APPENDIX
aim of this short note is to prove a theorem on $C_1 (N, p, q)$ summability under very general

\textbf{INTRODUCTION : For any sequence $\{p_n\}$, $\{q_n\}$ and $\{S_n\}$, we write}

$$t_n^{p, q} = R_n^{-1} \sum_{k=0}^{n} p_{n-k} q_k S_k$$

re

$$R_n = \sum_{k=0}^{n} p_{n-k} q_k \neq 0 \text{ for all } n$$

generalised Nörlund transform $(\langle N, p, q \rangle$ transform) of the sequence $\{S_n\}$ is the sequence

$$S_n \rightarrow S(N, p, q)$$

\(f(x)\) be a periodic function with period $2\pi$ and integrable in the sense of Lebesgue over an
val $(-\pi, \pi)$. Let the Fourier series of $f(x)$ be

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x) \quad \ldots (1.1)$$

\text{series conjugate to (1.1) is}

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=0}^{\infty} B_n(x) \quad \ldots (1.2)$$

write

$$\Psi(t) = f(x+t) - f(x-t) - l, \text{ where } l \text{ is a constant.}$$

$$p(1/t) = p(1/t), \text{ } [1/t] \text{ denotes the integral part of } 1/t$$

$$\Psi(t) = \int_{0}^{t} |\psi(u)| \text{ } du.$$  

\text{1969, DIKSHIT PROVED THE FOLLOWING :}

\textbf{THEOREM A : If}

$$\Psi(t) = \left[ \int_{0}^{t} \frac{1}{p_t} \right],$$

\rightarrow 0, \text{ then the sequence } \{n B_n(x)\} \text{ is summable } (C, 1), (N, p_n) \text{ to the value } \frac{l}{\pi}.$$

\text{object of the present paper is to study } C_1 \cdot (N, p, q) \text{ summability of a sequence of Fourier
ficients under less stringent conditions on the sequence } \{p_n\} \text{ and under more general
litions on the function. Namely, we prove the following :}
THEOREM: Let \( p(u) \) be a monotonic, non-increasing strictly positive for \( u \geq 0 \), \( p_n = j \) and \( q(u) \) be a monotonic nondecreasing, strictly positive for \( u \geq 0 \), \( q_n = q(n) \),

\[
R(u) = \int_0^u p(x) q(u-x) \, dx
\]

and

\[
q_n \int_0^u \frac{\lambda(u)}{u} \, du = 0(R_n), \text{ as } n \to \infty
\]

where \( \lambda(t) \) is a suitable positive, non-decreasing function of \( t \), such that

\[
\lambda(n) = o(P_n), \text{ as } n \to \infty.
\]

If

\[
\Psi(t) = 0 \left[ \frac{\Delta(1/t)}{P(1/t)} \right], \text{ as } t \to +0
\]

then the sequence \( \{n B_n(x)\} \) is summable \( C_1(N, p, q) \) to the value \( \frac{1}{\pi} \).

3. WE REQUIRE THE FOLLOWING LEMMA TO PROVE OUR THEOREM:

**LEMMA.** (Dikshit 1969): If \( \{p_n\} \) is non-negative and non-increasing then as \( n \to \infty \),

\[
\sum_{v=0}^{n} \frac{1}{P_v} \sum_{k=0}^{v} (v-k) p_k \exp \{i(v-k) t \} = 0(r^{-2}) + 0 \left( \frac{r^{-1} P_v}{P_{n+1}} \right)
\]

uniformly in \( 0 < t \leq \pi \).

4. PROOF OF THE THEOREM:

We know that

\[
n B_n(x) = \frac{1}{\pi} \int_0^\pi \left[ f(x+t) - f(x-t) \right] u \sin nt \, dt
\]

\[
= \frac{1}{\pi} \int_0^\pi \psi(t) n \sin nt \, dt + \frac{1}{\pi} \left[ 1 - (-1)^n \right]
\]

Therefore, if \( \{t_n(x)\} \) denotes the \( \{N, p, q\} \) mean of \( \{n B_n(x)\} \) then

\[
t_n(x) - \frac{1}{\pi} = \frac{1}{\pi R_n} \int_0^\pi \psi(t) \left( \sum_{k=0}^{n} p_{n-k} k \sin kt \right) dt - \frac{1}{\pi R_n} \sum_{k=0}^{n} p_{n-k} (-1)^k
\]

Since \( \{p_n\} \) is non-negative and non-increasing and \( R_n \to \infty \), as \( n \to \infty \) we have by an application of Abel's lemma

\[
t_n(x) - \frac{1}{\pi} = 0(1) + \frac{1}{\pi R_n} \int_0^\pi \psi(t) \left( \sum_{k=0}^{n} p_{n-k} k \sin kt \right) dt
\]

Denoting the \( C_1 \), \( \{N, p, q\} \) mean of \( \{n B_n(x)\} \) by \( t'_n(x) \), we write

\[
t'_n(x) - \frac{1}{\pi} = 0(1) + \frac{1}{\pi(n+1)} \int_0^\pi \psi(t) \left( \sum_{v=0}^{n} \frac{1}{P_v} \sum_{k=0}^{v} p_{v-k} k \sin kt \right) dt
\]

by the regularity of the \( (C, 1) \) mean.

In order to prove the theorem, it is sufficient to show that as \( n \to \infty \)

\[
1 = -\frac{1}{n+1} \int_0^\pi \psi(t) g(n, t) \, dt = 0(1)
\]
\[ g(n, t) = \sum_{v=0}^{n} \frac{1}{P_v} \sum_{k=0}^{v} p_k (v-k) \sin (v-k) t \]

is a sufficient small, such that \(0 < \delta < \pi\), let us write

\[
I = \frac{1}{n+1} \int_0^{1/n} \psi(t) g(n, t) dt + \frac{1}{n+1} \int_{1/n}^{\delta} \psi(t) g(n, t) dt + \frac{1}{n+1} \int_{1/n}^{\pi} \psi(t) g(n, t) dt
\]

\[= I_1 + I_2 + I_3, \text{ say.} \quad \ldots \quad (4.2)\]

and

\[ g(n, t) \leq \sum_{v=0}^{n} \frac{1}{P_v} \sum_{k=0}^{v} p_k (v-k) \leq An^2 \]

have by the hypothesis (2.4)

\[
I_1 = 0 \left( n \int_0^{1/n} \left| \psi(t) \right| dt \right)
\]

\[= 0 \left( n \cdot 0 \left( \frac{\lambda(n)}{n^P_n} \right) \right) \]

\[= 0(1), \text{ as } n \to \infty. \quad \ldots \quad (4.3)\]

\[ \text{By virtue of lemma for the interval } \delta \leq t \leq \pi, \text{ we have} \]

\[
\frac{1}{n} g(n, t) = 0 \left( \frac{1}{n} \right) + 0 \left( 0 \right) + 0(1), \text{ as } n \to \infty
\]

since the \((C, 1)\) mean is regular and \(P_n \to \infty\), as \(n \to \infty\). Thus as \(n \to \infty\)

\[
I_3 = \frac{1}{n+1} \int_{1/n}^{\pi} \psi(t) g(n, t) dt = 0(1) \quad \ldots \quad (4.4)
\]

\[ \text{illy, to prove that } I_2 = 0(1), n \to \infty, \text{ we observe that by lemma} \]

\[
2 = \left[ \frac{1}{n+1} \int_{1/n}^{\delta} \psi(t) g(n, t) dt \right]
\]

\[= 0 \left[ \frac{1}{n} \int_{1/n}^{\delta} t^{-2} \left| \psi(t) \right| dt \right] + 0 \left[ \frac{1}{n} \int_{1/n}^{\delta} \left| \psi(t) \right| \left( \frac{1}{P_n} \sum_{v=0}^{n} \frac{1}{P_v} \right) dt \right] + 0 \left[ \frac{\theta_n}{R_n} \int_{1/n}^{\delta} \frac{\left| \psi(t) \right|}{t} dt \right]
\]

\[= 0(1) + 0(1) + 0(1), \text{ say} \quad \ldots \quad (4.5)\]

\[
i_{2.1} = \frac{1}{n} \left[ t^{-2} \psi(t) \right]_{1/n}^{\delta} + \frac{2}{n} \left[ \psi(t) \right]_{1/n}^{\delta} dt
\]

\[= 0 \left[ \frac{1}{n} \left( \frac{\lambda(1/t)}{t^2 P_{1/n}} \right) \right]_{1/n}^{\delta} + 0(1) + 0 \left( \sum_{k=1}^{n-1} \Psi(1/u) u du \right)\]

\[= 0(1) + 0 \left( \frac{\lambda(n)}{P_n} \right) + 0 \left( \sum_{k=1}^{n-1} \Psi(1/u) u du \right)
\]
\[ \int_k^{k+1} \Psi(1/u) \, du \leq \Psi(1/k) \]

\[ = 0 \left[ \frac{\lambda(k)}{P_k} \right] \]
\[ = 0(1), \text{ as } k \to \infty. \]

So

\[ I_{2.1} = 0(1) + \frac{2}{n} \sum_{k=1}^{n-1} [O(1)] \]
\[ = 0(1) \]

as \( n \to \infty \), by the hypothesis (2.3) and (2.4).

Let

\[ V(n, \nu) = P_v \sum_{k=v}^{n} \left( \frac{1}{P_k} \right) \]

and \( n^{-1} < m^{-1} < (m-1)^{-1} \),

\( m \) is any integer, then we write

\[ I_{2.2} = \frac{1}{n} \left( \sum_{v=m}^{n-1} \int_{(v+1)^{-1}}^{v^{-1}} \frac{\psi(t)}{t} \, V(n, \tau) \, dt \right) \]
\[ = I_{2.2.1} + I_{2.2.2}, \]

say.

In the interval \((v+1)^{-1} < t \leq v^{-1}\), we have \( V(n, \tau) = V(n, \nu) \)

\[ \int_{(v+1)^{-1}}^{v^{-1}} V(n, \tau) \left( \frac{\psi(t)}{t} \right) \, dt = V(n, \nu) \left[ \left( \frac{\psi(t)}{t} \right) \right]_{(v+1)^{-1}}^{v^{-1}} + \int_{(v+1)^{-1}}^{v^{-1}} V(n, \tau) \left( \frac{\psi(t)}{t^2} \right) \, dt \]

Further by (2.3) and (2.4), we have

\[ = \sum_{v=m}^{n-1} V(n, \nu) \left[ \nu \Psi \left( \frac{1}{\nu} \right) - (v+1) \Psi \left( \frac{1}{v+1} \right) \right] \]
\[ = \sum_{v=m}^{n-1} \Delta_v \left[ \nu \Psi \left( \frac{1}{\nu} \right) - \sum_{v=m}^{n-1} (v+1) \Psi \left( \frac{1}{v+1} \right) \right] \]
\[ = m \Psi \left( \frac{1}{m} \right) V(n, m) - n \nu \Psi \left( \frac{1}{n} \right) \Psi \left( \frac{1}{n+1} \right) \]
\[ + \sum_{v=m}^{n-1} \frac{\lambda(v+1)}{P_{v+1}} \sum_{k=v}^{n-1} \frac{1}{P_k} \]
\[ = 0 \left[ \frac{m \cdot \frac{1}{m} \lambda(m) \sum_{k=m}^{n-1} \frac{1}{P_k}}{P_m} \right] + O(1) + 0 \left[ \sum_{v=m}^{n-1} \frac{1}{P_{v+1}} \sum_{k=v}^{n-1} \frac{1}{P_k} \right] \]
\[ = 0 \left[ \frac{P_m}{P_m} \sum_{k=m}^{n-1} \frac{1}{P_k} \right] + O(1) + 0 \left[ \frac{1}{P_k} \sum_{v=m}^{n-1} P_{v+1} \right] + 0(n) \]
\[ = 0(n), \text{ as } n \to \infty. \]

combining (4.8) and (4.9), we get

\[ I_{2.2.1} = 0(1) + \int_{n^{-1}}^{m^{-1}} \frac{\psi(t)}{t^2} \, V(n, \tau) \, dt \]... (4).
Similarly it follows that

\[ I_{2,2,2} = 0(1) + \int_{-n}^{n} (\delta \Psi(t) \frac{1}{t^2} \mathbf{V}(n, \tau)) dt \]  

Thus, we have from (4.10) and (4.11)

\[ I_{2,2} = 0(1) + \int_{-n}^{n} (\frac{1}{n^2} \mathbf{V}(n, \tau)) dt \]

\[ = 0(1) + \frac{1}{n} \int_{-n}^{n} (\Psi(1) \mathbf{V}(n, u)) du \]

\[ = 0(1) + \frac{1}{n} \sum_{k=1}^{n} \mathbf{V}(n, k) \Psi(1/k) \]

\[ = 0(1) + O(1) \]

\[ = 0(1) + O(1), \text{ as } n \to \infty. \]  

\[ I_{2.3} = \frac{q_n}{R_n} \int_{1/n}^{\delta} \left( \frac{\Psi(t)}{t} \right)^{1/n} dt \]

\[ = \frac{q_n}{R_n} \left[ (\Psi(t) \frac{P(1/t)}{t})^{1/n} \right] + \frac{q_n}{R_n} \int_{1/n}^{\delta} (\Psi(t) \frac{P(1/t)}{t^2}) dt + \frac{q_n}{R_n} \int_{1/n}^{\delta} \Psi(t) \frac{1}{t} dP(1/t) \]

\[ = I_{2.3,1} + I_{2.3,2} + I_{2.3,3}, \text{ say} \]  

\[ I_{2.3,1} = \frac{q_n}{R_n} O(\lambda(1/n))^{1/n} \]

\[ = 0 \left[ \frac{q_n \lambda(n)}{R_n} \right] \]

\[ = 0(1), \text{ as } n \to \infty. \]  

\[ I_{2.3,2} = \frac{q_n}{R_n} \int_{1/n}^{\delta} (\Psi(t) \frac{P(1/t)}{t^2}) dt \]

\[ = \frac{q_n}{R_n} \left[ \int_{1/n}^{\delta} \frac{(1/t) P(1/t)}{t^2} dt \right] \]

\[ = \frac{q_n}{R_n} \left[ \int_{1/n}^{\delta} \frac{\lambda(x)}{x} dx \right] \]

\[ = 0(1), \text{ as } n \to \infty. \]
Combining (4.5)–(4.16), we find that
\[ I_2 = 0(1) \] \hspace{1cm} \text{(6.16)}

Now from (4.2), (4.3), (4.4) and (4.17), we get (4.1). This completes the proof of the theorem.

**COROLLARY :** If we put \( \lambda(u) = 1 \) and \( q_n = 1 \) for all \( n \), we get the result of Diskhit (1969).

**REFERENCES :**