Chapter - 5

ON $(N, p_n)$

SUMMABILITY OF 
LEGENDRE SERIES
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5.1 The Legendre series associated with a Lebesgue integrable function in the interval defined by $-1 \leq x \leq 1$, is

\[ f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \]  

where

\[ a_n = \left( n + \frac{1}{2} \right) \int_{-1}^{1} f(x) P_n(x) dx \]

and the n-th Legendre polynomial $P_n(x)$ is defined by the following expression

\[ \frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x) z^n \]

Let \( \{P_n\} \) be a sequence of constants, real or complex, such that

\[ P_n = p_0 + p_1 + p_2 + p_3 + \ldots + p_n = \sum_{m=0}^{n} p_m \neq 0 \]

we use the following notation

\[ \psi(t) = \psi(\theta, t) = f(\cos(\theta - t) - f(\cos\theta)) \]

\[ \Psi(t) = \int_{0}^{1} \psi(u) du \]

\[ \phi(t) = f(x+t) + f(x-t) - 2f(x) \]
\[
\Phi(t) = \int_0^t \phi(u) \, du
\]

\[
N_n(t) = \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{\sin \left( (n-k+1)t \right)}{\sin \left( \frac{t}{2} \right)}
\]

where \( \tau \) denotes the integral part of \( \frac{t}{2} \)

5.2 A series \( \sum_{n=0}^{\infty} U_n \) with partial sums \( S_n = \sum_{m=0}^{n} U_m \) is said to be summable by Nörlund method defined by the sequence \( \{p_n\} \) or simply \( (N,p_n) \) to the sum \( S \), if

\[
\lim_{n \to \infty} N_n^{(p)}(s) = s
\]

where the sequence \( \{N_n^{(p)}(s)\} \) of Nörlund means is defined by the sequence to sequence transformation.

\[
N_n^{(p)}(s) = \sum_{k=0}^{n} \frac{P_{n-k}S_k}{P_n} = \sum_{k=0}^{n} \frac{P_{n-k}U_k}{P_n} = \sum_{k=0}^{n} \frac{P_kS_{n-k}}{P_n}
\]

The regularity condition for the summation of series by Nörlund means are given by

\[
\lim_{n \to \infty} \frac{P_n}{P_{n+1}} = 0 \quad \text{and} \quad \sum_{k=0}^{n} |P_k| = O \left( |P_n| \right), \quad \text{as} \quad n \to \infty
\]

When \( P_n \geq 0 \) condition (5.2.3) are necessary and sufficient for regularity. Moreover if \( p_n \geq 0 \) and \( P_n \to \infty \), then \( (N,p_n) \) must be regular.

In this chapter we take \( \{p_n\} \) to be a real, non-negative and monotonic, non-increasing sequence. Such that \( P_n \to \infty \), as \( n \to \infty \).

5.3 Pati [39] has established the following theorem on the Nörlund summability of a Fourier series.
Theorem: A: If \((N, p_n)\) be a regular Nörlund method defined by a real, non-negative, monotonic, non-increasing sequence of coefficients \(\{p_n\}\) such that

\[
P_n = \sum_{v=0}^{n} P_v \to \infty, \text{ as } n \to \infty \text{ and}
\]

(5.3.1.) \(\log n = O(P_n), \text{ as } n \to \infty\)

Then if

(5.3.2.) \(|\phi(t)| = \int_{0}^{t} |\phi(u)| \, du = o \left( \frac{t}{P_t} \right), \text{ as } t \to 0^+\)

The Fourier series of a function \(f(t)\) is summable \((N, p_n)\) to \(f(x)\) at the point \(t=x\).

Here in this chapter we have generalized the theorem A for Legendre series in the following form:-

Theorem: Let \(\{p_n\}\) be a non-negative monotonic, non-increasing sequence of real constant and \(P_n \to \infty\) and let \(\lambda(t)\) be a positive integrable monotonic non-increasing function of \(t\) such that

(5.3.3.) \(\int_{1}^{n} \frac{\lambda(u)}{u} \, du = O(P_n), \text{ as } n \to \infty\)

If

(5.3.4.) \(\int_{0}^{t} |f(x \pm u) - f(x)| \, du = o \left( \frac{\lambda(1/t)}{P_t} \right), \text{ as } t \to 0\)

Then the Legendre series is summable \((N, P_n)\) to \(f(x)\) at an interval point \(x\) of the interval \((-1+\varepsilon, 1+\varepsilon), \varepsilon > 0\).

5.4 For the proof of our theorem, we shall use the following lemma;

Lemma (1): (Sansone [52]).
\begin{align*}
\sum_{v=0}^{n} (2v + 1) P_v(x)P_v(y) &= (n + 1) Q_n(x,y) \hspace{1cm} (5.4.1.)
\end{align*}

where,
\begin{align*}
Q_n(x,y) &= \frac{P_{n+1}(y) P_n(x) - P_n(y) P_{n+1}(x)}{y-x}
\end{align*}

This identity is known Christoffel's formula of summation.

\textbf{Lemma (2)} \ (Szegö [56])

For \(0 < \xi < v < \pi - \xi\)
\begin{align*}
P_n(\cos v) &= \sqrt{\frac{2}{\pi n \sin v}} \cos \left( (n + 1)v - \frac{\pi}{4} \right) + O\left(\frac{n^3}{2}\right) \hspace{1cm} (5.4.2.)
\end{align*}

\textbf{Lemma (3)} : under the condition (5.1.2.), we have
\begin{align*}
\int_{0}^{t} |f(\cos(\theta - v)) - f(\cos \theta)| \, dv &= \lim \left[ \frac{\xi}{t} \frac{1}{P_t} \right]_{t \to +0} \hspace{1cm} (5.4.3.)
\end{align*}

where \(x = \cos \theta, x + u = \cos \phi \) and \(\theta - \phi = v\)

The proof of the lemma follows on the lines of Foa [10].

\textbf{Lemma (4)} :\ (Mc Fadden [33])

If \(\{p_n\}\) is a non negative and non increasing sequence, then for
\(0 \leq a < b \leq \infty, 0 \leq t \leq \pi\) and for any \(n\) and \(a\)
\begin{align*}
\left| \sum_{k=a}^{b} p_k e^{(n-k)t} \right| &< p_t \hspace{1cm} (5.4.4.)
\end{align*}

\textbf{5.5 PROOF OF THE THEOREM} : - The \(n^{th}\) partial sum of the series (5.1.1.) is
\begin{align*}
S_n(x) &= \sum_{v=0}^{n} q_v P_v(x) = \frac{n+1}{2} \int_{-1}^{1} f(y) Q_n(x,y) \, dy \hspace{1cm} \text{by Lemma (5.4.1)}
\end{align*}
putting $f(y) = 1$, if can be easily seen that

$$1 = \frac{n+1}{2} \int_{-1}^{1} Q_n(x, y) \, dy$$

Therefore

$$S_n(x) - f(x) = \frac{n+1}{2} \int_{-1}^{1} \left[ f(y) - f(x) \right] Q_n(x, y) \, dy$$

From the definition, we have

$$N_n^{(p)}(s) - f(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_k \left[ S_{n-k}(x) - f(x) \right]$$

be regularity condition

Let us take a positive $S$ less than 1 and consider it as the sum of two other positive number $\mu$ and $\delta$. Let $d$ be another positive number such that $0 < d < \mu$ and $\mu x$, $\mu x'$ be two continuous function of $x$ within $(-1, 1)$ which lies within the limits $d < \mu x < g$, $d < \mu x' < g$

therefore $f_0 - 1 + s \leq x \leq 1 - s$, we have

$$N_n^{(p)}(x) - f(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_k \left( \frac{n-k+1}{2} \right) \int_{-1}^{1} \int_{x-kx}^{x+lx} \int_{x+lx'}^{x+lx} \left[ f(y) - f(x) \right] Q_{n-k}(x, y) \, dy + O(1)$$

Hobson [20] has shown that uniformly for $-1 + s \leq x \leq 1 - s$.

(5.5.1.)

$$\lim_{n \to \infty} A_{n-k}^x = 0, \quad \lim_{n \to \infty} C_{n-k}^x = 0$$

Now we suppose

$x = \cos \theta, \ y = \cos \phi, \ 0 < \theta < \pi, \ 0 < \phi < \pi, \ 1 - \delta = \cos \rho$
1 - (\mu + \delta) = 1 - s = \cos(\rho + \sigma), \rho > 0, \sigma > 0, \text{ we have } 0 < \rho < \pi/2, \text{ and } 0 < \sigma < \pi/2.

Thus, if \( \eta \) denotes the minimum of \( [\arccos \mu - \arccos (\mu + u)] \) for \( u \) in \((-1, 1-\mu)\), we have on the lines of Sansone \([52]\).

\[
B_{n-k} (\cos \theta) = \left( \frac{n-k+1}{2} \right)^{\theta+\eta} \int_{\theta-\eta}^{\theta+\eta} \left[ f(\cos \phi) - f(\cos \theta) \right] \sin \phi \theta_{n-k} (\cos \theta, \cos \phi) \, d\phi
\]

in which \( \rho + \sigma \leq \theta < \pi - (\rho + \sigma) : 0 < \eta \leq \sigma \).

Using Lemma (5.4.2) and working on the lines of Szegö \([56]\), for \( \alpha = \beta = 0 \), we get after some simplification.

\[
B_{n-k} (\cos \theta) = \frac{1}{2\pi \sqrt{\sin \theta}} \int_{\theta-\eta}^{\theta+\eta} \left[ f(\cos \phi) - f(\cos \theta) \right] \sqrt{\sin \phi} \left\{ \sin \left( (\frac{n-k+1}{2}) (\theta - \phi) \right) + \frac{\sin \left( (\frac{n-k+1}{2}) (\theta + \phi) \right)}{\sin \frac{1}{2} (\theta - \phi)} + \frac{\sin \left( (\frac{n-k+1}{2}) (\theta + \phi) \right)}{\sin \frac{1}{2} (\theta + \phi)} + O \left( \frac{1}{(n-k)^2} \right) \right\} \, d\phi + O(1)
\]

Hence.

\[
N_n^{(s)}(x) - f(x) = \frac{1}{2\pi \sqrt{\sin \theta}} \sum_{k=0}^{n-1} \int_{\theta-\eta}^{\theta+\eta} \left[ f(\cos \phi) - f(\cos \theta) \right] \sqrt{\sin \phi} \left\{ \sin \left( (\frac{n-k+1}{2}) (\theta - \phi) \right) + \frac{\sin \left( (\frac{n-k+1}{2}) (\theta + \phi) \right)}{\sin \frac{1}{2} (\theta - \phi)} + \frac{\sin \left( (\frac{n-k+1}{2}) (\theta + \phi) \right)}{\sin \frac{1}{2} (\theta + \phi)} + O \left( \frac{1}{(n-k)^2} \right) \right\} \, d\phi + O(1)
\]

(5.5.3.) \( = L_1 + L_2 + L_3 \)

putting \( \theta - \phi = t \) we get
\[ L_1 = \frac{1}{\pi \sqrt{\sin \theta}} \sum_{k=0}^{n-1} \int_0^\theta [f(\cos(\theta - t)) - f(\cos \theta)] \frac{\sqrt{\sin(\theta - t)} \sin (n-k+1)t}{\sin \frac{t}{2}} \, dt \]

\[ L_1 = \frac{1}{\pi \sqrt{\sin \theta}} \int_0^\theta [f(\cos(\theta - t)) - f(\cos \theta)] \sqrt{\sin(\theta - t)} \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{\sin (n-k+1)t}{\sin \frac{t}{2}} \, dt \]

\[ L_1 = O \left[ \int_0^\frac{\pi}{n} |\psi(t)| |N_n(t)| \, dt \right] \]

\[ L_1 = O \left[ \left( \frac{1}{n} \int_0^\frac{\pi}{n} |\psi(t)| |N_n(t)| \, dt \right)^\frac{1}{n} \right] \]

(5.5.4.) \[ L_1 = 1 + J \]

Now uniformly in \( 0 < t \leq \frac{1}{n} \)

\[ N_n(t) = \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{\sin (n-k+1) t}{\sin \left( \frac{t}{2} \right)} \]

\[ N_n(t) = O \left[ \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{2 \sin^2 \left( \frac{t}{2} \right)}{\sin \left( \frac{t}{2} \right)} \right] \]

\[ N_n(t) = O(n) \text{ as } n \to \infty \]

Therefore

\[ I = O \int_0^\frac{1}{n} |\psi(t)| |N_n(t)| \, dt \]

\[ = O \left[ n \int_0^\frac{1}{n} |\psi(t)| \, dt \right] \]
\( = O \left[ n \circ \left( \frac{\lambda(n)}{nP_n} \right) \right] \)

(5.5.5.) \( 1 = o(1) \) as \( n \to \infty \)

Again, for

\[
\frac{1}{n} \leq t \leq n
\]

\[
N_n(t) = O \left[ \frac{1}{P_n \sin \left( \frac{t}{2} \right)} \sum_{k=0}^{n-1} p_k \sin(n-k+1)t \right]
\]

\[
N_n(t) = O \left[ \frac{p_1}{t P_n} \right]
\]

By virtue of Lemma (5.4.4.)

Therefore

\[
J = O \left[ \frac{1}{P_n} \left\{ \psi(t) \frac{P_1}{t} \right\}^\eta \right] + O \left[ \frac{1}{P_n \frac{1}{n}} \psi(t) \frac{P_1}{t^2} dt \right] + O \left[ \frac{1}{P_n \frac{1}{n}} \psi(t) \frac{dP_1}{t} \right]
\]

(5.5.6.) \( J = J_1 + J_2 + J_3 \)

\[
J_1 = O \left[ \frac{1}{P_n} \left\{ \psi(t) \frac{P_1}{t} \right\}^\eta \right]
\]

\[
= O \left( \frac{1}{P_n} \circ \left( \frac{\lambda(1)}{t} \right) \right)^\eta
\]
\[ J_2 = \mathcal{O}\left(\frac{1}{P_n}\right) \int_1^n \psi(t) \frac{P_n}{t^2} dt \]

\[ = \mathcal{O}(1) + \mathcal{O}\left(\frac{1}{P_n} \sum_{k=1}^{n-1} \int \psi\left(\frac{1}{u}\right) P_{[u]} du\right) \]

But

\[ \int_k^{k+1} \psi\left(\frac{1}{u}\right) P_{[u]} du \leq \psi\left(\frac{1}{k}\right) P_k = \mathcal{O}\left(\frac{\lambda(k)}{k}\right) \]

so,

\[ J_2 = \mathcal{O}(1) + \mathcal{O}\left(\frac{1}{P_n} \sum_{k=1}^{n-1} \mathcal{O}\left(\frac{\lambda(k)}{k}\right)\right) \]

\[ = \mathcal{O}(1) + \mathcal{O}\left(\frac{1}{P_n} \int_1^n \lambda(k) du\right) \]

\[ = \mathcal{O}(1) + \mathcal{O}(1) \]

(5.5.7.) \[ J_2 = \mathcal{O}(1), \text{ as } n \to \infty \]

(5.5.8.) \[ J_3 = \mathcal{O}(1), \text{ as } n \to \infty \]

\[ J_3 = \mathcal{O}\left(\frac{1}{P_n} \int_1^n \psi(t) \frac{dP_t}{t}\right) \]

\[ J_3 = \mathcal{O}\left(\frac{1}{P_n} \int_1^n \psi\left(\frac{1}{u}\right) u \ dP_{[u]}\right) \]

\[ J_3 = \mathcal{O}(1) + \mathcal{O}\left[\frac{1}{P_n} \sum_{k=1}^{n-1} K_p \psi\left(\frac{1}{k}\right)\right] \]
\[ J_3 = o(1) + O\left[ \frac{1}{P_n} \sum_{k=1}^{n-1} p_k \psi\left( \frac{1}{k} \right) \right] \]

\[ J_3 = o(1) + O\left[ \frac{1}{P_n} \cdot \left( \int_{1}^{\lambda(u)} \frac{1}{u} \, du \right) \right] \]

\[ J_3 = o(1) + o(1) \]

(5.5.9.) \[ J_3 = o(1), \text{ as } n \to \infty \]

From (5.5.7.), (5.5.8) and (5.5.9) we get

(5.5.10) \[ J = o(1), \text{ as } n \to \infty \]

From (5.5.5) and (5.5.10) we get

(5.5.11) \[ L_1 = o(1), \text{ as } n \to \infty \]

working on lines of Szegö [56] it can be easily seen that

(5.5.12) \[ L_2 = o(1), L_3 = o(1) \text{ as } n \to \infty \]

combining (5.5.11) and (5.5.12) we obtain that

\[ N^{(p)}_n(s) - f(x) = o(1) \text{ as } n \to \infty \]

This completes the proof of the theorem.