CHAPTER 2
A One Sector Neoclassical Growth Model with Overlapping-Generations

In this chapter we consider a Solow-type one-sector neoclassical growth model. But unlike Solow, we assume that the savings propensity is not an arbitrarily given constant. Savings behaviour is determined by optimizing households in an overlapping-generations framework.

The overlapping-generations framework with a Modigliani-Brumberg type of life cycle savings function was introduced in the context of a neoclassical growth model by Peter Diamond (1965), who examined the effect of public debt on the competitive dynamic equilibrium of this kind. In a later paper Cass and Yaari (1967) used a similar framework to show that there may be no, one, or several steady state growth paths and the steady state path need not be an efficient one. However neither Diamond nor Cass and Yaari analysed the stability question in detail; while Diamond simply assumed that the conditions for stability would generally be satisfied, Cass and Yaari admitted their inability to demonstrate in the general case the asymptotic convergence of all competitive equilibria.\(^1\) It was Stephen Marglin (1984) who first pointed out the possible sources of instability in a neoclassical growth model with life-cycle savings. More recently an extensive discussion about the nature of equilibrium in this kind of a neoclassical growth model was provided by Galor and Ryder (1989). But Galor and Ryder have mainly concentrated on the problem of existence; the question of stability has been more or less ignored.\(^2\) In contrast, the present work focuses on the problem of stability alone; we get rid of

\(^1\) Taking a specific production function of the form: \(f(k)=A-B \log k\) \((A,B>0)\) Cass and Yaari have shown that in this specific case the stability property holds. Then on the basis of this example they conjecture that this property holds for the general case as well. However, the stability property holds in this case because the specific production function that they have taken satisfies certain conditions, which are not necessarily satisfied by all other production functions with standard neoclassical properties. We shall come back to this point later.

\(^2\) Galor and Ryder did derive a set of sufficiency condition for the existence of a unique and globally stable non-trivial equilibrium. But in a phase plane analysis, their stability condition simply implies concavity of the phase curve. They did not analyse under what conditions (in terms of the production as well as utility function) the phase curve would indeed be concave.
the existence question by assuming that an equilibrium always exists. In fact, our work, in a way, is complementary to that of Galor and Ryder. They have shown that Inada conditions are not sufficient for the existence of a non-trivial equilibrium. We, on the other hand, ignore the existence question and show that concavity of the production function is not sufficient for stability in this kind of a framework. We derive certain sufficient conditions for \textit{instability} even with a concave production function. In doing so, we follow Marglin and generalise his result to a certain extent.

The structure of this chapter is as follows: section II describes the general framework. We discuss the stability condition in section III. In section IV, we have carried out the analysis with specific utility functions and derived certain sufficiency conditions for instability in terms of a general production function. In section V, we take a CES production function and examine the stability of the dynamic system for different elasticities of substitution.

\textbf{II. General Framework:}

We consider an overlapping-generations neoclassical growth model \textit{à la} Diamond, where each individual lives for two periods — working in the first period of his life and being retired in the second. At the end of the first period, he earns a wage income of which he consumes a part and saves and invests the rest. At the end of the second period he gets back his invested capital with the interest, which he consumes entirely. Let generation \( t \) refer to those people who are born at the beginning of period \( t \). We assume that people belonging to the same generation are identical in every respect. Let \( c_i^1 \) and \( [c_i^2]^e \) denote respectively the current and expected future consumption of the representative individual of generation \( t \). The representative individual of generation \( t \) maximizes his lifetime utility subject to his budget constraint. The optimization exercise of the representative member of the \( t \)-th generation is given below:

\[
\text{Max. } U \left( c_i^1, [c_i^2]^e \right) \text{ subject to } c_i^1 + s_i^t = w_i
\]
where $s_t$ is his savings in the current period. As he invests all his savings in the present period and gets that back in the next period with interest, his anticipated future consumption is $[c_t^2]^e = \{1 + [i_{t+1}]^e\}^t s_t$, $[i_{t+1}]^e$ being the expected rate of return in the next period. We further assume that people have *static* expectation about future so that $[i_{t+1}]^e = i_t$.

The assumption about expectation is one of the important aspects in which our framework differs from that of Galor and Ryder. Galor and Ryder have assumed perfect foresight or rational expectation instead of static expectation and the rationale given to justify this assumption is the following. With static expectation even though there is a well-defined growth path following from any initial configuration of capital endowment, this path does not ‘coincide with the actual course of the economy unless the economy is in long run steady state. For this reason, it is customary to assume that the agents are capable of predicting the future course of the economy and that they adopt these predictions as their expectations. Such “rational” or “perfect foresight” expectations are independent of past observations and must be self-fulfilling.\(^3\) The logic of this argument is, however, not very clear to us. If “steady state” is indeed viewed as the dynamic counterpart of static equilibrium (wherein lies its significance in dynamic theory), then it is perfectly plausible to have a number of non-equilibrium growth paths where expectations are not fulfilled and an “equilibrium” or “steady state” growth path such that along this path and only along this growth path expectations are fulfilled. In fact, there seems to be no obvious economic logic in assuming perfect foresight in a model of long run dynamic behaviour unless the economy is *already* in a steady state. As Grandmont has pointed out, “the hypothesis of perfect foresight is not generally warranted (when) out of “long run” equilibria and in order to study correctly the evolution of any economy, one should portray the traders as learning the dynamics of prices on the transition path. ... In particular, perfect foresight appears to be a plausible *outcome* of learning when a trader’s environment is repetitive enough, but seems to be far less acceptable out of such special

\(^3\) Galor and Ryder (1989), pp. 364.
circumstances." One can incorporate such a learning process by introducing an expectation function that depends on the current and the past values of the variable(s). Various kinds of adaptive expectation framework allow for such a learning process. The assumption of adaptive expectations, where past values of the variables enter the expectation function, though intuitively more appealing than the assumption of static expectation, generates complex dynamics, which are difficult to handle. Therefore in our model we have assumed static expectation. In doing so we may have lost some generality, but we gain in terms of expositional and analytical simplicity. Another point to be noted here is that under certain conditions, any equilibrium that is unstable under static expectations, will be unstable under perfect foresight as well (as has been shown in Appendix A.1).

Given the above assumptions, the utility maximisation exercise of the representative member of generation $t$ can be written as

$$\text{Max. } U \left( c_t^1, [c_t^2]^e \right) \text{ subject to } c_t^1 + \frac{[c_t^2]^e}{1 + i} = w_t$$

In equilibrium, the expected future consumption must equal the actual consumption of the next period, i.e., in equilibrium, $[c_t^2]^e = c_t^2$.

On the production side, technology is represented by a production function $F(L,K)$, which uses two factors of production – labour $(L)$ and capital $(K)$ – to produce a single commodity that can be used as a consumable as well as capital. The production function satisfies all the standard properties of a neoclassical production function, namely, it is continuous, concave, and exhibits constant returns to scale. However, it does not necessarily satisfy the Inada conditions.

At any point of time $t$, the economy functions in the following way: at time $t$, there is a certain stock of capital $(K_t)$ and certain amount of labour $(L_t)$ which, along with the production technology, give certain gross output $Y_t = F(K_t, L_t)$. If capital depreciates at a constant rate $\delta$, then the net income (after depreciation) is given by $Y_t - \delta K_t$, which is to be distributed among the young and the old generations as wage income and profit income (net) respectively. In the discussion

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that follows, we in fact assume that $\delta = 1$. (This assumption is really a simplifying one; our basic results do not change if we assume any other constant value of $\delta$). The assumption that the rate of depreciation is one implies that in our model there is no fixed capital; all capital is circulating capital.

Labour force grows at a constant rate $n$ such that $L_t = (1 + n)^t L_0$. Note that here the term "labour force" does not imply the entire population. At any point of time $t$, it is only the young generation (generation $t$) which constitutes the working population at that point of time. The entire population at time $t$ is given by $(L_t + L_{t-1})$, which also grows at the same rate $n$.

There is perfect competition so that the wage rate and the rate of interest are equal to the marginal products of labour and capital respectively.

Finally, following the neoclassical tradition, we assume that at every point of time the existing labour force and capital stock are fully employed.

III. Stability Analysis: The General Case

In this model all the saving is done by the young generation alone. The old generation in fact dissaves; they consume their entire capital stock. The aggregate savings of the workers today constitute tomorrow's capital stock. Let $s_t$ denote the savings of the working household. Then the basic dynamic equation of the model is given by

$$K_{t+1} = L_t s_t$$  \hspace{1cm} (2.1)

Now let us define a new variable $k_t$, which is the capital-labour ratio. Then

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{L_t s_t}{(1 + n) L_t}$$

or, $$k_{t+1} - k_t = \frac{s_t - (1 + n) k_t}{(1 + n)} = \Delta k$$  \hspace{1cm} (2.2)

Thus at any point of time, change in the capital-labour ratio depends on the difference between $s$ and $(1+n) k$. So if we can express $s$ as a function of $k$, then
(2.2) will give us a difference equation in \( k \), which we can solve to get the time path of the capital-labour ratio.

As was mentioned before, the savings of the younger generation is determined by the following utility maximisation exercise:

\[
\text{Max. } U \left( c_i^1, \left[ c_i^2 \right]^e \right) \text{ subject to } c_i^1 + \frac{[c_i^2]^e}{1 + i} = w_i.
\]

The first order conditions:

\[
\frac{\partial U}{\partial c_i^1} = (1 + i) \quad (2.3)
\]

\[
c_i^1 + \frac{[c_i^2]^e}{1 + i} = w \quad (2.4)
\]

From (2.3) and (2.4), we can express \( c_i^1 \) and \( [c_i^2]^e \) as functions of \( w \) and \( i \). Let \( c_i^1 = \psi(w,i) \) and \( [c_i^2]^e = \xi(w,i) \). Then savings of the working household is

\[
s = w - \psi(w,i) \quad (2.5)
\]

Now in our neoclassical set up, \( w \) and \( i \) are functions of \( k \) alone. In each period, the wage rate and the rate of interest adjust to ensure full employment of both the factors. Therefore, the wage rate is equal to the marginal product of labour at full employment and gross rate of return is equal to the marginal product of capital at full employment. In other words,

\[
w = \frac{\partial F(L,K)}{\partial L}
\]

and

\[
i = \frac{\partial F(L,K)}{\partial K} - 1
\]

The production function is homogeneous of degree one. Let \( f(k) \) represent output per worker. Then

\[
w = f(k) - kf'(k)
\]

and

\[
i = f'(k) - 1
\]

Therefore from (2.5),

\[
s = w(k) - \psi(w(k),i(k)) = \phi(k) \text{ (say)}
\]

Hence from (2.2),
\[ \Delta k = \frac{\phi(k) - (1 + n)k}{1 + n} \]  

(2.6)

This is the fundamental equation of our model. Note that the system will be viable only if \( i \geq 0 \). For any \( i < 0 \), no investment would be forthcoming; the working household would simply hold on to their savings (without making it available for production in the next period in the form of investment) and consume it in the next period. Since the entire capital stock of the economy comes from the savings and subsequent investment of the working households, with \( i < 0 \), no production would take place. Therefore, a necessary condition for the viability of the system is \( i \geq 0 \), i.e., \( f'(k) \geq 1 \). This poses an upper limit on the capital-labour ratio. If \( f'(k) = 1 \) at \( k = \bar{k} \), then the feasible range of \( k \) is given by \([0, \bar{k}]\).

As in Solow, we can now carry out the analysis for long run steady state equilibrium in terms of the fundamental equation (eq. 2.6). In steady state, the capital-labour ratio remains constant. Therefore the equilibrium condition is given by

\[ \phi(k) = (1 + n)k \]  

(2.7)

Apart from the requirement that the per capita capital stock remain unchanged in steady state, there is another condition for equilibrium in terms of expectation fulfillment: in equilibrium, the anticipated future consumption of each working household must equal the actual consumption in the next period. Therefore we have the following additional equilibrium condition:

\[ \{1 + [i_{t+1}]^e\} \phi(k_t) = \{c_t^e\}^e = c_t^e = \{1 + i_{t+1}\} \phi(k_t) \]

Now, given the assumption of static expectation, \([i_{t+1}]^e = i_t\). Also \( i_t \) is a function of \( k_t \); therefore, constancy of \( k \) in steady state implies constancy of \( i \) as well. Hence in steady state, \( i_t = i_{t+1} \). This, together with the assumption of static expectation, implies that in steady state the above equilibrium condition will always be fulfilled. So we need not consider this condition separately.

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5 In the general case with any \( \delta \geq 0 \), the feasibility condition is given by \( f'(k) \geq \delta \).
From the fundamental equation we find that when 
\[ \phi(k) > (1 + n)k, \Delta k > 0; \] so \( k \) rises over time. On the other hand, when 
\[ \phi(k) < (1 + n)k, \Delta k < 0; \] so \( k \) falls over time. Therefore we can characterise the long run equilibrium in terms of the two functions \( \phi(k) \) and \( (1+n)k \). If we plot \( k \) along the horizontal axis, then \( (1+n)k \) is an upward sloping straight line (with slope greater than unity) passing through the origin. A possible diagrammatic representation of the \( (1+n)k \) function is shown in Figure 2.1 below. The shape of \( \phi(k) \) function is more complicated. Note that 
\[
\frac{d\phi}{dk} = \frac{dw}{dk} - \frac{d\psi(w,i)}{dk} = \{-f''(k)\}(1-\psi_w)k + \psi_i \]  
(2.8)
By the concavity property of the production function \( f''(k) < 0 \). Therefore
\[
\frac{d\phi}{dk} >, =, or < 0 \text{ according as } \{(1-\psi_w)k - \psi_i\} >, =, \text{ or } < 0.
\]
Let us now look at the two partial derivatives \( \psi_w \) and \( \psi_i \). We assume that consumption in both periods is a normal good. Hence as wage rate rises, *ceteris paribus* consumption rises in both periods. Thus \( 0 < \psi_w < 1 \). However, the sign of \( \psi_i \) is ambiguous. As \( i \) rises, the relative price of future consumption falls inducing people to substitute future consumption for present consumption (Substitution Effect). At the same time, this fall in the price of future consumption increases the choice set available to people which enables them to consume more in both periods of their life (Income Effect). So what happens to current consumption (\( \psi \)) as \( i \) increases depends on the relative strength of the two effects. When income effect dominates the substitution effect, \( \psi_i < 0 \). On the other hand, when substitution effect outweighs the income effect, \( \psi_i > 0 \). To put it differently, a low degree of substitutability between present and future consumption implies that \( \psi_i > 0 \) and a

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\( ^6 \) From the budget equation, \( \psi(w,i) + \frac{\xi(w,i)}{1+i} = w \). Differentiating both sides with respect to \( w \):
\[
\psi_w + \frac{\xi_w}{1+i} = 1 \Rightarrow 0 < \psi_w < 1.
\]
Figure 2.1
high degree of substitutability implies that \( \psi_i < 0 \). We first consider the case where the degree of substitutability between present and future consumption is low.

(i) low degree of substitutability between present and future consumption:

In this case, \( 0 < \psi_w < 1 \) and \( \psi_i > 0 \). Therefore \( (1 - \psi_w)k + \psi_i > 0 \) for all \( k \geq 0 \). Hence \( \frac{d\phi}{dk} > 0 \). Also note that \( \phi(0) = 0 \). Thus for \( \psi_i > 0 \), \( \phi(k) \) is an upward sloping curve passing through the origin. However the curvature of \( \phi(k) \) is not known. Figure 2.2 shows four cases: in all four \( \phi(k) \) is positively sloped and passes through the origin. In case (a), we have a stable equilibrium; in case (b), an unstable one; and in cases (c) and (d), no equilibrium exists at all. If we ignore the problem of existence and assume that an equilibrium always exists, then the equilibrium will be stable if \( \phi''(k^*) < 0 \) and unstable if \( \phi''(k^*) > 0 \).

In the above analysis, we have assumed that the sign of \( \phi''(k) \) remains the same for all feasible values of \( k \). However, even if \( \phi'' \) changes sign as the value of \( k \) changes, we can still conclude that a sufficient condition for \( k^* \) to be an unstable equilibrium is \( \phi''(k^*) > 0 \) (as shown in Figure 2.3).

Next consider the case where the degree of substitutability between present and future consumption is high.

(ii) high degree of substitutability between present and future consumption:

When the degree of substitutability between present and future consumption is high, the stability analysis becomes more complicated. In this case, \( \psi_i < 0 \). Therefore, from equation (2.8),

\[
\frac{d\phi}{dk} = \frac{dw}{dk} - \frac{d\psi(w, i)}{dk} = (-f''(k))\{(1 - \psi)k - |\psi_i|\} \tag{2.9}
\]
Figure 2.2
Figure 2.3
Therefore,

\[ \frac{d\phi}{dk} \begin{cases} >, =, \text{or} < \text{ according as } k <, = \text{or} > \frac{|\psi|}{1-\psi_w} \end{cases} \]

Let \( \frac{|\psi|}{1-\psi_w} = m \) (a constant). Then for all positive values of \( k \), the \( \phi(k) \) function falls, reaches a minimum at \( k = m \) and then rises again. But \( \phi(0) = 0 \). Therefore, the \( \phi(k) \) function is discontinuous at \( k = 0 \). This case generates various interesting possibilities depicted in Figure 2.4 below. We may have no, unique or multiple equilibria. Moreover, the dynamic system may show monotonic convergence, oscillatory convergence, oscillatory divergence, and regular or irregular cyclical movements depending on the parameter values.

There is however no reason why \( \frac{|\psi|}{1-\psi_w} \) should be take a constant value for any positive \( k \). If \( \frac{|\psi|}{1-\psi_w} = g(k) \), where \( g \) is some function of \( k \), then nothing conclusive can be said about the shape of the \( \phi(k) \) curve; \( \phi(k) \) will reach a minimum or a maximum depending on the slope and curvature of the \( g(k) \) function. We consider below three mutually exclusive (but not exhaustive) cases.

**Case (i):** \( g'(k) > 0 \) and \( g''(k) > 0 \)

Let us first assume that the function \( g(k) \) has a fixed point, that is, \( g(k) = k \) for some \( k \). \( g(k) \) is a convex function of \( k \); therefore if such a fixed point exists, then it will be unique. Let us denote this fixed point by \( \hat{k} \). Then for \( k < \hat{k} \), \( k > g(k) \); so \[ \frac{d\phi}{dk} > 0. \]

Again for \( k > \hat{k} \), \( k < g(k) \); hence \[ \frac{d\phi}{dk} < 0. \] At \( k = \hat{k} \), \( k = g(k) \); therefore \[ \frac{d\phi}{dk} = 0. \] Thus
Figure 2.4
\( \phi(k) \) reaches a maximum at \( k = \hat{k} \). Figure 2.5(a) and 2.5(b) below show the equilibrium value of \( k \) in this case. Figure 2.5(c) depicts the possibility when \( g(k) \) does not have any fixed point and it lies above the 45° line for all values of \( k \). In the latter case, the \( \phi(k) \) curve is a decreasing function of \( k \). Once again, the dynamic movements of the system will be characterised by either oscillatory convergence, or oscillatory divergence, or regular cycles or even chaotic movements.\(^7\)

**Case (ii):** \( g'(k) > 0 \) and \( g''(k) < 0 \)

Here \( g(k) \) is concave in \( k \). Using similar logic as in the previous case, we now show that the \( \phi(k) \) curve in this case either reaches a minimum, or is an increasing function of \( k \) for all values of \( k \). The possible equilibrium configurations has been shown in Figure 2.6.

**Case (iii):** \( g'(k) < 0 \)

Once again the \( \phi(k) \) curve will reach a minimum point here at some positive \( k \) value. Figure 2.7 depicts this case. Since the dynamic behaviour of the economy in cases (ii) and (iii) are similar to the various possibilities already discussed, we do not analyse them separately. The crux of the matter is, the economic system will be characterised by more complex dynamics if we assume a high degree of substitutability between present and future consumption. In fact this assumption allows for possibilities like cobwebs or even chaotic movements and is no more stabilizing than the assumption of low degree of substitutability (unless we take very specific parameter values).

Thus we find that in a one-sector neoclassical growth model if we assume that saving is determined by optimizing households in an overlapping-generations framework, then the strong stability result of Solow no longer holds. The concavity of the production function is not sufficient to guarantee stability here. Also, contrary to the Galor and Ryder result, the assumption that \( \psi_i < 0 \) is neither necessary nor sufficient for stability here.

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\(^7\) That chaotic movements can occur in this case is clear from figure 5(b). If we measure \( k_i \) along the horizontal axis and \( k_{i+1} \) along the vertical axis, then the phase diagram in this case will be similar to the phase diagram of the famous logistic equation \( x_{i+1} = Ax_i(1-x_i) \) which is known to generate chaotic movement for certain parameter values.
Figure 2.5
Figure 2.6

Figure 2.7
However, the assumption of high degree of substitutability between present and future consumption is a questionable one. Consumption, by its very nature, is time specific; hence, consumption tomorrow cannot be a good substitute of consumption today. Therefore, in the subsequent analysis we ignore this case.

In the discussion so far, the equilibrium value(s) of the capital-labour ratio was assumed to lie within the feasible range \([0, \bar{k}]\). There is however no reason why it would necessarily be so. Things worsen if the stable equilibrium value happens to lie outside the feasible region. In that case, even if other conditions for stability are satisfied, the economy cannot attain the stable equilibrium value. Suppose we start from a \(k_0\) such that \(k_0 < \bar{k} < k^*\). If \(k^*\) is stable, then over time \(k\) will start moving towards \(k^*\). However, it cannot go beyond \(\bar{k}\). If \(k > \bar{k}\), then net return to investment becomes negative; as a result capital stock forthcoming for next period’s production becomes zero and consequently, the capital-labour ratio in the next period goes to zero. Thus we have an additional condition for stability, namely: \(k^* \leq \bar{k}\).

IV. Specific Utility Functions:

In this section, we carry out the stability analysis with some specific forms of the utility function. We have seen earlier that a sufficient condition for instability in the general case is: \(\phi' > 0\) and \(\phi'' > 0\). We now consider certain specific utility functions and derive a sufficient condition for instability in terms of a general production function. The first utility function that we consider is of Cobb-Douglas variety:

\[
U(c^1, c^2) = (c^1)^\beta (c^2)^{1-\beta}; \quad 0 < \beta < 1
\]

Maximizing this utility function for the young generation, subject to their budget constraint, we get:

\[
c^1 = \psi(w, i) = \beta w
\]

Therefore, the per capita savings function, \(\phi\), associated with this utility function is:

\[
\phi(k) = w - \psi(w, i) = (1 - \beta) w(k)
\]
Hence,
\[
\phi'(k) = (1 - \beta) \frac{dw}{dk} = (1 - \beta) [-k f''(k)] > 0
\]
and
\[
\phi''(k) = (1 - \beta) [-k f'''(k) - f''(k)]
\]
From (2.14), \(\phi'' > 0\) if

- either \(f'''(k) \leq 0\)
- or \(f'''(k) > 0\) and \(\frac{k f''(k)}{-f''(k)} < 1\)

Note that \(f'(k)\) is nothing but the marginal product of capital (gross). Therefore, the first of the two conditions implies that as \(k\) increases, the marginal product of capital falls either at a constant rate or at a decreasing rate. On the other hand, the second condition implies that the marginal product of capital is falling at an increasing rate, but the elasticity of the rate at which the marginal product is falling is less than one. Thus, with a Cobb-Douglas utility function, any of the above two conditions is sufficient for the instability of the economic system.

The next set of specific utility functions that we consider belongs to the CRRA family which exhibit constant elasticity of marginal utility. Marglin in his formulation has used a Cobb-Douglas utility function. But it is generally believed that the utility functions of the CRRA family are more suitable to represent intertemporal choices and this utility function has been most commonly used in growth literature to analyse intertemporal behaviour. Let us first consider the simplest utility function of this family, namely, the logarithmic utility function:

\[
U(c^1, c^2) = \log c^1 + \log c^2
\]

The corresponding \(\psi\) function and \(\phi\) function in this case, are given by:

\[
c^i = \psi(w, i) = \frac{1}{2} w
\]
\[
\phi(k) = w - \psi(w, i) = \frac{1}{2} w(k)
\]

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8 The term CRRA implies constant relative risk aversion.
9 For a discussion on the choice of intertemporal utility function, see Chakravarty (1969), pp. 22-28.
These two functions are very similar to the forms that $\psi$ and $\phi$ take in the Cobb-Douglas case (given by eq. (2.11) and (2.12) respectively); the only difference lies in the constant term. Therefore, the sufficiency condition for instability in terms of the production function will also be the same.

The fact that Cobb-Douglas utility function and logarithmic utility function give us similar results for stability is not surprising. In both the cases, the income effect of a price change (that is, the income effect of a change in the interest rate – which is the relative price of future consumption in terms of present consumption here) exactly matches the corresponding substitution effect. Hence changes in the interest rate do not have any effect on the current consumption or current savings decisions. As a result, the behaviour of the savings function depends entirely on the wage rate.

Let us now consider another member of the CRRA family with elasticity of marginal utility greater than unity:

$$U(c^1, c^2) = \frac{(c^1)^{1-\sigma}}{1-\sigma} + \frac{(c^2)^{1-\sigma}}{1-\sigma} \quad (2.18)$$

Once again, maximizing this utility function for the young generation, subject to the budget constraint, we get:

$$c^1 = \psi(w, i) = \frac{w}{1-(1+i)^{\sigma}} \quad (2.19)$$

Therefore,

$$\phi(k) = w(k) - \psi(w(k), i(k)) = w(k) \left[ 1 - \frac{1}{1+(1+i(k))^{\sigma}} \right] \quad (2.20)$$

We have assumed that $\sigma$ is greater than unity. It can be easily shown that $\sigma > 1$ implies $\psi_i > 0$, i.e., the degree of substitutability between present and future consumption is low. Now from (2.20),

$$\phi'(k) = (1-\psi_w) \frac{dw}{dk} - \psi_i \frac{di}{dk} \quad (2.21)$$
and \[ \phi''(k) = (1 - \psi_w) \frac{d^2w}{dk^2} - \psi_{ww} \left( \frac{dw}{dk} \right)^2 - \psi_i \frac{d^2i}{dk^2} - \psi_{ii} \left( \frac{di}{dk} \right)^2 - 2 \psi_{iw} \frac{dw}{dk} \frac{di}{dk} \] \(\text{(2.22)}\)

Again, from (2.19),

\[ \psi_w = \frac{1}{1 + (1 + i)^\sigma}; \quad 0 < \psi_w < 1 \] \(\text{(2.23)}\)

\[ \psi_i = \frac{\sigma - 1}{\sigma} \cdot \frac{w(1 + i)^{1 - 2\sigma}}{2^{(1 - \sigma)^2}} > 0 \quad \left[ \because \sigma > 1, \text{by assumption} \right] \quad \text{(2.24)}\]

\[ \psi_{ww} = 0 \] \(\text{(2.25)}\)

\[ \psi_{ii} = \frac{\sigma - 1}{\sigma} \cdot \frac{(1 - 2\sigma)}{(1 + (1 + i)^\sigma)^{1 - 3\sigma}} \cdot \left\{ \frac{1 - (1 + i)^\sigma}{2^{(1 - \sigma)^2}} \right\} < 0 \] \(\text{(2.26)}\)

\[ \psi_{iw} = \frac{\sigma - 1}{\sigma} \cdot \frac{1 - 2\sigma}{(1 + (1 + i)^\sigma)^{1 - 2\sigma}} > 0 \] \(\text{(2.27)}\)

From equations (2.22)-(2.27), simplifying, we get,

\[ \phi''(k) = (-f''(k)) \left[ (1 - \psi_w)k + \psi_i \right] + \{-f''(k)(1 - \psi_i) + \{f''(k)\}^2 (2\psi_{iw} k - \psi_{ii}) \} \] \(\text{(2.28)}\)
From the above equation we find that
\[ \phi''(k) > 0 \text{ if } f''(k) \leq 0. \]

Thus, in all the three cases that we have discussed so far, a sufficient condition for instability is that \( f''(k) \leq 0 \) (either for all values of \( k \) or at least at the equilibrium point).

Therefore, if the utility function is isoelastic (either Cobb-Douglas, or CRRA type) with elasticity of marginal utility (\( \sigma \)) greater than or equal to one, and the production function is such that \( f''(k) \leq 0 \), then the equilibrium (if it exists) will always be unstable even if the production function \( f(k) \) satisfies all the standard neoclassical properties.

As we have seen before, \( f''(k) \leq 0 \) implies that as \( k \) increases, the gross marginal product of capital falls either at a constant rate or at a decreasing rate. In the first case, \( f''(k) = 0 \) and in the second case, \( f''(k) < 0 \). The possible shapes of the marginal product curve (of capital) in these two cases have been shown in Figure 2.8 (see next page).

It is not difficult to find an example where the marginal product curve satisfies the above condition. Let us consider the following production technology:

\[ f(k) = k(A - Bk) ; \quad A, B > 0 \quad (2.29) \]

This production function satisfied all the standard neoclassical properties: it is continuous, concave and exhibits constant returns to scale. In this case, for all \( k < \frac{A}{2B} \), \( f'(k) = A - 2Bk > 0 \)

\[ f''(k) = -2B < 0 \]

and \( f'''(k) = 0 \)

where \( k = \frac{A}{2B} \) is the capital saturation point. Beyond this point, marginal product of capital becomes negative; so the economy will never produce with a capital-labour ratio greater than \( \frac{A}{2B} \). If technology is represented by a production function of this form, then the resulting equilibrium will always be unstable.
Figure 2.8
The production function given by (2.29) does not, however, satisfy the Inada conditions. But Inada conditions are neither necessary nor sufficient for stability even in Solow model (though, in Solow, they are sufficient for the existence of a non-trivial steady-state). In fact, stability in Solow model is guaranteed by the concavity property of the production function, just as the concavity property of the \( \phi(k) \) function would ensure stability in our case. Since we are not concerned with the question of existence here, we ignore the Inada conditions. In any case, Inada conditions are too restrictive; apart from the Cobb-Douglas production function and its linear transformations, there is no other well-known production function that satisfies this property.\(^{10}\)

At this juncture, let us go back to Cass and Yaari. The specific production function that they have used is of the following form:

\[
 f(k) = k(A - B \log k); \quad A, B > 0
\]

Hence, \( f'(k) = A - B - B \log k > 0 \) for \( k < e^{\frac{1-A}{B}} \).

\[
 f''(k) = -\frac{B}{k} < 0
\]

and \( f'''(k) = \frac{B}{k^2} > 0 \)

Thus this production function satisfies the necessary condition for stability.\(^ {11}\) But on the basis of this, one cannot claim that the same would be true for any neoclassical production function (as example (2.29) shows). Hence the Cass-Yaari stability result is derived under special assumption which does not necessarily hold in the general case.

\(^{10}\) Galor and Ryder have shown that Inada conditions are not sufficient even for existence of equilibrium in this kind of an overlapping-generations framework.

\(^{11}\) Note that if \( f''(k) \leq 0 \) is a sufficient condition for instability then \( f''(k) > 0 \) is a necessary condition for stability.
V. Specific Production Function:

In this section, we take a specific production function with constant elasticity of substitution and analyse the dynamic behaviour for different ranges of elasticity values and for the different utility functions discussed above. The CES production function that we consider for this purpose takes the following per capita form:

\[ f(k) = A \left[ \alpha k^{-\rho} + (1-\alpha) \right]^{\frac{1}{\rho}}; \quad A > 0, \rho > -1, 0 < \alpha < 1 \]  \quad (2.31)

Therefore,

\[ i(k) = f'(k) - 1 = A \alpha k^{-\rho-1} \left[ \alpha k^{-\rho} + (1-\alpha) \right]^{\frac{1}{\rho}} - 1 \] \quad (2.32)

and

\[ w(k) = f(k) - k f'(k) = A (1-\alpha) \left[ \alpha k^{-\rho} + (1-\alpha) \right]^{\frac{1}{\rho}} \] \quad (2.33)

We analyse below the stability of the system under inelastic and elastic factor substitution possibilities.

Case (a): Elasticity of factor substitution is less than unity:

With a CES production function, inelastic factor substitutability implies that \( \rho > 0 \). Let us first consider a utility function of the Cobb-Douglas type. (This is the example considered by Marglin.) Then, from the equilibrium condition in the general case (eq. (2.7)) and from the \( \phi(k) \) function with this specific utility function (eq. (2.12)), we can derive the following equilibrium condition in terms of \( k \):

\[ (1+n)k = \phi(k) = (1-\beta) w \] \quad (2.34)

Now, we can write the \( \phi(k) \) function as:

\[ \phi(k) = (1-\beta) \frac{w}{f(k)} f(k) = s_w \omega f(k) \] \quad (2.35)

where \( s_w \) is the propensity to save out of wages (a constant here, given by \( 1-\beta \)) and \( \omega \) is the share of wages in national income. From (2.34) and (2.35), we can write the equilibrium condition in this case as:
\[(1+n)k = s_{w} \omega f(k)\]
\[\Rightarrow \frac{f(k)}{k} = \frac{(1+n)}{s_{w} \omega(k)} \quad (2.36)\]

We can now analyse the nature of the equilibrium in terms of the Swan diagram.

With CES production function, the LHS of (2.36) becomes:
\[
\frac{f(k)}{k} = A\left[\alpha + (1-\alpha) k^{\rho}\right]^{\frac{1}{\rho}} \quad (2.37)
\]

It is easy to see that the LHS of (2.36) is a negative function of \(k\). Also, with \(\rho > 0\),
\[
\lim_{k \to 0} \frac{f(k)}{k} = A(\alpha)^{\frac{-1}{\rho}} \quad \text{(a positive constant)} \quad \text{and} \quad \lim_{k \to \infty} \frac{f(k)}{k} = 0.
\]

On the other hand, with this production function, the share of wages
\[
\omega = \frac{(1-\alpha)}{(\alpha k^{-\rho} + 1-\alpha)} \quad (2.38)
\]

Therefore, with \(\rho > 0\), the share of wages is an increasing function of \(k\). It can be easily shown that in this case, \(\lim_{k \to 0} \omega(k) = 0\) and \(\lim_{k \to \infty} \omega(k) = 1\). Thus the RHS of (2.26) is a decreasing function of \(k\) and also,
\[
\lim_{k \to 0} RHS = \infty \quad \text{and} \quad \lim_{k \to \infty} RHS = \frac{(1+n)}{s_{w}} = \frac{(1+n)}{(1-\beta)} \quad \text{(a positive constant)}.
\]

If we plot the LHS and RHS of (2.36) with \(k\) on the horizontal axis, we find that in the neighbourhood of the origin, the RHS lies above the LHS, and as \(k\) approaches infinity, the LHS goes to zero while the RHS approaches a positive limiting value. Therefore in this case, if there is any point of intersection between the curves (representing the LHS and the RHS of (2.26)), then it must exists in pairs, thus giving (at least) two equilibrium points – the first one unstable and the second one stable.

It is obvious that with a CES production function with inelastic factor substitution possibilities, the logarithmic utility function will also give the same result, since in this case nothing but a constant term changes in the RHS of (2.36). (The workers' savings propensity is half in this case, instead of \((1-\beta)\)). What
happens if the utility function is CRRA-type with $\sigma > 1$? From (2.20), in the latter case, we can write the equilibrium condition in a way similar to (2.36):

$$\frac{f(k)}{k} = \frac{(1 + n)}{s_w(k) \omega(k)}$$

(2.36a)

Only the workers' savings propensity is not a constant here; it is a function of $k$, which takes the following form:

$$s_w = \frac{1}{\frac{1}{\sigma-1}} = \frac{1}{1 + \{f'(k)\}^{\sigma-1}} \frac{1}{\frac{1}{\sigma}}$$

(2.39)

When $\rho > 0$, like $\omega$, $s_w$ is also an increasing function of $k$. Also, it can be easily shown that in this case,

$$\lim_{k \to 0} s_w = \frac{1}{\frac{1}{\sigma-1}} \quad \text{(a positive constant)} \quad \text{and} \quad \lim_{k \to 0} s_w = 1$$

These limiting values of $s_w$, together with the limits of $\omega$ gives,

$$\lim_{k \to 0} RHS = \infty \quad \text{and} \quad \lim_{k \to \infty} RHS = \frac{(1 + n)}{s_w \omega} = (1 + n) \quad \text{(a positive constant)}.$$  

Hence, once again we find that, either there does not exist any equilibrium, or they exists in pairs, alternatively unstable and stable – starting with an unstable one.

Therefore, when technology is represented by a CES production function with elasticity less than one and the utility function is isoelastic (either Cobb-Douglas or CRRA with $\sigma \geq 1$), either there does not exist any equilibrium at all, or the equilibrium values exist in pairs with the first one unstable and the second one stable. This has been shown in Figures 2.9 (a) and 2.9 (b) below. In this case, if an economy starts with a per capita capital stock below $k^*$, it will face economic retrogression over time.

Next we consider the case where the production function exhibits elastic substitution possibilities.
Figure 2.9

(a) \( \frac{(1+n)}{s_{\infty}} \omega \) vs. \( k \)

(b) \( \frac{f(k)}{k} \) vs. \( k^* \) and \( k'' \)

Figure 2.10

\( \frac{(1+n)}{s_{\infty}} \omega \) vs. \( k^* \)
Case (b): Elasticity of factor substitution is greater than unity:

Elastic substitution possibilities in CES production function implies that $-1 < \rho < 0$. In this case once again the LHS of (2.36) is a decreasing function of $k$. But now

$$\lim_{k \to 0} \frac{f(k)}{k} = \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{f(k)}{k} = A(\alpha)^{-1}.$$

Thus for very low values of $k$, the LHS take infinitely large values, and it approaches a positive constant as $k$ goes to infinity.

On the other hand, the share of wages is now a decreasing function of if $k$ with the limits: $\lim_{k \to 0} \omega(k) = 1$ and $\lim_{k \to \infty} \omega(k) = 0$. Therefore, if the utility function is either Cobb-Douglas or logarithmic, then the RHS of (2.36) is an increasing function of $k$, going from a positive constant value to infinity. Thus, there will always be a unique point of intersection between the LHS and the RHS. Moreover, the equilibrium represented by this unique intersection point will be stable.

Therefore, when technology is represented by a CES production function with elasticity greater than one and the utility function is either Cobb-Douglas or logarithmic, there always exists a unique, stable equilibrium. This has been shown in Figure 2.10 (see the preceding page).

The case of CRRA utility function (with $\sigma > 1$) is not so straightforward. In this case, $s_w$ and $\omega$ move in the opposite direction. When the elasticity of substitution (in the production front) is greater than one, from (2.39), $s_w$ is still an increasing function of $k$. Moreover, with $-1 < \rho < 0$,

$$\lim_{k \to 0} s_w = 0 \quad \text{and} \quad \lim_{k \to \infty} s_w = \frac{1}{\sigma - 1} \left( a \text{ positive constant} \right)$$

Therefore, in this case, the RHS approaches infinity for both high and low values of $k$. Let $s(k) = s_w(k) \omega(k)$ be the aggregate saving propensity in the economy (savings out of per capita income). Now, as $k$ increases, $s_w$ increases while $\omega$ decreases. Therefore,
\[
\frac{ds(k)}{dk} >, =, \text{ or } < 0 \text{ according as } \frac{1}{s_w} \frac{ds_w}{dk} >, =, \text{ or } < 0 \frac{d\omega}{\omega \, dk} \]  \hspace{1cm} (2.40)
\]

Simplifying the latter part of (2.40), it can be shown that

\[
\frac{ds(k)}{dk} >, =, \text{ or } < 0 \text{ according as }
\]

\[
k <, =, \text{ or }> \left[ \frac{(\sigma - 1)}{\sigma} \left( \frac{1 - \alpha}{\alpha} \right) \frac{(1 + \rho)}{(-\rho)} \frac{\frac{(\sigma - 1)}{\sigma} f(k)}{1 + \left\{ \frac{(\sigma - 1)}{\sigma} f(k) \right\}^\frac{1}{\rho}} \right] = h(k) \]  \hspace{1cm} (say)
\]

With \( \sigma > 1, 0 < \alpha < 1, \) and \(-1 < \rho < 0, h'(k) < 0. \) So \( h(k) \) has a unique fixed point.

Let us denote this by \( \tilde{k} \). Then the \( s(k) \) function behaves in the following way: it reaches a maximum at \( k = h(k) = \tilde{k} \) and approaches zero as \( k \) tends to either zero or infinity. Consequently, the RHS of equation (2.36a) (which is the inverse of \( s(k) \) multiplied by a constant term \( 1 + n \)) reaches a minimum at \( k = \tilde{k} \) and its limiting values go to infinity at both ends (see Figure 2.11 below).

We can conceive of two mutually exclusive possibilities here:

(i) \( \lim_{k \to 0} \frac{d}{dk} \frac{f(k)}{k} > \lim_{k \to 0} \frac{d}{dk} \frac{(1+n)/s(k)}{s(k)} \), i.e., the \( f(k)/k \) curve starts above the \( (1+n)/s(k) \) curve.

(ii) \( \lim_{k \to 0} \frac{d}{dk} \frac{f(k)}{k} < \lim_{k \to 0} \frac{d}{dk} \frac{(1+n)/s(k)}{s(k)} \), i.e., the \( f(k)/k \) curve starts below the \( (1+n)/s(k) \) curve.

As \( k \) approaches infinity, the \( f(k)/k \) curve takes a finite positive value, whereas \( (1+n)/s(k) \) goes to infinity. Therefore, in case (i), there will be a unique point of intersection between LHS and RHS of (2.36a), giving a unique stable equilibrium.

But in case (ii), either there does not exist any point of intersection at all, or they exist in pairs with the first one unstable and the second one stable. These two cases
Figure 2.11
Figure 2.12

Figure 2.13
have been shown in Figure 2.12 and Figure 2.13 respectively (see the previous page). Therefore, unlike the other two utility functions, when the households' preferences are represented by a CRRA utility function with elasticity of marginal utility greater than unity, then even a sufficiently elastic production function may fail to restore stability.

The upshot of this whole discussion is that in a one sector neoclassical growth model, where savings decisions are undertaken by optimizing households in an overlapping-generations framework, serious problems may arise regarding stability, even when the production function satisfies all the standard neoclassical properties. Stability requires more stringent conditions not only on the production function, but also on the utility function.

Before we move on to the next chapter, let me re-emphasize that any equilibrium capital-labour ratio will be applicable if and only if it lies within the feasible range \([0, \tilde{k}]\). In the multiple equilibria cases discussed above, the instability problem will be accentuated if the stable equilibrium value lies outside the feasible range. In fact, we had run some computer simulations for a CES production function and CRRA utility function with specific parameter values \((n = 0.01, A = 5, \alpha = 0.5, \sigma = 2)\) to find out whether the stable equilibrium value lies within the feasible range.\(^{12}\) We have found that for a wide range of \(\rho\) values (between 0.9 to 3), the stable equilibrium value is greater than \(\tilde{k}\) and is therefore not applicable.

**Brief Summary of the chapter**

In this chapter we have analysed the stability property of a one-sector neoclassical growth model where the households optimally determine their savings in an overlapping-generations context. We have shown that in this case the strong stability result of the Solow model does not hold. The standard properties of the neoclassical production function can no longer guarantee the stability of the long

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\(^{12}\) The computer programme was written by Prof. J Subba Rao of School of Environmental Sciences, J.N.U., New Delhi.
run equilibrium. Stability requires more stringent conditions both on the utility function and the production function. With specific utility and production functions, we have shown that for stability we not only require relatively high degree of substitutability between present and future consumption, but also a high elasticity of factor substitution in the production front. We have also derived certain sufficient conditions for instability in this case.