Chapter - 7

UNIFORM NORLUND SUMMABILITY OF FOURIER SERIES AND OF CONJUGATE FOURIER SERIES
7.1 Two theorems on uniform Nörlund summability of Fourier series and of conjugate Fourier series are proved, which improve and extend the uniform harmonic summability theorems of Saxena [49].

7.2 In this chapter we let \( f : \mathbb{R} \to \mathbb{R} \) be \( 2\pi \) periodic and Lebesgue integrable on \([-\pi, \pi]\), and denote its Fourier series by

\[
\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

writing \( b_0 = 0 \), the conjugate Fourier series of \( f \) is

\[
\sum_{n=0}^{\infty} (a_n \sin nx - b_n \cos nx)
\]

We also let \( p \) denote a fixed positive integer, and \((q_n)_{n=0}^{\infty}\) denote a \( p \)-cyclically decreasing sequence of real numbers in the sense that \( q_n > q_{n+p} \) for all \( n \). We assume further that \( q_n > 0 \) for all \( n \) and the series \( \sum q_n \) is divergent.

For \( n=0, 1 \ldots \) and any real \( x \), we denote

\[
S_n(x) = \text{the } (n+1)\text{th partial sum of the series } (7.2.1).
\]
\( \bar{S}_n(x) = \) the \((n + 1)\)th partial sum of the series (7.2.2.)

\[ Q_n = q_0 + q_1 + \ldots + q_n. \]

\( \phi_x(t) = f(x + t) + f(x - t) - 2f(x), \)

\( \psi_x(t) = f(x + 2t) - f(x - 2t) \)

\[ \Phi_x(t) = \int_0^t |\phi_x(u)| \, du, \]

\[ \Psi_x(t) = \int_0^t |\psi_x(u)| \, du, \]

\[ \bar{f}_n(x) = -\frac{1}{\pi} \int_{\frac{n+1}{t}}^\infty \psi_x(t) \cot t \, dt, \]

\[ \tau = \left[ \frac{1}{t} \right], \]

and whenever meaningful, we define

(7.2.3.) \( \bar{f}(x) = \lim_{n \to \infty} \bar{f}_n(x) \)

A series of function \( \sum_{n=0}^{\infty} u_n(x) \) is said to be summable \((N, q_n)\) uniformly on a set \( E \), to a function \( U \), if its sequence of Nörlund means, Nörlund [38], Hardy [11],

\[ \frac{1}{Q_n} \sum_{k=0}^{n} q_k (u_0(x) + u_1(x) + \ldots + u_{n-1}(x)) \]

converges uniformly on \( E \) to the values \( U(x) \).

In the particular case \( q_n = \frac{1}{n + 1} \), Saxena [49] proved that if

(7.2.4) \( \Phi_x(t) = 0 \left( \frac{t}{\log \frac{1}{t}} \right), \quad t \to 0^+ \)
uniformly on a set E where f is bounded, then the series (1.2.1) is summable 
\( N, \frac{1}{n+1} \) uniformly on E to f. He also proved a similar result for the conjugate 
series (7.2.2).

7.3 The object of this chapter is to improve and extend the results of 
Saxena to our general \((q_n)\), with more elegant method of proof.

We observe that, when \( q_n = \frac{1}{n+1} \), then \((q_n)\) is strictly monotonic 
decreasing, (so 1. cyclically decreasing), \( q_n \to 0, Q_n \to \infty, \frac{\log n}{Q_n} \to 1 \) and, as 
can be easily seen, the condition (7.2.4) implies

\[ \sum_{k=1}^{\infty} Q_k \Phi_{k} = o \left( Q_n \right), n \to \infty \]

**Theorem 1.** Let \( \{q_n\} \) be a sequence such that \( q_n > 0, q_n \downarrow \) and \( Q_n \to \infty \) and \( \lambda(t) \) 
be a positive monotonic, non-decreasing function such that,

(7.3.1) \[ \lambda(n) \log n = O \left( Q_n \right), \quad (n \to \infty) \]

If

(7.3.2) \[ \Phi_x(t) = \int_{0}^{t} |\phi_x(u)| \, du = o \left( \frac{\lambda(1)}{Q_t} \right), (t \to 0^+) \]

uniformly in a set E in which f is bounded, then the series (7.2.1) is summable 
\((N, q_n)\) uniformly in E to the sum f(x).

**Theorem 2:** If the sequence \( \{q_n\} \) and function \( \lambda(t) \) be same as in theorem 1, 
then if

(7.3.3) \[ \Psi_x(t) = \int_{0}^{t} \left| \psi_x(u) \right| \, du = o \left( \frac{t \lambda(1)}{Q_t} \right), t \to 0^+ \]
uniformly in a set $E$ then the series (7.2.2) is summable $(N, q_n)$ uniformly in $E$ to the sum $f(x)$, provided that the limit in (7.2.3) exists uniformly in $E$.

7.4. The following lemma is required in proof of our theorems.

Lemma: (Mcfadden [33]).

For $0 < a < b < \infty$, $0 \leq t \leq \pi$ and any $n$,

$$\left| \sum_{k=a}^{b} q_k e^{(n-k)t} \right| \leq C Q,$$

where $C$ is an absolute constant.

7.5 PROOF OF THE THEOREM 1

Let $\delta$ be a fixed positive number less than $\frac{1}{2}$, we have

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_0^\delta \phi_x(t) \frac{\sin \left( \frac{n + \frac{1}{2}}{2} \right) t}{\sin \frac{1}{2} t} \, dt + o(1)$$

where $o(1)$ is a magnitude that tends to zero for any $x$ and uniformly in $E$ (i.e. in a set on which $f$ is bounded).

Therefore

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_0^\delta \phi_x(t) \frac{\sin \left( \frac{n - \frac{1}{2}}{2} \right) t}{\sin \frac{1}{2} t} \, dt + o(1)$$

Now,

$$\frac{1}{Q_n} \sum_{r=0}^{n} q_r [S_{n,r}(x) - f(x)]$$

$$= \frac{1}{Q_n} \sum_{r=0}^{n} q_r \frac{1}{2\pi} \int_0^\delta \phi_x(t) \frac{\sin \left( \frac{n - \frac{1}{2}}{2} \right) t}{\sin \frac{1}{2} t} \, dt + o(1)$$

uniformly in $E$. 


In order to prove our theorem, we have to show that under our assumption

\[ I_1 = o(1) \text{ and } I_2 = o(1), \quad n \to \infty \]

uniformly in the set \( E \).

Since \( |\sin \theta| \leq |\theta| \) and \( \sin \frac{1}{2}t \geq \frac{t}{\pi} \) for \( 0 \leq t \leq \pi \), clearly

(7.5.3) \[ N_n(t) = O(n) \]

uniformly, \( 0 < t \leq \frac{1}{n} \) and making use of the Lemma, uniformly in \( 0 < t \leq \pi \)

(7.5.4) \[ N_n(t) = O\left[ \frac{Q_n}{tQ_n} \right] \]

Now,

\[ I_1 = O(n) \int_{0}^{1} |\phi_{\epsilon}(t)| dt \]
(97)

\[ I_1 = o\left( \frac{1}{n} \frac{\lambda(n)}{Q_n} \right) \]

(7.5.5) \[ I_1 = o(1), \ n \to \infty \]

uniformly in \( E \)

Also

\[ I_2 = \int_{\frac{1}{n}}^{\delta} \phi_x(t)N_n(t)dt \]

\[ I_2 = O\left( \frac{1}{Q_n} \right) \int_{\frac{1}{n}}^{\delta} |\phi_x(t)| \frac{Q_t}{t} dt \]

\[ = O\left( \frac{1}{Q_n} \right) \left[ \Phi_x(t) \frac{Q_t}{t} \right]_{\frac{1}{n}}^{\delta} + \int_{\frac{1}{n}}^{\delta} \Phi_x(t) \frac{Q_t}{t^2} dt - \int_{\frac{1}{n}}^{\delta} \Phi_x(t) \frac{1}{t} dQ_t \]

\[ = o(1) + o\left( \frac{1}{Q_n} \right) \int_{\frac{1}{n}}^{\delta} \frac{\lambda(u)}{u} du + O\left( \frac{1}{Q_n} \right) \int_{\frac{1}{n}}^{\delta} \Phi_x\left( \frac{1}{u} \right) udQ_{[u]} \]

\[ = o(1) + o\left( \frac{\lambda(n)}{Q_n} \right) \int_{1}^{u} \frac{du}{u} + O\left( \frac{1}{Q_n} \right) \sum_{k=1}^{\log n} k Q_{\Phi_x\left( \frac{1}{k} \right)} \]

\[ = o(1) + o\left( \frac{\lambda(n) \log n}{Q_n} \right) + O\left( \frac{1}{Q_n} \right) \sum_{k=1}^{\log n} Q_k \Phi_x\left( \frac{1}{k} \right) \]

\[ = o(1) + o(1) + o\left( \frac{1}{Q_n} \sum_{k=1}^{\log n} \frac{\lambda(k)}{k} \right) \]

\[ = o(1) + o\left( \frac{1}{Q_n} \int_{1}^{\delta} \frac{\lambda(u)}{u} du \right) \]
\[ o(1) + o\left(\frac{\lambda(n) \log n}{Q_n}\right) \]

\[ = o(1) + o(1) \]

(7.5.6) \[ I_2 = o(1), \text{ as } n \to \infty \]

uniformly in E.

Now from (7.5.5) and (7.5.6) we get (7.5.2) which completely the proof of the theorem 1.

**PROOF OF THEOREM 2**: We know that

\[ \overline{S}_n(x) = -\frac{1}{2} \int_0^{\pi/2} \psi_x(t) \frac{\{\cos - \cos (2n+1)t\}}{\sin t} dt \]

Therefore

\[ \overline{S}_n(x) - f(x) = \frac{1}{2} \int_0^{\pi/2} \psi_x(t) \frac{\cos (2n+1)t}{\sin t} dt \]

Now,

\[ \frac{1}{Q_n} \sum_{v=0}^{n} q_{n-v} \left\{ \overline{S}_v(x) - \bar{f}(x) \right\} \]

\[ = \frac{1}{Q_n} \sum_{v=0}^{n} q_{n-v} \int_0^{\pi/2} \psi_x(t) \frac{\cos (2v+1)t}{\sin t} dt \]

\[ = \int_0^{\pi/2} \psi_x(t) \frac{1}{\pi Q_n} \sum_{v=0}^{n} q_{n-v} \frac{\cos (2v+1)t}{\sin t} dt \]

\[ = \int_0^{\pi/2} \psi_x(t) M_n(t) dt \]
Where,

$$M_n(t) = \frac{1}{\pi Q_n} \sum_{v=0}^{n} q_{n-v} \cos \frac{(2v+1)t}{\sin t} dt$$

In order to prove the theorem, we have to show that

$$\int_{0}^{\pi} \psi_x(t)M_n(t)dt = o(1) \text{ uniformly in } E, \text{ as } n \to \infty$$

Write,

$$\int_{0}^{\pi} \psi_x(t)M_n(t)dt = \left( \int_{0}^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\frac{2\pi}{n}} + \cdots + \int_{\frac{(n-1)\pi}{n}}^{\pi} \right) \psi_x(t)M_n(t)dt$$

(7.5.7) \[= P_1 + P_2 + P_3, \text{ say}\]

Since by our assumption \(f_n(x) \to f(x)\) uniformly in \(E\)

(7.5.8) \[-\frac{1}{\pi} \int_{0}^{\pi} \psi_x(t) \cot t dt = o(1) \text{ uniformly in } E, \text{ as } n \to \infty\]

Also for all \(t\) such that \(0 < t \leq \pi\)

$$-\frac{1}{\pi Q_n} \sum_{v=0}^{n} q_{n-v} \frac{\cos t - \cos(2v+1)t}{\sin t}$$

$$= O\left( \frac{1}{Q_n} \sum_{v=0}^{n} q_{n-v} \sum_{k=1}^{v} |\sin kt| \right)$$

$$= O\left( \frac{n}{Q_n} \sum_{v=0}^{n} q_{n-v} \right)$$

(7.5.9) \[= O(n)\]

Making use of the estimates in (7.5.8) and (7.5.9), we have
\[
P_i = \int_0^\infty \psi_x(t) \left\{ -\frac{1}{\pi Q_n} \sum_{v=0}^n q_{n-v} \frac{\cos t - \cos(2v+1)t}{\sin t} \right\} dt
\]
\[
+ \frac{1}{\pi Q_n} \sum_{v=0}^n q_{n-v} \int_0^\infty \psi_x(t) \cot t \ dt
\]

\[
P_1 = O \left[ n \psi \left( \frac{1}{n} \right) \right] + o(1)
\]

\[
P_1 = o \left[ n \frac{\lambda(n)}{Q_n} \right]
\]

(7.5.10) \quad P_1 = o(1), \quad n \to \infty

uniformly in \( E \), by the hypothesis of the theorem and the regularity of the method of summation. By virtue of the Lemma, we have clearly

\[
M_n(t) = O \left[ \frac{Q_n}{t Q_n} \right]
\]

so that,

\[
P_2 = O \left[ \frac{1}{Q_n} \int \psi_x(t) \frac{Q_n}{t} \ dt \right]
\]

(7.5.11) \quad P_2 = o(1), \quad n \to \infty

uniformly in \( E \), as in the proof of theorem 1.

Finally

\[
P_3 = \frac{1}{\pi Q_n} \sum_{v=0}^n q_{n-v} \int_0^\infty f(x+2t) \frac{\cos(2v+1)t}{\sin t} \ dt - \frac{1}{\pi Q_n} \sum_{v=0}^n q_{n-v} \int_0^\infty f(x-2t) \frac{\cos(2v+1)t}{\sin t} \ dt
\]

(7.5.12) \quad = o(1) \ as \ n \to \infty
Uniformly in $E$, by virtue of a known result (Hardy and Rogosinski [14], theorem 31) and the regularity of the method of summation.

Hence from (7.5.7), (7.5.10), (7.5.11) and (7.5.12) we have

$$\frac{1}{Q_n} \sum_{v=0}^{n} q_{n-v} \left[ S_v(x) - f(x) \right] = o(1)$$

uniformly in $E$. This completes the proof.