CHAPTER-5

CERTAIN RESULTS INVOLVING ORDINARY HYPERGEOMETRIC SERIES AND CONTINUED FRACTIONS

5.1 Introduction :

In this chapter, we shall make use of the following result due to Gauss [1], viz.;

$$\frac{{}_2F_1\left[\begin{matrix}a, & b+1; & z\\ & c+1 & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix}a, & b; & z\\ & c & \end{matrix}\right]}=\frac{1}{1-}\,\frac{a(c-b)z/c(c+1)}{1-}\,\frac{(b+1)(c-a+1)z/(c+1)(c+2)}{1-}$$

$$\times\frac{(a+1)(c-b+1)z/(c+2)(c+3)}{1-}\,\frac{(b+2)(c-a+2)z/(c+3)(c+4)}{1-\ldots}, \qquad (1)$$

in order to establish certain results involving ordinary hypergeometric series and continued fractions. We shall also be in need of following recurrence relations for ${}_2F_1[a, \ b; \ c, \ z]$ due to Gauss.

$$\{c-2a+(a-b)z\}{}_2F_1[a, \ b; \ c; \ z]+a(1-z){}_2F_1[a+1, \ b; \ c; \ z]$$

$$=(c-a){}_2F_1[a-1, \ b; \ c; \ z], \qquad (2)$$

$$(b-a){}_2F_1[a, \ b; \ c; \ z]+a\,{}_2F_1[a+1, \ b; \ c; \ z]$$

$$=b\,{}_2F_1[a, \ b+1; \ c; \ z], \qquad (3)$$

$$(c-a-b)\,{}_2F_1[a,\ b;\ c;\ z]+a(1-z)\,{}_2F_1[a+1,\ b;\ c;\ z]$$

$$=(c-b)\,{}_2F_1[a,\ b-1;\ c;\ z], \tag{4}$$

$$c\{a+(b-c)z\}F_1[a,\ b;\ c;\ z]+(c-a)(c-b)z\,{}_2F_1[a,\ b;\ c+1;\ z]$$

$$=ac(1-z)\,{}_2F_1[a+1,\ b;\ c;\ z], \tag{5}$$

$$(c-a-1)\,{}_2F_1[a,\ b;\ c;\ z]+a\,{}_2F_1[a+1,\ b;\ c;\ z]$$

$$=(c-1)\,{}_2F_1[a,\ b;\ c-1;\ z], \tag{6}$$

$$(c-a-b)\,{}_2F_1[a,\ b;\ c;\ z]+b(1-z)\,{}_2F_1[a,\ b+1;\ c;\ z]$$

$$=(c-a)\,{}_2F_1[a-1,\ b;\ c;\ z], \tag{7}$$

$$(b-a)(1-z)\,{}_2F_1[a,\ b;\ c;\ z]+(c-b)\,{}_2F_1[a,\ b-1;\ c;\ z]$$

$$=(c-a)\,{}_2F_1[a-1,\ b;\ c;\ z], \tag{8}$$

$$c(1-z)\,{}_2F_1[a,\ b;\ c;\ z]+(c-b)z\,{}_2F_1[a,\ b;\ c+1;\ z]$$

$$=c\,{}_2F_1[a-1,\ b;\ c;\ z], \tag{9}$$

$$\{a-1+(1+b-c)z\}\,{}_2F_1[a,\ b;\ c;\ z]+(c-a)\,{}_2F_1[a-1,\ b;\ c;\ z]$$

$$=(c-1)(1-z)\,{}_2F_1[a,\ b;\ c-1;\ z], \tag{10}$$

$$\{c-2b+(b-a)z\}\,{}_2F_1[a,\ b;\ c;\ z]+b(1-z)\,{}_2F_1[a,\ b+1;\ c;\ z]$$

$$=(c-b)\,{}_2F_1[a,\ b-1;\ c;\ z], \tag{11}$$

$$c\{b+(a-c)z\}\,{}_2F_1[a,\ b;\ c;\ z]+(c-a)(c-b)z\,{}_2F_1[a,\ b;\ c+1;\ z]$$

$$=bc(1-z)\,{}_2F_1[a,\ b+1;\ c;\ z], \tag{12}$$

$$(c-b-1)\ {}_2F_1[a, \quad b; \quad c; \quad z]+b\ {}_2F_1[a, \quad b+1; \quad c; \quad z]$$

$$=(c-1)\ {}_2F_1[a, \quad b; \quad c-1; \quad z], \tag{13}$$

$$c(1-z)\ {}_2F_1[a, \quad b; \quad c; \quad z]+(c-a)z\ {}_2F_1[a, \quad b; \quad c+1; \quad z]$$

$$=c\ {}_2F_1[a, \quad b; \quad c-1; \quad z], \tag{14}$$

$$\{b-1+(1+a-c)z\}\ {}_2F_1[a, \quad b; \quad c; \quad z]+(c-b)\ {}_2F_1[a, \quad b-1; \quad c; \quad z]$$

$$=(c-1)(1-z)\ {}_2F_1[a, \quad b; \quad c-1; \quad z], \tag{15}$$

$$c\{c-1+(1+a+b-2c)z\}\ {}_2F_1[a, \quad b; \quad c; \quad z]$$

$$+(c-a)(c-b)z\ {}_2F_1[a, \quad b; \quad c+1; \quad z]$$

$$=c(c-1)(1-z)\ {}_2F_1[a, \quad b; \quad c-1; \quad z], \tag{16}$$

5.2 Main Results:

In this section, we shall establish our main results :

(i) From (5.1.6) we have :

$$(c-a-1)\ {}_2F_1[a, \quad b; \quad c; \quad z]+a\ {}_2F_1[a+1, \quad b; \quad c; \quad z]$$

$$=(c-1)\ {}_2F_1[a, \quad b; \quad c-1; \quad z]$$

Now, replacing c by c + 1 in above result and then dividing throughout by

${}_2F_1[a, \quad b; \quad c; \quad z]$, we get :

$$\frac{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c+1 & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]}=\frac{c}{c-a}-\frac{a}{c-a}\frac{{}_2F_1\left[\begin{matrix} a+1, & b; & z \\ & c+1 & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]} \tag{17}$$

Using (5.1.1) after inchanging a and b in (5.2.17) we get :

$$\frac{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c+1 & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]}$$

$$= \frac{c}{c-a} - \frac{\frac{a}{c-a}}{1-} \frac{b(c-a)z/c(c+1)}{1-} \frac{(a+1)(c-b+1)z/(c+1)(c+2)}{1-}$$

$$\times \frac{(b+1)(c-a+1)z/(c+2)(c+3)}{1-} \frac{(a+2)(c-b+2)z \ (c+3)(c+4)}{1-....} \qquad (18)$$

(ii) From (5.1.13) we have :

$$(c-b+1)\ {}_2F_1[a, \ b; \ c; \ z] + b\ {}_2F_1[a, \ b+1; \ c; \ z]$$

$$= (c-1)\ {}_2F_1[a, \ b; \ c-1; \ z]$$

Now replacing c by c + 1 in above result and then dividing throughout by ${}_2F_1[a, \ b; \ c; \ z]$, we get :

$$\frac{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c+1 & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]} = \frac{c}{c-b} - \frac{b}{c-b} \frac{{}_2F_1\left[\begin{matrix} a, & b+1; & z \\ & c+1 & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]}$$

Using (5.1.1) in (5.2.19) we get :

$$\frac{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c+1 & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]}$$

$$= \frac{c}{c-b} - \frac{\frac{b}{c-b}}{1-}\,\frac{a(c-b)z/c(c+1)}{1-}\,\frac{(b+1)(c-a+1)z/(c+1)(c+2)}{1-}$$

$$\times \frac{(a+1)(c-b+1)z/(c+2)(c+3)}{1-}\,\frac{(b+2)(c-a+2)z/(c+3)(c+4)}{1-\ldots} \qquad (20)$$

(iii) From (5.1.14) we have :

$$c(1-z)\ {}_2F_1[a, \ b; \ c; \ z] + (c-a)z\ {}_2F_1[a, \ b; \ c+1; \ z]$$

$$= c\ {}_2F_1[a, \ b-1; \ c; \ z]$$

Now, replacing b by b + 1 in above result and then dividing throughout by ${}_2F_1[a, \ b; \ c; \ z]$, we get :

$$\frac{{}_2F_1\left[\begin{matrix} a, & b-1; & z \\ & c & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]} = (1-z) + \frac{(c-a)}{c} z \frac{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c+1 & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]}$$

Using (5.1.1) in (5.2.21) we get :

$$\frac{{}_2F_1\begin{bmatrix} a, & b-1; & z \\ & c & \end{bmatrix}}{{}_2F_1\begin{bmatrix} a, & b; & z \\ & c & \end{bmatrix}}$$

$$= 1 - \frac{az/c}{1-}\,\frac{b(c-a)z/c(c+1)}{1-}\,\frac{(a+1)(c-b+1)z/(c+1)(c+2)}{1-}$$

$$\times\frac{(b+1)(c-a+1)z/(c+2)(c+3)}{1-}\,\frac{(a+1)(c-b+2)z/(c+3)(c+4)}{1-\ldots} \qquad (22)$$

(iv) From (5.1.19) we have :

$$c(1-z)\ {}_2F_1[a, \quad b; \quad c; \quad z] + (c-b)z\ {}_2F_1[a, \quad b; \quad c+1; \quad z]$$

$$= c\ {}_2F_1[a-1, \quad b; \quad c; \quad z]$$

Now, dividing ${}_2F_1[a, \quad b; \quad c; \quad z]$, in above result, we get :

$$\frac{{}_2F_1\begin{bmatrix} a-1, & b; & z \\ & c & \end{bmatrix}}{{}_2F_1\begin{bmatrix} a, & b; & z \\ & c & \end{bmatrix}} = (1-z) + \frac{(c-b)}{c}z\frac{{}_2F_1\begin{bmatrix} a, & b; & z \\ & c+1 & \end{bmatrix}}{{}_2F_1\begin{bmatrix} a, & b; & z \\ & c & \end{bmatrix}} \qquad (23)$$

Using (5.2.22) after inchanging a and b in (5.2.23) we have :

$$\frac{{}_2F_1\begin{bmatrix} a-1, & b; & z \\ & c & \end{bmatrix}}{{}_2F_1\begin{bmatrix} a, & b; & z \\ & c & \end{bmatrix}}$$

$$= 1 - \frac{bz/c}{1-}\,\frac{a(c-b)z/c(c+1)}{1-}\,\frac{(b+1)(c-a+1)z/(c+1)(c+2)}{1-}$$

$$\times \frac{(a+1)(c-b+1)z/(c+2)(c+3)}{1-} \frac{(b+1)(c-a+2)z/(c+3)(c+4)}{1-\ldots} \tag{24}$$

(v) from (5.1.12) we have :

$$c\{b+(a-c)z\}\,{}_2F_1[a, \quad b; \quad c; \quad z] + (c-a)(c-b)z\,{}_2F_1[a, \quad b; \quad c+1; \quad z]$$

$$= bc(1-z)\,{}_2F_1[a, \quad b+1; \quad c; \quad z]$$

Now, replacing b by b-1 in above result and then dividing throughout by

${}_2F_1[a, \quad b; \quad c; \quad z]$, we get :

$$\frac{{}_2F_1\left[\begin{matrix} a, & b-1; & z \\ & c+1 & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]} = \frac{c(b-1)(1-z)}{(c-a)(c-b+1)}$$

$$-\frac{c\{b-1+(a-c)z\}}{(c-a)(c-b+1)z} \frac{{}_2F_1\left[\begin{matrix} a, & b-1; & z \\ & c+1 & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]} \tag{25}$$

Using (5.2.22) in (5.2.25), we get :

$$\frac{{}_2F_1\left[\begin{matrix} a, & b-1; & z \\ & c+1 & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]} = 1 - \frac{az/(c-b+1)}{1-} \frac{b(c-a)z/c(c+1)}{1-}$$

$$\times \frac{(a+1)(c-b+1)z/(c+1)(c+2)}{1-} \frac{(b+1)(c-a+1)z/(c+2)(c+3)}{1-}$$

$$\times \frac{(a+1)(c-b+2)z/(c+3)(c+4)}{1-\ldots} \tag{26}$$

(vi) From (5.1.14) we have :

$$c(1-z)\ {}_2F_1[a,\ \ b;\ \ c;\ \ z] + (c-a)z\ {}_2F_1[a,\ \ b;\ \ c+1;\ \ z]$$

$$= c\ {}_2F_1[a,\ \ b-1;\ \ c;\ \ z]$$

Now replacing b by b-1 in above result and then dividing throughout by

${}_2F_1[a,\ \ b;\ \ c;\ \ z]$, we get :

$$\frac{{}_2F_1\left[\begin{matrix} a, & b+1; & z \\ & c & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]} = \frac{1}{(1-z)} - \frac{(c-a)}{c(1-z)} z \frac{{}_2F_1\left[\begin{matrix} a, & b+1; & z \\ & c+1 & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]} \tag{27}$$

Using (5.1.1) in (5.2.27) we have :

$$\frac{{}_2F_1\left[\begin{matrix} a, & b+1; & z \\ & c & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]}$$

$$= \frac{1}{1-z} \frac{(c-a)z/c(1-z)}{1-} \frac{a(c-b)z/c(c+1)}{1-} \frac{(b+1)(c-a+1)z/(c+1)(c+2)}{1-}$$

$$\times \frac{(a+1)(c-b+1)z/(c+2)(c+3)}{1-} \frac{(b+1)(c-a+2)z/(c+3)(c+4)}{1-\ldots} \tag{28}$$

(vii) Now, replacing b by a in (5.2.28), we get :

$$\frac{{}_2F_1\begin{bmatrix} a+1, & b; & z \\ & c & \end{bmatrix}}{{}_2F_1\begin{bmatrix} a, & b; & z \\ & c & \end{bmatrix}} = \frac{1}{1-z} - \frac{(c-b)z/c(1-z)}{1-} \frac{b(c-a)z \;\; c(c+1)}{1-}$$

$$\times \frac{(a+1)(c-b+1)z/(c+1)(c+2)}{1-} \frac{(b+1)(c-a+1)z/(c+2)(c+3)}{1-} \times$$

$$\times \frac{(a+1)(c-b+2)z/(c+3)(c+4)}{1-\ldots} \qquad (29)$$

(viii) From (5.1.4) we have :

$$(c-a-b)\; {}_2F_1[a, \;\; b; \;\; c; \;\; z] + a(1-z)\; {}_2F_1[a+1, \;\; b; \;\; c; \;\; z]$$

$$= (c-b)\; {}_2F_1[a, \;\; b-1; \;\; c; \;\; z]$$

Now, replacing b by b+1 in above result and then dividing throughout by

${}_2F_1[a, \;\; b; \;\; c; \;\; z]$, we get :

$$\frac{{}_2F_1\begin{bmatrix} a+1, & b+1; & z \\ & c & \end{bmatrix}}{{}_2F_1\begin{bmatrix} a, & b; & z \\ & c & \end{bmatrix}} = \frac{(c-b-1)}{a(1-z)} - \frac{(c-a-b-1)}{a(1-z)} \frac{{}_2F_1\begin{bmatrix} a, & b+1; & z \\ & c & \end{bmatrix}}{{}_2F_1\begin{bmatrix} a, & b; & z \\ & c & \end{bmatrix}} \qquad (30)$$

Using (5.2.28) in (5.2.30) we get :

$$\frac{{}_2F_1\left[\begin{matrix} a+1, & b+1; & z \\ & c & \end{matrix}\right]}{{}_2F_1\left[\begin{matrix} a, & b; & z \\ & c & \end{matrix}\right]}$$

$$= \frac{a-(c-b-1)z}{a(1-z)^2} - \frac{(c-a)(c-a-b-1)z/ac(1-z)^2}{1-} \frac{a(c-b)z/c(c+1)}{1-}$$

$$\times \frac{(b+1)(c-a+1)z/(c+1)(c+2)}{1-} \frac{(a+1)(c-b+1)z/(c+2)(c+3)}{1-}$$

$$\frac{(b+1)(c-a+2)z/(c+3)(c+4)}{1-\ldots}. \tag{31}$$

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