq^i)(1-cq^i)}{(1-eq^{2i})(1-eq^{2i+1})(1-fq^i)},$$

$$\beta_i = \frac{(1-ef/abc)}{(1-fq^i)},$$

$$\gamma_i = \frac{-(1-eq^{i+1/a})(1-eq^{i+1/b})(1-eq^{i+1/c})fq^i}{(1-eq^{2i+1})(1-eq^{2i+2})(1-fq^i)}$$

Denis [3] and Bhagirathi [1] have obtained a number of results involving bilateral basic-hypergeometric series and continued fractions.

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