CHAPTER 6

CERTAIN PARTITION THEOREMS

6.1 Introduction :

In this chapter, an attempt has been made to establish certain new partition theorems similar to Roger's- Ramanujan theorems by adopting the pattern of Hirschorn [1], Subbarao [1], Subbarao and Agarwal [1] and Singh, S.N. [3]. We consider the following identities due to Slater [5];

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q)_{2n}}=\frac{1}{(q^2,q^3,q^4,q^5,q^{11},q^{12},q^{13},q^{14};q^{16})_\infty} \tag{1}$$

[Slater[5];(83)P.160]

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q)_{2n}}=\frac{1}{(q,q^3,q^4,q^5,q^7,q^9,q^{11},q^{12},q^{13},q^{15}q^{16},q^{17},q^{19};q^{20})_\infty} \tag{2}$$

[Slater[5];(79)P.160]

$$\sum_{n=0}^{\infty}\frac{q^{2(2n+1)}}{(q;q)_{2n+1}}=\frac{1}{(q,q^3,q^5,q^7,q^9,q^{11},q^{13},q^{15};q^{16})_\infty} \tag{3}$$

[Slater[5];(84)P.161]

$$\sum_{n=0}^{\infty}\frac{q^{2(2n-1)}}{(q;q)_{2n}}=\frac{1}{(q,q^3,q^5,q^7,q^9,q^{11},q^{13},q^{15};q^{16})_\infty} \tag{4}$$

[Slater[5];(85)P.161]

$$\sum_{n=0}^{\infty}\frac{q^{2n(2n+1)}}{(q;q)_{2n+1}}=\frac{1}{(q,q^4,q^6,q^7,q^9,q^{10},q^{12},q^{15};q^{16})_\infty} \quad (5)$$

[Slater[5];(86)P.161]

We shall prove and establish the following partition theorems :

Theorem- 1

The number of partition of n into parts = 2, 3, 4, 5, 11, 12, 13 or 14 (mod 16) equals the number of n into even number of parts $\geq K$, where 2K is the number of parts in the partition.

Proof :

Let us consider a partition of n as

$$n = a_1 + a_2 + \ldots + a_k + a_{k+1} + \ldots + a_{2k} \quad (6)$$

where $a_1 \geq k,\ a_2 \geq k, \ldots a_{2k} \geq k$.

So, $a_1 + a_2 + \ldots + a_{2k} \geq 2k^2$.

Now, consider the following partition :

$$n - 2k^2 = (a_1 - k) + (a_2 - k) + \ldots + (a_{2k} - k),$$

which is a partition of n-$2k^2$ into at most 2k parts.

It is generated by, $\frac{q^{2k^2}}{(q;q)_{2k}}$ and so the series $\sum_{k=0}^{\infty}\frac{q^{2k^2}}{(q;q)_{2k}}$ gives the total number of partitions of type (7) or total number of partitions of n type (6) due to 1-1 correspondence between partitions of type (6) and (7). Hence the theorem-1 proved in view of the identity (6.1.1).

Theorem-2

The number of partitions of n into parts $\equiv 2,3,4,5,11,12,13$ or $14 \pmod{16}$ equals the number of partitions of n into even number of parts, say 2k such that in the first half part of the partition each part $\geq (k+1)$, and in the second half part of the partition each part $\geq (k-1)$.

Proof :

Let us consider a partition of n as

$$n = a_1 + a_2 + \ldots + a_k + a_{k+1} + \ldots + a_{2k} \tag{6}$$

where

$a_1 \geq (k+1), a_2 \geq (k+1), \ldots a_k \geq (k-1).$ and

$a_{k+1} \geq (k-1), a_{k+2} \geq (k-1), \ldots a_{2k} \geq (k-1).$

So, $a_1 + a_2 + \ldots + a_k + a_{k+1} + \ldots + a_{2k} \geq k(k+1) + k(k-1) \geq 2k^2.$

Now consider the following partition :

$$n - 2k^2 = \{a_1 - (k+1)\} + \ldots + \{a_{2k} - (k+1)\}$$

$$+ \{a_{k+1} - (k-1)\} + \ldots + \{a_{2k} - (k-1)\},$$

which is a partitiion of n - $2k^2$ into at most 2k parts.

It is generated by, $\dfrac{q^{2k^2}}{(q;q)_{2k}}$ and so the series $\sum_{k=0}^{\infty} \dfrac{q^{2k^2}}{(q;q)_{2k}}$ gives the total number of partition of type (9) or the total number of partitions of n of type (8) due to 1-1 correspondence between partitions of type (8) and (9). Hence the theorem 2, proved in view of the identity (6.1.1).

Theorem- 3

The number of partitions of n into parts = 1, 3, 4, 5, 7, 9, 11, 13, 15, 16, 17 or 19 (mod 20) equals the number of partitions of n into even number of parts, 2k, such that in the first half part of the partition each part $\geq (k-1)$.

Proof :

Let us consider a partition of n as

$$n = a_1 + a_2 + \ldots + a_k + a_{k+1} + a_{k+2} + \ldots + a_{2k} \tag{10}$$

where,

$a_1 \geq (k-1), a_2 \geq (k-1), \ldots a_k \geq (k-1).$ and

$a_{k+1} \geq 1,\ a_{k+2} \geq 1, \ldots a_{2k} \geq 1.$

So, $a_1 + a_2 + \ldots + a_k + a_{k+1} + \ldots + a_{2k} \geq k(k-1) + k \geq k^2.$

Now consider the following partition :

$$n - k^2 = \{a_1 - (k-1)\} + \{a_2 - (k-1)\} \ldots + \{a_k - (k-1)\}$$

$$+ \{a_{k+1} - 1\} + \ldots + \{a_{2k} - 1\},$$

which is a partitiion of n - k^2 into at most 2k parts.

It is generated by, $\frac{q^{2k^2}}{(q;q)_{2k}}$ and so the series $\sum_{k=0}^{\infty}\frac{q^{2k^2}}{(q;q)_{2k}}$ gives the total number of partition of type (11) or the total number of partitions of n of type (10) due to 1-1 correspondence between partitions of type (10) and (11). Hence the theorem 3, proved in view of the identity (6.1.2).

Theorem- 4

The number of partitions of n into parts $\equiv 1, 3, 5, 7, 9, 11, 13$ or 15 (mod 16) equals the number of partitions of n into even number of parts say, 2k+1, such that each part is $\geq k$.

Proof :

Let us consider a partition of n as

$$n = a_1 + a_2 + ... + a_{2k+1} \tag{12}$$

in which each part is $\geq k$.

$a_1 \geq k, a_2 \geq k, ... a_{2k+1} \geq k.$

So, $a_1 + a_2 + ... + a_{2k+1} \geq k(2k+1)$

Now consider the following partition :

$$n - k(2k+1) = (a_1 - k) + (a_2 - k) + ... + (a_{2k+1} - k)$$

which is a partitiion of n - k(2k+1) into at most (2k+1) parts.

It is generated by, $\frac{q^{k(2k+1)}}{(q;q)_{2k+1}}$ and so the series $\sum_{k=0}^{\infty}\frac{q^{2k(k+1)}}{(q;q)_{2k+1}}$ gives the total number of partition of type (13) or the total number of partitions of type (12) due to 1-1 correspondence between partitions of type (12) and (13). Hence the theorem 4 proved in view of the identity (6.1.3).

Theorem- 5

The number of partitions of n into parts $\equiv 1, 3, 5, 7, 9, 11, 13$ or $15 \pmod{16}$ equals the number of partitions of n into even number of parts, 2k, such that in the first half part of the partition each part is k and in the second half part of the partition each part is $\geq (k-1)$.

Proof :

Let us consider a partition of n as

$$n = a_1 + a_2 + \ldots + a_k + a_{k+1} + a_{k+2} + \ldots + a_{2k} \tag{14}$$

where

$$a_1 \geq k, a_2 \geq k, \ldots a_{2k+1} \geq k.$$

and $\quad a_{k+1} \geq (k-1), a_{k+2} \geq (k-1), \ldots, a_{2k} \geq (k-1).$

This gives that

$$a_1 + a_2 + \ldots + a_k + a_{k+1} + \ldots + a_{2k} \geq k^2 + k(k-1) \geq k(2k-1).$$

Now, consider the following partition :

$$n - k(2k-1) = (a_1 - k) + (a_2 - k) + \ldots + (a_k - k)$$

$$+\{a_{k+1} - (k-1)\} + \ldots + \{a_{2k} - (k-1)\}, \tag{15}$$

which is a partitiion of n - k(2k-1) into at most 2k parts.

It is generated by, $\frac{q^{k(2k-1)}}{(q;q)_{2k}}$ and so the series $\sum_{k=0}^{\infty}\frac{q^{k(2k-1)}}{(q;q)_{2k}}$ gives the total number of partition of type (15) or the total number of partitions of n of type (14) due to 1-1 correspondence between partitions of type (14) and (15). Hence the theorem 5 proved in view of the identity (6.1.4).

Theorem- 6

The number of partitions of n into parts $\equiv$ 1, 4, 6, 7, 9, 10, 12 or 15 (mod 16) equals the number of partitions of n into odd number of parts says 2k+1, such that in the first half part of the partition each part is $\geq$ k and in the second half part of the partition each part is $\geq (k+1)$. Partitions are taken in ascending order.

Proof :

Let us consider a partition of n as

$$n = a_1 + a_2 + \ldots + a_k + a_{k+1} + a_{k+2} + \ldots + a_{2k+1} \qquad (16)$$

where

$$a_1 \geq k, a_2 \geq k, \ldots a_{2k+1} \geq k.$$

and $\quad a_{k+2} \geq (k+1), \ldots, a_{2k+1} \geq (k+1).$

This gives that

$$a_1 + a_2 + \ldots + a_{k+1} + \ldots + a_{2k+1} \geq 2k(k+1).$$

Now, consider the following partition :

$$n - 2k(k-1) = a_1 - k + \ldots + a_{k+1} - k + (a_{k+2} - k + 1) + \ldots + (a_{2k+1} - k - 1), \quad (17)$$

which is a partitiion of n - 2k into at most 2k+1 parts.

It is generated by, $\frac{q^{k(2k+1)}}{(q;q)_{2k+1}}$ and so the series $\sum_{k=0}^{\infty} \frac{q^{2k(k+1)}}{(q;q)_{2k+1}}$ gives the total number of partition of type (17) or the total number of partitions of n of type (16) due to 1-1 correspondence between partitions of type (16) and (17). Hence the theorem 6, proved in view of the identity (6.1.5).

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