Chapter - 6

ON UNIFORM MATRIX SUMMABILITY OF A FOURIER SERIES
ON UNIFORM MATRIX SUMMABILITY OF A FOURIER SERIES

6.1 Let \( \{p_n\} \) be a sequence of numbers such that

\[
p_n = p_0 + p_1 + \ldots + p_n; (P_{-1} = p_{-1} = 0)
\]

Let \( f \in L(-\pi, \pi) \) and be periodic with period \( 2\pi \) and

\[
(6.1.1.) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)
\]

be the Fourier series of the function \( f \). We shall use following notations.

\[
\phi(t) = \phi(x, t) = f(x + t) + f(x - t) - 2S,
\]

\( s \) being constant.

\[
\Phi(t) = \int_{-t}^{t} |\phi(u)| du
\]

\[
A_{n, \tau} = \sum_{k=0}^{\tau} a_{n,n-k} \text{ where } \tau = \text{Integral part of } \frac{1}{t} = \left[ \frac{1}{t} \right]
\]

\[
M_n = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\sin \left( \frac{n-k+1}{2} \right) t}{\sin(t/2)}
\]

6.2 Let \( T = (a_{n,k}) \) be an infinite triangular matrix satisfying the Silverman-Toeplitz \[1913\] condition of regularity, i.e.

\[
\sum_{k=0}^{n} a_{n,k} \rightarrow 1, \quad \text{as } n \rightarrow \infty
\]
\[ a_{nk} = 0 \text{ for } k > n \]

and

\[
\sum_{k=0}^{n} |a_{nk}| \leq m, \text{ a finite constant}
\]

Let

\[
\sum_{m=0}^{\infty} u_m(x) \text{ be an finite series}
\]

such that

\[
U_k(x) = u_0(x) + u_1(x) + \cdots + u_k(x)
\]  

If there exists a function \( U = U(x) \) such that

\[
t_n(x) = \sum_{k=0}^{n} a_{nk} \{ U_k(x) - U \}
\]

\[
= \sum_{k=0}^{n} a_{n,n-k} \{ U_{n,n-k}(x) - U \}
\]

\[ = o(1), \text{ as } n \to \infty \]

uniformly in a set \( E \) in which \( U = U(x) \) is bounded, then we say that the series

\[
\sum u_m(x)
\]

is summable (T) uniformly in set \( E \) to the sum \( U \).

6.3 Siddiqi [55] proved the following theorem:

**Theorem A.** If

\[
(6.3.1) \quad \Phi(t) = \int_{0}^{t} |\phi(u)| du = o\left(\frac{t}{\log 1/t}\right), \text{ as } t \to +0
\]

then the Fourier series (6.1.1), at \( t=x \) is summable harmonic 1.7 to \( f(x) \).

Dealing with topics in this group, in 1961 Pati [39] has generalized the above Theorem A for the Nörlund summability of a Fourier series in the following form;
Theorem B.

If \((N,p_n)\) be a regular Nörlund method defined by a real non-negative, monotonic, non-increasing sequence of coefficients \(\{p_n\}\) such that

\[
P_n = \sum_{v=0}^{n} p_v \to \infty, \text{ as } n \to \infty
\]

and

\[
(6.3.2) \quad \log n = O(p_n), \text{ as } n \to \infty
\]

Then if,

\[
\Phi(t) = \int_{0}^{t} |\phi(u)|\,du = o\left(\frac{t}{P_t}\right), \text{ as } t \to +0
\]

the Fourier series (6.1.1) is summable \((N,p_n)\) to \(f(x)\) at the point \(x\).

We in this chapter, have improved and extended the above result under very general conditions by establishing the following:

Theorem. Let \((a_{nk})\) be an infinite triangular matrix such that the elements \((a_{nk})\) are non-negative, and non-decreasing with \(k\) and if

\[
(6.3.3) \quad \Phi(t) = \int_{0}^{t} |\phi(u)|\,du = o\left(\frac{1}{t} \frac{1}{P_t}\right), \text{ as } t \to +0
\]

uniformly in a set \(E\), where \(\in (t)\) is a positive function of \(t\) such that

(i) \(P_n \to \infty, \text{ as } n \to \infty\)

(ii) \(\in (n) \log n = O(p_n), \text{ as } n \to \infty\)

Then the Fourier series (6.1.1) is summable \((T)\) uniformly in \(E\) to the sum \(f(x)\).
6.4 We shall require following lemmas for the proof of our theorem.

**Lemma (4.1)** For \(0 < t < \frac{1}{n}\) and under the condition of our theorem on \((a_{nk})\),
\[ M_n(t) = O(n) \]

**Proof of Lemma:**
\[
M_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\sin\left(n - k + \frac{1}{2}\right) t}{\sin(t/2)}
\]
\[
= \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} (2n-2k+1) \left| \sin\left(\frac{t}{2}\right) \right|
\]
\[
= O(2n+1) \sum_{k=0}^{n} |a_{n,n-k}|
\]
\[
= O(2n+1)|A_{n,n}|
\]
\[
= O(n) O(1)
\]
(6.4.1)
\[= O(n) \]

**Lemma (4.2)** (Mc Fadden[33]). If \(\{p_n\}\) be a non-negative, non-increasing sequence, then for \(0 \leq t \leq \pi\), \(0 \leq a \leq b \leq \infty\) and any \(n\),
\[
(6.4.2) \quad \sum_{k=a}^{b} p_k e^{i(n-k)} = O(p_r)
\]

**Lemma (4.3)**: For \(\frac{1}{n} < t \leq \delta < \pi\),
\[
M_n(t) = O\left(\frac{A_{n,t}}{t}\right)
\]
Proof of Lemma.

Since for \( \frac{1}{n} < t \leq \delta < \pi, \sin \left( \frac{t}{2} \right) < t. \)

Therefore,

\[
M_n(t) = \left| \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\sin \left( \frac{n-k+1}{2} \right) t}{\sin \left( \frac{t}{2} \right)} \right|
\]

\[
= O \left( \frac{1}{t} \right)^{\text{Imaginary part of } \left| \sum_{k=0}^{n} a_{n,n-k} e^{(n-k+1)\pi^2}} \right)
\]

\[
= O \left( \frac{1}{t} \right)^{\sum_{k=0}^{n} a_{n,n-k} e^{(n-k+1)\pi^2}}
\]

\[
= O \left( \frac{A_{n,t}}{t} \right) \text{ by lemma (6.4.2)}
\]

which proves the lemma.

6.5 PROOF OF THE THEOREM:

Following Titchmarsh ([57]); p. 403) we have

\[
S_n(x) - f(x) = \frac{1}{2\pi} \int_0^\delta \phi(t) \frac{\sin \left( \frac{k+1}{2} \right) t}{\sin \left( \frac{t}{2} \right)} dt + o(1)
\]

uniformly in E

Then

\[
t_n(x) = \sum_{k=0}^{n} a_{n,n-k} \{S_{n-k}(x) - f(x)\}
\]
\[
= \frac{1}{2\pi} \int_0^\infty \left( \sum_{k=0}^{n} a_{n,n-k} \frac{\sin \left( \frac{n-k+\frac{1}{2}}{2} t \right)}{\sin \left( \frac{t}{2} \right)} \right) \phi(t) dt + o(1)
\]

\[
= \left( \int_0^1 + \int_1^{\delta} \right) M_n(t) \phi(t) + o(1) \quad \text{uniformly in } E.
\]

6.5.1

\[
= I_1 + I_2 + o(1) \quad \text{uniformly in } E.
\]

Now

\[
I_1 = \int_0^1 \phi(t) M_n(t) dt \quad \text{uniformly in } E
\]

\[
|I_1| \leq \int_0^1 |\phi(t)||M_n(t)| dt \quad \text{uniformly in } E
\]

\[
|I_1| = O(n) \int_0^1 \phi(t) dt \quad \text{uniformly in } E \text{ by lemma (6.4.1)}
\]

\[
= O(n) \alpha \left( \frac{\epsilon(n)}{nP_n} \right), \quad \text{by (6.3.3)}
\]

\[
= o \left( \frac{\epsilon(n)}{P_n} \right)
\]

(6.5.2)

\[
= o(1), \quad \text{as } n \to \infty
\]

\[
|I_2| = O(1) \int_1^{\delta} |\phi(t)| \frac{A_{n,\tau}}{t} dt
\]

\[
= \left\{ \frac{A_{n,\tau}}{t} \phi(t) \right\} \frac{\delta}{1} + \int_1^{\delta} \frac{A_{n,\tau}}{t^2} \phi(t) dt + \int_1^{\delta} \frac{\phi(t)}{t} d(A_{n,\tau})
\]

\[
= o \left( \frac{\epsilon(1)}{P_1} \right) + \int_1^{\delta} \frac{A_{n,\tau}}{t^2} \phi(t) dt + \int_1^{\delta} \frac{A_{n,\tau}}{t} \phi(t) dt
\]
\[
\begin{align*}
&= \frac{A_{n/\delta} \in \left(\frac{1}{\delta}\right)}{P_\delta} + \frac{A_{n,n} \in (n)}{P_n} \sum_{k=1}^{n} \frac{A_{n,u} \in (u)}{uP_u} du + \int_{1/\delta}^1 a_{n-y} \in (y) \ dy \\
&= o(1) + o(1) + o(1) + o(1) + o(e(n)) v P_n + \sum_{k=1}^{n} \frac{a_{n,n-k} \in (k)}{P_k k^2},
\end{align*}
\]

by mean value theorem.

\[
= o(1) + o(1) + o(1) + o(1) + o\left(\frac{\in (n)}{P_n} \right) + o\left(\frac{\in (n)}{P_n} \sum_{k=1}^{n} \frac{a_{n,n-k}}{k^2} \right),
\]

by hypothesis of the theorem.

\[
= o(1) + o(1) + o(1) + o\left(\frac{\in (n)}{P_n} \right) O\left(\frac{A_{n,n}}{n^2} \right),
\]

\[
= o(1) + o(1) + o(1) + o(1) + o(1), \text{ as } n \to \infty
\]

(6.5.3) \quad = o(1), \text{ uniformly in } E, \text{ as } n \to \infty

Combining (6.5.1), (6.5.2) and (6.5.3), we have

\[
t_n(x) = \sum_{k=0}^{n} a_{n,n-k} \{S_{n-k}(x) - f(x)\}
\]

\[
= o(1), \text{ as } n \to \infty, \text{ uniformly in } E.
\]

This completes the proof of our theorem.