CHAPTER – 3

On (N,p,q) Summability Of Fourier Series & Its Conjugate Series

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ON (N, p, q) SUMMABILITY OF FOURIER SERIES AND ITS CONJUGATE SERIES

3.1 For any sequence $\{p_n\}$, $\{q_n\}$ and $\{s_n\}$ we write

$$t_n^{p,q} = R_n^{-1} \sum_{k=0}^{n} p_{n-k}\, q_k\, S_k$$

where

$$R_n = \sum_{k=0}^{n} p_{n-k} q_k \; (\neq 0, \text{for all } n).$$

Generalized Nörlund transfrom ((N,p,q) transform) of the sequence $\{s_n\}$ is sequence $\{t_n^{p,q}\}$. If $t_n^{p,q} \to S$ as $n \to \infty$, then sequence $\{s_n\}$ is summable by generalized Norlund method (N,p,q) to S and is denoted by (Browien (1958))

$S_n \to S\ (N,p,q)$.

Let $f(x)$ be a periodic function with period 2π and integrable in the sense of Lebesgue in $(-\pi, \pi)$.

The Fourier series of $f(x)$ is given by

(3.1.1) $$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(x)$$

and the conjugate series of (3.1.1) is given by

(3.1.2) $$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x).$$

We write

$$\phi(t) = \phi(t,x) = f(x+t) + f(x-t) - 2 f(x)$$

$$\psi(t) = \psi(t,x) = f(x+t) - f(x-t)$$

$$P\left(\frac{1}{t}\right) = P\left[\frac{1}{t}\right] = P_{\tau}, \left[\frac{1}{t}\right] = \text{Integral part of } \frac{1}{t}$$

3.2 MAIN RESULTS :

For related results see Fejzer (1904), Lebesgue (1905), Hardy (1913), Siddiqi (1948), Pati (1961), Singh (1963), Hirokawa (1963), Izumi (1968) etc. In this chapter we establish the following two theorems.

THEOREM 1 :

If (N,p,q) be regular Generalized Nörlund method define by real non-negative monotonic, non- increasing sequence of coefficients $\{p_n\}$ and real, non- negative monotonic, non-decreasing sequence of coefficients $\{q_n\}$ such that

(3.2.1) $$R_n = \sum_{v=0}^{n} p_v q_{n-v} \to \infty, \text{ as } n \to \infty$$

(3.2.2) $$\text{and } q_n \int_1^n \frac{\lambda(u)}{u} = O\ (R_n), \quad \text{as } n \to \infty$$

where $\lambda(t)$ is a suitable positive, non- decreasing function of t such that

(3.2.3) $$\lambda(n) = O\ (P_n), \quad \text{as } n \to \infty.$$

(3.2.4) $$\text{If } \Phi\ (t) = \int_0^t |\ \phi(u)\ | du = 0 \left[\frac{t\lambda\left(\frac{1}{t}\right)}{P_{\left(\frac{1}{t}\right)}}\right] \text{ as } t \to 0$$

then Fourier series (3.2.1) is summable (N,p,q) to the value f (t) at x = t.

THEOREM 2 :

Let regular generalized Nörlund method (N,p,q) be defined as in theorem 1 and (3.2.1), (3.2.2) and (3.2.3) hold good. If

$$(3.2.5) \qquad \int_0^t |\psi(u)|\, du = 0\left[\frac{t\lambda\left(\frac{1}{t}\right)}{P_{\left(\frac{1}{t}\right)}}\right], \text{ as } t \to 0$$

then the conjugate series (3.2.2) is summable (N,p,q) to the value

$$\frac{1}{2\pi} \int_0^\pi \psi(t) \cot \frac{1}{2}\, t\, dt$$

at all points, where the integral exists in the sense of Lebesgue.

3.3 LEMMAS :

For the proof of the theorem we require the following lemmas

LEMMA 1

For $0 < t < \pi$,

$$\left| \sum_{k=0}^{\alpha} p_k \frac{\sin\left(n - k + \frac{1}{2}\right) t}{\sin \frac{1}{2}\, t} \right| = O\left(\frac{P\left(\frac{1}{t}\right)}{t} \right)$$

and

$$\left| \sum_{k=0}^{n} p_k \frac{\cos\left(n - k + \frac{1}{2}\right) t}{\sin \frac{1}{2}\, t} \right| = O\left(\frac{P\left(\frac{1}{t}\right)}{t} \right).$$

For the proof of the lemma, [Singh [46] (1963)]

LEMMA 2

If $\pi \geq t > 0$,

$$N(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}$$

then

$$|N(t)| = O\left(\frac{q_n P\left(\frac{1}{t}\right)}{R_k}\right)$$

PROOF

Since for a fixed n, q_{n-k} is non- increasing with k and

$$\left|\sum_{k=0}^{\alpha} p_k \frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right|$$

increases as $n \to \infty$ using mean value theorem, we have

$$|N(t)| \le \frac{q_k}{R_n}\left|\sum_{k=0}^{n} p_k \frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right|$$

$$= O\left(\frac{q_n P\left(\frac{1}{t}\right)}{R_n t}\right).$$

LEMMA 3

If $\pi \ge t > 0$

$$\overline{N}(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}$$

then

$$\overline{N}(t) = O\left(\frac{q_n P\left(\frac{1}{t}\right)}{R_n t}\right).$$

Proof of lemma 3 runs parallel to the proof of lemma 2.

3.4 PROOF OF THEOREM 1 :

If $S_n(x)$ denotes the n^{th} partial sum of the series (3.1.2), we have

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin \frac{1}{2} t} dt.$$

Therefore we have

$$t_n^{p,q}(x) - f(x) = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\sin(n-k+1)t}{\sin \frac{1}{2} t} dt$$

or, $t_n^{p,q}(x) - f(x) = \int_0^\pi \phi(t) N(t) dt.$

In order to establish the theorem, we have to show that

$$= \left[\int_0^\pi |\phi(t)| \, |N(t)| \, dt\right] = O(1) \text{ as } n \to \infty.$$

For $0 < \delta < \pi$ we write

$$I = \int_0^\pi \phi(t) N(t) dt$$

$$= \left[\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_\delta^\pi\right] \phi(t) N(t) dt$$

$$= I_1 + I_2 + I_3 \text{ say}$$

Now, uniformly in $0 < t < \frac{1}{n}$

$$\left| N(t) \right| \leq \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{\sin\left(n - k + \frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

$= O(n)$.

Hence

$$I_1 = O(n) \int_0^{\frac{1}{n}} \left| \phi(t) \right| dt$$

$$= O(n)\ 0\left(\frac{\lambda(n)}{n P_{(n)}}\right)$$

$$= 0\left(\frac{\lambda(n)}{P_{(n)}}\right)$$

$$= 0(1) \qquad \text{by } (3.2.3).$$

By lemma 2, we have

$$I_2 \leq \frac{Mq_n}{R_n} \int_{\frac{1}{n}}^{\delta} \left| \phi(t) \right| \frac{P\left(\frac{1}{t}\right)}{t} dt$$

where M is a positive constant, which may be different at each occurrence.

Given $\varepsilon > 0$, let δ be chosen so that

$$\left| \Phi(t) \right| \leq \frac{\in t\lambda\left(\frac{1}{t}\right)}{P_{\left(\frac{1}{t}\right)}}, \quad 0 < t \leq \delta.$$

Thus

$$|I_2| \leq \frac{Mq_n}{R_n}\left[\Phi(t)\frac{P\left(\frac{1}{t}\right)}{t}\right]_{\frac{1}{n}}^{\delta} + \frac{Mq_n}{R_n}.\int_{\frac{1}{n}}^{\delta}\Phi(t)\frac{P\left(\frac{1}{t}\right)}{t^2}dt$$

$$+\frac{Mq_n}{R_n}\int_{\frac{1}{n}}^{\delta}\Phi(t)\frac{d\,P\left(\frac{1}{t}\right)}{t}$$

the M's may be different

$$= I_{2.1} + I_{2.2} + I_{2.3} \quad \text{say.}$$

If M (δ) denotes a constant depending on δ, we see that for fixed δ,

$$|I_{2.1}| = \frac{M(\delta)\,q_n}{R_n} + 0\left(\frac{q_n}{R_n}\ \frac{\lambda(n)\,P_n}{P_n}\right)$$

$$= \frac{M(\delta)\,q_n}{R_n} + 0\left(\frac{q_n\lambda(n)}{R_n}\right)$$

$$= 0(1) + 0(1) \quad \text{as } n \to \infty$$

$$= 0(1) \quad \text{as } n \to \infty.$$

Now,

$$|I_{2.2}| \leq \frac{M \in q_n}{R_n}\int_{\frac{1}{n}}^{\delta}\Phi(t)\frac{P\left(\frac{1}{t}\right)}{t^2}dt$$

$$= \frac{M \in q_n}{R_n} \; 0 \left[\int_{\frac{1}{n}}^{\delta} \frac{\lambda\left(\frac{1}{t}\right) P\left(\frac{1}{t}\right)}{P\left(\frac{1}{t}\right)} \left(\frac{1}{t}\right) dt \right].$$

$$= \frac{M \in q_n}{R_n} \; 0 \left(\int_{\frac{1}{n}}^{\delta} \frac{\lambda\left(\frac{1}{t}\right)}{t} \, dt \right)$$

$$= \frac{M \in q_n}{R_n} \; 0 \left(\int_{\frac{1}{\delta}}^{n} \frac{\lambda(x)}{x} \, dx \right)$$

$$= 0\,(1) \qquad \text{as } n \to \infty .$$

Next,

$$| I_{2.3} | \le \frac{M \in q_n}{R_n} \int_{\frac{1}{n}}^{S} \Phi(t) \, \frac{dP\left(\frac{1}{t}\right)}{t}$$

$$= \frac{M \in q_n}{R_n} \int_{\frac{1}{\delta}}^{n} \Phi\left(\frac{1}{u}\right) u \, d\, P(u)$$

$$= 0\;(1) + \frac{M \in q_n}{R_n} \sum_{k=1}^{n-1} k \, p_k \Phi\left(\frac{1}{k}\right)$$

$$= 0\;(1) + \frac{M \in q_n}{R_n} \sum_{k=1}^{n-1} P_k \Phi\left(\frac{1}{k}\right)$$

$$= 0\;(1) + \frac{M \in q_n}{R_n} \sum_{k=1}^{n-1} 0 \left(P_k \frac{\lambda(k)}{k P_k} \right)$$

$$= 0\;(1) + \frac{M \in q_n}{R_n} \sum_{k=1}^{n-1} 0 \left(\frac{\lambda(k)}{k} \right)$$

$$= 0\ (1) + \frac{M \in q_n}{R_n}\ 0\left(\int_1^n \frac{\lambda(u)}{u}\,du\right)$$

$$= 0\ (1) + 0\ (1) \qquad \text{as } n \to \infty$$

$$= 0\ (1) \qquad \text{as } n \to \infty$$

by Riemann-Lebesgue theorem and regularity conditions of method of summability.

PROOF OF THE THEOREM 2 :

If $\overline{S}_n(x)$ denotes the n^{th} partial sum of the series $\Sigma B_n(x)$. Then we have

$$\overline{S}_n(x) = \frac{1}{2\pi}\int_0^{\eta} \frac{\cos\frac{1}{2}t - \cos\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt$$

Hence

$$t_n^{p,q} - \frac{1}{2\pi}\int_0^{\pi} \psi(t)\cot\frac{t}{2}\,dt$$

$$= -\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2\pi}\int_0^{\pi} \psi(t)\,\frac{\cos\,(n-k+\frac{1}{2})\,t}{\sin\frac{t}{2}}\,dt$$

$$= -\int_0^{\pi} \psi(t)\,\overline{N}(t)dt.$$

In order to establish the theorem, we have to show that under our assumptions

$$\left[\int_0^{\pi} \psi(t)\,\overline{N}(t)dt\right] = 0\ (1), \quad \text{as } n \to \infty$$

Let

$$\int_0^{\pi} \psi(t)\,\overline{N}(t)dt = \left[\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi}\right]\psi(t)\,\overline{N}(t)dt$$

$= J_1 + J_2 + J_3$ (say).

Since the conjugate function exists, therefore

$$\frac{1}{2\pi}\int_0^{\frac{1}{n}} \psi(t) \cot \frac{t}{2} dt = 0\ (1),\ \text{as } n \to \infty$$

Hence

$$\frac{1}{2\pi}\int_0^{\frac{1}{n}} \psi(t) \cot \frac{t}{2} dt - J_1$$

$$= \frac{1}{2\pi}\int_0^{\frac{1}{n}} \psi(t) \left[\cot \frac{t}{2} - \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k} \frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin \frac{t}{2}} \right] dt$$

$$= \frac{1}{2\pi R_n}\int_0^{\frac{1}{n}} \psi(t) \sum_{k=0}^{n} p_k q_{n-k} \left[\sum_{m=0}^{n-k} 2 \sin mt \right] dt$$

$$\leq \frac{1}{\pi R_n}\int_0^{\frac{1}{n}} |\psi(t)| \sum_{k=0}^{n} p_k q_{n-k} (n-k) dt$$

$$\leq M_{(n)}\int_0^{\frac{1}{n}} |\psi(t)|\, dt = O(n).\left(\frac{1}{n}\ \frac{\lambda(n)}{P_n}\right) = o\left(\frac{\lambda(n)}{P_n}\right)$$

$$= 0\ (1),\ \text{as } n \to \infty.$$

Therefore

$$J_1 = 0\ (1).$$

Now for $\frac{1}{n} \leq t < \delta$ by lemma 3,

$$J_2 \leq \frac{Mq_n}{R_n}\int_{\frac{1}{n}}^{\delta} |\psi(t)|\, t \frac{P\left(\frac{1}{t}\right)}{t} dt$$

$= 0\ (1)$, as in case of I_2

Also,

$J_3 = 0(1)$, as $n \to \infty$

by Riemann- Lebesgue theorem and regularity of the method of summability.