CHAPTER – 2

On $K^{\lambda,\mu,\nu}$ *Summability Of A Triple Fourier Series*

CHAPTER – 2

ON $K^{\lambda,\mu,\nu}$ SUMMABILITY OF A TRIPLE FOURIER SERIES

2.1 Let f (x, y, z) be a periodic function with period 2π in each case which is summable in the cube $(-\pi,-\pi,-\pi,\pi,\pi,\pi)$.

Then the Fourier series of f (x, y, z) is given be -

$$(2.1.1) \qquad f(x,y,z) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{l,m,n} \begin{bmatrix} a_{l,m,n} \cos lx \cos my \cos nz \\ + b_{l,m,n} \cos lx \cos my \sin nz \\ c_{l,m,n} \cos lx \sin my \cos nz + \\ d_{l,m,n} \cos lx \sin my \sin nz + \\ e_{l,m,n} \sin lx \cos my \cos nz + \\ f_{l,m,n} \sin lx \cos my \sin nz + \\ g_{l,m,n} \sin lx \sin my \cos nz + \\ h_{l,m,n} \sin lx \sin my \sin nz \end{bmatrix}$$

Where

$$l,m,n = \begin{cases} \frac{1}{8}, \; l = m = n = 0 \\ \frac{1}{4}, \; l > 0, \; m = n = 0; \; m > 0, \; l = n = 0; \; n > 0, \; l = m = 0 \\ \frac{1}{2}, \; l, \; m > 0, \; n = 0; \; l, n > 0, \; m = 0; \; m, n > 0, \; l = 0 \\ 1, \; l, m, n \geq 1 \end{cases}$$

and

$$a_{l,m,n} = \frac{1}{\pi^3} \iiint_C f(x,y,z) \cos lx \cos my \cos nz \, dx \, dy \, dz$$

with similar expressions of other coefficients, where C denotes fundamental cube $(-\pi,-\pi,-\pi;\ \pi.\pi,\pi)$.

Let us define for l=0, 1, 2, 3, ..., the number $\begin{bmatrix} l \\ p \end{bmatrix}$, for $0 \le p \le l$

$$\prod_{v=0}^{l-1} (x+v) = x(x+1)\,(x+2)\,\ldots\ldots\ldots\ (x+l-1) = \frac{\lfloor(x+l)}{\lfloor x}$$

$$= \sum_{p=0}^{l} \begin{bmatrix} l \\ p \end{bmatrix} x^p. \tag{2.1.2}$$

Similarly

$$\prod_{v=0}^{m-1} (y+v) = y(y+1)\,(y+2)\ldots\ldots\ \ldots\ (y+m-1) = \frac{\lfloor(y+m)}{\lfloor y}$$

$$= \sum_{q=0}^{m} \begin{bmatrix} m \\ q \end{bmatrix} y^q. \tag{2.1.3}$$

and

$$\prod_{v=0}^{n-1} (z+v) = z(z+1)\,(z+2)\ \ldots\ldots\ \ldots\ (z+n-1) = \frac{\lfloor(z+n)}{\lfloor z}$$

$$= \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} z^r. \tag{2.1.4}$$

The numbers $\begin{bmatrix} l \\ p \end{bmatrix}$, $\begin{bmatrix} m \\ q \end{bmatrix}$ and $\begin{bmatrix} n \\ r \end{bmatrix}$, are known as the absolute value of stirling numbers of first kind.

Let $\{S_{l,m,n}\}$ be sequence of partial sums of triple infinite series

$\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{l,m,n}$.

Let us consider

(2.1.5)
$$S_{l,m,n}^{\lambda,\mu,\nu} = \frac{\lceil\lambda}{\lceil(\lambda+l)} \frac{\lceil\mu}{\lceil(\mu+m)} \frac{\lceil\nu}{\lceil(\nu+n)} \sum_{p=0}^{l} \sum_{q=0}^{m} \sum_{r=0}^{n} \begin{bmatrix} l \\ p \end{bmatrix} \begin{bmatrix} m \\ q \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} . \lambda^p \mu^q \nu^r S_{p,q,r}$$

to denote the (l, m, n)th $K^{\lambda,\mu,\nu}$ mean of order $(\lambda,\mu,\nu) > 0$ to the sum S and we can write

(2.1.6)
$$S_{l,m,n}^{\lambda,\mu,\nu} \to S\ (K^{\lambda,\mu,\nu}),\ \text{as}\ (l,m,n) \to \infty.$$

The method $K^{\lambda,\mu,\nu}$ is regular for $(\lambda,\mu,\nu) > 0$ and this case will be supposed through out this paper.

$$\phi\ (x,y,z) = \phi\ (u,v,w;\ x,y,\ )$$

$$= \frac{1}{8}[f\ (u+x,\ v+y,\ w+z) + f(u+x,\ v+y,\ w-z)$$

$$+ f\ (u+x,\ v-y,\ w+z) + f\ (u-x,\ v+y.\ w+z)$$

$$+ f\ (u+x,\ v-y,\ w-z) + f\ (u-x,\ v+y,\ w-z)$$

$$+ f\ (u-x,\ v-y,\ w+z) + f\ (u-x,\ v-y,\ w-z)$$

$$- 8f\ (u,v,w)\ ]$$

$$\therefore\ \Phi\ (x,y,z) = \int_0^x dh \int_0^y ds \int_0^z |\ \phi(h.s,t)\ |\ dt$$

$$K_l\ (x) = \frac{\sum_{p=0}^{l} \begin{bmatrix} l \\ p \end{bmatrix} \lambda^p\ \text{Sin}\left(p+\frac{1}{2}\right)x}{\lceil\lambda+l\ \ \text{Sin}\left(\frac{x}{2}\right)}$$

$$K_m(y) = \frac{\sum_{q=0}^{m} \begin{bmatrix} m \\ q \end{bmatrix} \mu^q \operatorname{Sin}\left(q+\frac{1}{2}\right) y}{\overline{|(\mu + m)} \operatorname{Sin}\left(\frac{y}{2}\right)}$$

$$K_n(z) = \frac{\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \nu^r \operatorname{Sin}\left(r+\frac{1}{2}\right) z}{\overline{|(\nu + n)} \operatorname{Sin}\left(\frac{z}{2}\right)}$$

2.2 KNOWN THEOREM :

A theorem on $K^{\lambda,\mu}$ summability of double Fourier series was established by Shaym Lal (1997) in the following from :

If $\Phi(x,y) = \int_0^x ds \int_0^y |\phi(s,t)| \, dt$

$$= 0 \left[\frac{x}{\log\left(\frac{1}{x}\right)} \frac{y}{\log\left(\frac{1}{y}\right)} \right], \text{ as } (x,y) \to +0$$

then the double Fourier series of the function f (x,y) is summable $K^{\lambda,\mu}$ $((\lambda,\mu) > 0)$ to the sum f (u,v) at x = u, y = v.

2.3 MAIN THEOREM :

Let $\alpha(x)$, $\beta(y)$ and γ (z are three non-negative functions of x,y and z and $\{p_l\}$ $\{p_m\}$ and $\{p_n\}$ be a non-negative monotonic, non-increasing sequence of real constants such that

$$P_l = \sum_{i=0}^{l} p_i \to \infty \quad \text{as } l \to \infty,$$

$$P_m = \sum_{j=0}^{m} p_j \to \infty \quad \text{as } m \to \infty,$$

and $P_n = \sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$.

Let $\alpha(x)$, $\beta(y)$, $\gamma(z)$ be a positive monotonic, non-increasing function of x and y such that

$\alpha(l)\ \log l = O(P_l)$ as $l \to \infty$

$\beta(m)\ \log m = O(P_m)$ as $m \to \infty$

$\gamma(n)\ \log n = O(P_n)$ as $n \to \infty$

If $\Phi(x, y, z) = \int_0^x dh \int_0^y ds \int_0^z \left| \phi(h,s,t) \right| dt$

(2.3.1) $$= 0 \left[\frac{x\alpha\left(\frac{1}{x}\right) y\beta\left(\frac{1}{y}\right) z\gamma\left(\frac{1}{z}\right)}{P_{\left(\frac{1}{x}\right)} P_{\left(\frac{1}{y}\right)} P_{\left(\frac{1}{z}\right)}} \right] \quad \text{as } (x,y,z) \to +$$

the triple Fourier series (2.1.1) of the function f (x, y, z) is summable $K^{\lambda,\mu,\nu}$ $((\lambda,\mu,\nu) > 0)$ to the sum f (u, v, w) at x = u, y = v, z = w

2.4 LEMMAS :

For the proof of the theorem, the following lemmas are required :

Lemma (2.4.1) Vučkovič (1965), Let $\lambda > 0$ and $0 < x < \frac{\pi}{2}$, then

$$\frac{\operatorname{Im} \overline{\left|\lambda e^{ix} + 1\right.}}{\left|(\lambda \cos x + 1) \sin \frac{x}{2}\right.} = \frac{\left|\sin(\lambda \log l . \sin x)\right|}{\sin \frac{x}{2}} + 0(1) \text{ as } l \to \infty,$$

uniformly in x

Lemma (2.4.2), Let $\mu > 0$ and $0 < y < \frac{\pi}{2}$, then

$$\frac{\text{Im}\ \overline{|\mu e^{iy}+m}}{\overline{|(\mu\cos y+1).\ \sin\frac{y}{2}}} = \frac{|\sin\ (\mu\log m.\sin y)|}{\sin\ \frac{y}{2}} + 0\ (1)\ \text{as}\ m\to\infty,$$

uniformly in y.

Lemma (2.4.3), Let $\nu > 0$ and $0 < z < \frac{\pi}{2}$, then

$$\frac{\text{Im}\overline{|(\nu e^{iz}+n)}}{\overline{|(\nu\cos z+1)\sin\frac{z}{2}}} = \frac{|\sin\ (\nu\log n.\sin z)|}{\sin\ \frac{z}{2}} + 0(1)\ \text{as}\ n\to\infty,$$

uniformly in z.

2.5 PROOF OF THE THEOREM :

Let $S_{p,q,r}$ (u,v,w) denote the (p,q,r) th partial sum of the series (2.1.1) as x = u, y = v, z = w, then we have

$$S_{p,q,r}\text{-}f\,(u,v,w) = \frac{1}{8\pi^3}\int_0^\pi \int_0^\pi \int_0^\pi \phi\,(x,y,z)$$

$$\frac{\text{Sin}\left(p+\frac{1}{2}\right)x\ .\ \text{Sin}\left(q+\frac{1}{2}\right)y\ .\ \text{Sin}\left(r+\frac{1}{2}\right)z}{\text{Sin}\frac{x}{2}\ .\ \text{Sin}\frac{y}{2}\ .\ \text{Sin}\frac{z}{2}}\,dxdydz$$

Now

$$\frac{\overline{|\lambda}}{\overline{|\lambda+l}}\ \frac{\overline{|\mu}}{\overline{|\mu+m}}\ \frac{\overline{|\nu}}{\overline{|\nu+n}} \sum_{p=0}^{l}\sum_{q=0}^{m}\sum_{r=0}^{n} \begin{bmatrix} l \\ p \end{bmatrix} \begin{bmatrix} m \\ q \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix}$$

$$\lambda^p \mu^q \nu^r \left\{S_{p.q.r} - f(u,v,w)\right\}$$

$$= \frac{\overline{|\pi}\,\overline{|\mu}\,\overline{|\nu}}{8\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \phi\,(x,y,z) \sum_{p=0}^{l} \begin{bmatrix} l \\ p \end{bmatrix} \frac{\lambda^p\ \sin\left(p+\frac{1}{2}\right)x}{\overline{|\lambda+l}\ \sin\ \frac{x}{2}}$$

$$\sum_{q=0}^{m} \begin{bmatrix} m \\ q \end{bmatrix} \frac{\mu^{q} \sin\left(q+\frac{1}{2}\right) y}{\Gamma\mu+m \; \sin \frac{y}{2}} \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \frac{\nu^{r} \sin\left(r+\frac{1}{2}\right) z}{\Gamma\nu+n \; \sin \frac{z}{2}} dxdydz$$

$$S_{p,q,r}^{\lambda,\mu,\nu} - f(u,v,w) = \frac{\Gamma\lambda \, \Gamma\mu \, \Gamma\nu}{8\pi^{3}} \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \phi\,(x,y,z)$$

$$K_l(x)\, K_m\,(y)\, K_n(z)\, dxdydz$$

$$= 0 \left[\left\{ \int_0^{\frac{1}{l}} \int_0^{\frac{1}{m}} \int_0^{\frac{1}{n}} + \int_{\frac{1}{l}}^{\pi} \int_{\frac{1}{m}}^{\pi} \int_{\frac{1}{n}}^{\pi} \right\} \left| \phi\,(x,y,z) \right| \right.$$

$$\left. \left| K_l(x) \right| \left| K_m(y) \right| \left| K_n(z) \right| dxdydz \right] + o(1)$$

(2.5.1) $\quad = 0\,(\,I'\,) + 0\,(\,I''\,) + o(1)$ (say).

Now,

$$\left| K_l(x) \right| = 0 \left[\frac{\mathrm{Im}\left\{ e^{ix/2} \dfrac{\Gamma\left(\lambda e^{ix}+1\right)}{\Gamma\lambda e^{ix}} \right\}}{\Gamma(\lambda+1).\mathrm{Sin}\dfrac{x}{2}} \right]$$

$$= 0 \left[\frac{\mathrm{Im}\ \Gamma\left(\lambda e^{ix}+1\right)}{\Gamma\lambda+1.\mathrm{Sin}\ \dfrac{x}{2}} \right] + 0 \left[\frac{\mathrm{Re}\ \Gamma\lambda e^{ix}+1}{\Gamma\lambda+1} \right]$$

$$= 0 \left[\frac{\Gamma\lambda\cos x+1\ .\ \mathrm{Im}\ \Gamma\left(\lambda e^{ix}+1\right)}{\Gamma\lambda+1\ .\ \Gamma\lambda\cos x+1\ .\ \sin\ \dfrac{x}{2}} \right] + 0 \left[\frac{\lambda \cos x + 1}{\Gamma\lambda + 1} \right]$$

If, $0 < x < \frac{1}{l}$ then

$$\frac{\left|\overline{\lambda \cos x + l}\right.}{\left|\overline{\lambda + l}\right.} = \left[l^{-\lambda(1-\cos x)}\right]$$

$$= \left[e^{-\lambda(1-\cos x)\log l}\right]$$

$$= \left[e^{\frac{\lambda}{2}x^2 \log l}\right]$$

since, for $0 < x < \frac{1}{l}$, $0 < 1 - \cos x < \frac{x}{2}$

$$\therefore \; |K_l(x)| = 0\left[e^{\frac{-\lambda}{2}x^2 \log l} \frac{\operatorname{Im}\left|\overline{(\lambda e^{ix} + l)}\right.}{\left|\overline{(\lambda \cos x + l)}\right..\sin \frac{x}{2}}\right]$$

$$+ 0\left[e^{\frac{-\lambda}{2}x^2 \log l}\right], \text{ for } 0 < x < \frac{1}{l}.$$

Similarly

$$|K_m(y)| = 0\left[e^{\frac{-\mu}{2}y^2 \log m} \frac{\operatorname{Im}\left|\overline{(\mu e^{iy} + m)}\right.}{\left|\overline{(\mu \cos y + m)}\right..\sin \frac{y}{2}}\right]$$

$$+ 0\left[e^{\frac{-\mu}{2}y^2 \log m}\right], \text{ for } 0 < y < \frac{1}{m}$$

and

$$|K_n(z)| = 0\left[e^{\frac{-\nu}{2}z^2 \log n} \frac{\operatorname{Im}\left|\overline{(\nu e^{iz} + n)}\right.}{\left|\overline{(\nu \cos z + n)}\right..\sin \frac{z}{2}}\right]$$

$$+\,0\left[e^{\frac{-\nu}{2}z^2\log n}\right],\ \text{for } 0 < z < \frac{1}{n}.$$

Now

(2.5.2) $$I' = 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\left|\phi(x,y,z)\right|\left|K_l(x)\right|\left|K_m(y)\right|\left|K_n(z)\right|dxdydz\right]$$

Substitute the values of $\left|K_l(x)\right|\left|K_m(y)\right|\left|K_n(z)\right|$ in equation.

We will get eight terms and consider the terms are

$$I' = I'_{l_1,m_1,n_1} + I'_{l_1,m_1,n_2} + I'_{l_1,m_2,n_1} + I'_{l_2,m_1,n_1}$$

$$+\ I'_{l_1,m_2,n_2} + I'_{l_2,m_1,n_2} + I'_{l_2,m_2,n_1} + I'_{l_2,m_2,n_2}$$

Now consider the Ist term

$$I'_{l_1,m_1,n_1} = 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}x^2\log l}\ \frac{\text{Im}\sqrt{\lambda e^{ix}+l}}{\sqrt{\lambda\cos x + l.\sin\frac{x}{2}}}\right.$$

$$e^{-\frac{\mu}{2}y^2\log m}\ \frac{\text{Im}\sqrt{\mu e^{iy}+m}}{\sqrt{\mu\cos y + m.\sin\frac{y}{2}}}\ e^{-\frac{\nu}{2}z^2\log n}\ \frac{\text{Im}\sqrt{\nu e^{iz}+n}}{\sqrt{\nu\cos z + n.\sin\frac{z}{2}}}$$

$$\left.\left|\phi(x,y,z)\right|dxdydz\right]$$

$$= 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}x^2\log l}\ \frac{\left|\sin(\lambda\log l\sin x)\right|}{\sin\frac{x}{2}}\right.$$

$$e^{-\frac{\mu}{2}y^2\log m}\ \frac{\left|\sin(\mu\log m.\sin y)\right|}{\sin\frac{y}{2}}\ e^{-\frac{\nu}{2}z^2\log n}\ \frac{\left|\sin(\nu\log n.\sin z)\right|}{\sin\frac{z}{2}}$$

$\left.|\phi(x, y, z)|\, dxdydz\right]$

by Lemmas (2.4.1), (2.4.2) & (2.4.3)

$$= 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} \frac{|\sin(\lambda \log l\ x)|}{\sin\frac{x}{2}} .\right.$$

$$\frac{|\sin(\mu \log m.\ y)|}{\sin\frac{y}{2}} \frac{|\sin(\nu \log n.\ z)|}{\sin\frac{z}{2}}$$

$$\left.|\phi(x, y, z)|\, dxdydz\right]$$

$$= 0\left[\lambda \log l.\ \mu \log m.\ \nu \log n\right] . \int_0^{\frac{1}{l}} \int_0^{\frac{1}{m}} \int_0^{\frac{1}{n}} |\phi(x, y, z)|\, dxdydz$$

$$= 0\left[\lambda \log l.\ \mu \log m.\ \nu \log n.\ \frac{\alpha(l)}{lP_l} . \frac{\beta(m)}{m.P_m} . \frac{\gamma(n)}{nP_n}\right]$$

$$= 0\left[\lambda\, \mu\, \nu\, \frac{\alpha(l) \log l\, \beta(m) \log m.\ \gamma(n) \log n}{lP_l\ .\ mP_m\ .\ nP_n}\right]$$

$$= 0\left[\frac{\lambda\, \mu\, \nu\, P_l\, P_m\, P_n}{l\, m\, n\, P_l\, P_m\, P_n}\right]$$

$$= 0\left[\frac{\lambda\, \mu\, \nu}{lmn}\right]$$

$$= o(1) \qquad \text{as} \qquad (l, m, n) \to \infty.$$

2 nd term.

$$I'_{l_1,m_1,n_2} = 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}x^2\log l}\;\frac{|\sin(\lambda\log l\sin x)|}{\sin\frac{x}{2}}\right.$$

$$e^{-\mu/2y^2\log m}\frac{|\sin(\mu\log m.\sin y)|}{\sin\frac{y}{2}}.e^{-\nu/2z^2\log n}$$

$$\left.|\phi(x,y,z)|\,dxdydz\right]$$

$$= 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\frac{|\sin(\lambda\log l\,x)|}{\sin\frac{x}{2}}\frac{|\sin(\mu\log m.\,y)|}{\sin\frac{y}{2}}\;|\phi(x,y,z)|\,dxdydz\right.$$

$$= 0\left[\lambda\log l.\,\mu\log m.\,\frac{\alpha(l)}{lP_l}.\frac{\beta(m)}{mP_m}.\frac{\gamma(n)}{nP_n}\right]$$

$$= 0\left[\lambda\mu\,\frac{\alpha(l)\log l\,\beta(m)\log m\,\gamma(n)}{lP_l\,mP_m\,nP_n}\right]$$

$$= 0\left[\frac{\lambda\mu}{lmn}\;\frac{P_l\,P_m\,\gamma(n)}{P_l\,P_m\,P_n}\right]$$

$$= 0\left[\frac{\lambda\mu\gamma(n)}{lmn\,P_n}\right]$$

$$= o(1) \qquad \text{as } (l, m, n)\to\infty.$$

3rd term;

$$I'\,l_{l,m2,nl} = 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}x^2\log l}\;\frac{|\sin(\lambda\log l\sin x)|}{\sin\frac{x}{2}}\right.$$

$$e^{-\mu/2y^2\log m}.e^{-\nu/2z^2\log n}$$

$$\frac{\left|\sin\left(\nu \log n.\ \sin z\right)\right|}{\sin\frac{z}{2}} \left[\ \left|\phi\ (x,\ y,\ z)\right| dxdydz\ \right]$$

$$= 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} \frac{\left|\sin\left(\lambda \log l.\ x\right)\right|}{\sin\frac{x}{2}} . \frac{\left|\sin\left(\nu \log n.z\right)\right|}{\sin\frac{z}{2}}\right.$$

$$\left.\left|\phi(x,\ y,\ z)\right| dxdydz\right]$$

$$= 0\left[\lambda \log l.\ \nu \log n\right] \int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} \left|\phi\ (x,y,z)\right| dxdydz$$

$$= 0\left[\lambda \log l.\ \nu \log n.\ \frac{\alpha\,(l)}{lP_l} . \frac{\beta\,(m)}{mP_m} . \frac{\gamma\,(n)}{nP_n}\right]$$

$$= 0\left[\frac{\lambda\nu\alpha\,(l)\log l\ \beta\,(m)\ \gamma\,(n)\ \log n}{lmn\ P_l\ P_m\ P_n}\right]$$

$$= 0\left[\frac{\lambda\ \nu\ P_l\ \beta\,(m)\ P_n}{lmn\ P_l\ P_m\ P_n}\right]$$

$$= 0\left[\frac{\lambda\ \nu\ \beta\,(m)}{lmn\ P_m}\right]$$

$$= o\,(1) \qquad \text{as } (l,\ m,\ n) \rightarrow \infty$$

4th term.

$$I'_{l_2, m_1, n_1} = 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\lambda/2\, x^2 \log l} . e^{-\mu/2 y^2 \log m}\right.$$

$$\frac{\left|\sin\left(\mu \log m.\ \sin y\right)\right|}{\sin\frac{y}{2}} . e^{-\nu/2 z^2 \log n}$$

$$\frac{\left|\sin\ (\nu \log n.\ \sin z)\right|}{\sin \frac{z}{2}}$$

$$\left|\ \phi(x,\ y,\ z)\ \right|\ dxdydz\Big]$$

$$= 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\frac{\left|\sin\ (\nu \log m.y)\right|}{\sin \frac{y}{2}}\right]\frac{\left|\sin(\nu \log n.z)\right|}{\sin \frac{z}{2}}$$

$$\left|\ \phi(x,\ y,\ z)\ \right|\ dxdydz\Big]$$

$$= 0\ [\mu \log m.\ \nu \log n]\ \int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}}\phi\ (x,y,z)\ dxdydz\Big]$$

$$= 0\left[\mu \log m.\ \nu \log n\ \frac{\alpha(l)}{lP_l}\ \frac{\beta(m)}{mP_m}\ \frac{\gamma(n)}{nP_n}\right]$$

$$= 0\left[\frac{\mu\nu\alpha(l)\ \beta(m)\log m\ \gamma\ (n)\ \log n}{lmn\ P_l\ P_m\ P_n}\right]$$

$$= 0\left[\frac{\mu\ \nu\ \alpha(l)\ P_m\ P_n}{lmn\ P_l\ P_m\ P_n}\right]$$

$$= 0\left[\frac{\mu\ \nu\ \alpha\ (l)}{lmn\ P_l}\right]$$

$$= o\ (1) \qquad \text{as } (l,\ m,\ n) \to \infty$$

5th term,

$$I'_{l_1,m_2,n_2} = 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}x^2 \log l} - \frac{\left|\ \sin(\lambda \log l \sin x)\ \right|}{\sin \frac{x}{2}}\right.$$

$$e^{-\mu/2\, y^2 \log m} . e^{-\nu/2\, z^2 \log n} \; |\phi(x,y,z)| \, dxdydz \Big]$$

$$= 0 \left[\int_0^{\frac{1}{l}} \int_0^{\frac{1}{m}} \int_0^{\frac{1}{n}} \frac{|\sin(\lambda \log l\, x)|}{\sin \frac{x}{2}} \; |\phi(x,y,z)| \, dx\, dy\, dz \right]$$

$$= 0 \, [\lambda \log l] \int_0^{\frac{1}{l}} \int_0^{\frac{1}{m}} \int_0^{\frac{1}{n}} |\phi(x,y,z)| \, dxdydz$$

$$= 0 \left[\lambda \log l \; \frac{\alpha(l)}{lP_l} . \frac{\beta(m)}{mP_m} . \frac{\gamma(n)}{nP_n} \right]$$

$$= 0 \left[\frac{\lambda\, \alpha(l) \log l.\, \beta(m).\gamma(n)}{lmn\, P_l\, P_m\, P_n} \right]$$

$$= 0 \left[\frac{\lambda P_l\, \beta(m)\, \gamma(n)}{lmn\, P_l\, P_m\, P_n} \right]$$

$$= 0 \left[\frac{\lambda \beta(m)\, \gamma(n)}{lmn\, P_m\, P_n} \right]$$

$$= o(1) \qquad \text{as } (l, m, n) \to \infty.$$

6th term,

$$I'_{l_2, m_1, n_2} = 0 \left[\int_0^{\frac{1}{l}} \int_0^{\frac{1}{m}} \int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2} x^2 \log l} . e^{-\frac{\mu}{2} y^2 \log m} \right.$$

$$\frac{|\sin(\mu \log m. \sin y)|}{\sin \frac{y}{2}}$$

$$\left. . e^{-\nu/2\, z^2 \log n} \; |\phi(x,y,z)| \, dxdydz \right]$$

$$= 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} \frac{|\sin(\mu \log m.y)|}{\sin\frac{y}{2}} |\phi(x,y,z)| \; dxdydz\right]$$

$$= 0[\mu \log m] \, . \int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} |\phi(x,y,z)| \, dxdydz$$

$$= 0\left[\mu \log m. \frac{\alpha(l)}{lP_l} . \frac{\beta(m)}{mP_m} . \frac{\gamma(n)}{nP_n}\right]$$

$$= 0\left[\frac{\mu\alpha(l)\,\beta(m)\log m\,\gamma(n)}{lmn\; P_l\, P_m\, P_n}\right]$$

$$= 0\left[\frac{\mu\,\alpha(l)\,P_m\,\gamma(n)}{lmn\; P_l\, P_m\, P_n}\right]$$

$$= 0\left[\frac{\mu\,\alpha(l)\,\gamma(n)}{lmn\; P_l\, P_n}\right]$$

$$= o(1) \text{ as } (l, m, n) \to \infty$$

7th term,

$$I'_{l_2,m_2,n_1} = 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}x^2 \log l}\, e^{-\frac{\mu}{2}y^2 \log m}\, e^{-\frac{\nu}{2}z^2 \log n}\right.$$

$$\frac{|\sin(\nu \log n \,.\, \sin z)|}{\sin\frac{z}{2}}$$

$$\left.|\phi(x,y,z)| \, dxdydz\right]$$

$$= 0\left[\int_0^{\frac{1}{l}}\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} \frac{|\sin(\nu \log n.z)|}{\sin\frac{z}{2}} |\phi(x,y,z)| \, dxdydz\right]$$

$$= 0\left[\nu \log n\right] \int_0^{\frac{1}{l}} \int_0^{\frac{1}{m}} \int_0^{\frac{1}{n}} \left|\phi\,(x,y,z)\right| dxdydz$$

$$= 0\left[\nu \log n.\ \frac{\alpha\,(l)}{l\,P_l}\ \frac{\beta(m)}{mP_m}\ \frac{\gamma(n)}{nP_n}\right]$$

$$= 0\left[\frac{\nu\alpha(l)\ \beta\ (m)\ \gamma(n)\ \log n}{lmn\ P_l\ P_m\ P_n}\right]$$

$$= 0\left[\frac{\nu\alpha(l)\ \beta\ (m)\ P_n}{lmn\ P_l\ P_m\ P_n}\right]$$

$$= 0\left[\frac{\nu\alpha\ (l)\ \beta\ (m)}{lmn\ P_l\ P_m}\right]$$

$$= o\,(1) \qquad \text{as } (l, m, n) \to \infty.$$

8th term,

$$I'_{l_2,m_2,n_2} = 0\left[\int_0^{\frac{1}{l}} \int_0^{\frac{1}{m}} \int_0^{\frac{1}{n}} e^{-\frac{\lambda}{2}x^2 \log l} . e^{-\frac{\mu}{2}y^2 \log m} . e^{-\frac{\nu}{2}z^2 \log n}\right.$$

$$\left.\left|\phi\,(x,y,z)\right|\ dxdydz\right]$$

$$= 0\left[\int_0^{\frac{1}{l}} \int_0^{\frac{1}{m}} \int_0^{\frac{1}{n}} \phi\,(x,y,z)\ dxdydz\right]$$

$$= 0\left[\frac{\alpha(l)}{lP_l}.\ \frac{\beta(m)}{mP_m}.\ \frac{\gamma(n)}{nP_n}\right]$$

$$= 0\left[\frac{\alpha(l)\ \beta(m)\ \gamma(n)}{lmn\ P_l\ P_m\ P_n}\right]$$

$$= o\,(1) \quad \text{as } (l, m, n) \to \infty.$$

Thus we get that

(2.5.3) $I' = 0(1)$ as $(l,m,n) \to \infty$

For $\frac{1}{l} \le \delta_1 < x < \pi$, then

$$|K_{l(x)}| = 0 \left[\frac{m^{-\lambda(1-\cos\delta_1)}}{\sin\frac{\delta_1}{2}} \right]$$

$= o\,(1)$, as $l \to \infty$

Similary for,

$$\frac{1}{m} \le \delta_2 < y < \pi, \ |K_m(y)| = 0\ (1) \text{ as } m \to \infty$$

and $$\frac{1}{n} \le \delta_3 \le z < \pi, \ |K_n(z)| = 0\ (1) \text{ as } n \to \infty$$

$$\therefore |K_l(x)|\ |K_m(y)|\ |K_n(z)| = 0\ (1) \text{ as } (l,m,n) \to \infty.$$

Lastly when $\frac{1}{l} < x < \pi$, $\frac{1}{m} < y < \pi$, $\frac{1}{n} < z < \pi$

the $\phi\,(x,y,z)$ is bounded . then we have.

(2.5.4) $$I'' = 0 \left[\int_{\frac{1}{l}}^{\pi} \int_{\frac{1}{m}}^{\pi} \int_{\frac{1}{n}}^{\pi} |\phi\,(x,y,z)|\ dxdydz \right] = 0\ (1) \text{ as } (l,m,n) \to \infty$$

Therefore using (2.5.1), (2.5.3) and (2.5.4)

we get $$S_{p,q,r}^{\lambda,\mu,\nu} - f\,(u,v,w) = 0\ (1) + 0\ (1)$$

$$= 0(1) \text{ as } (l,m,n) \to \infty.$$

This completes the proof of the theorem.