CHAPTER-10

On The Degree Of Approximation Of the conjugate of a Function Belonging To The Class Lip (α,p) By (N, p_n) Means Of A Conjugate Series.

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ON THE DEGREE OF APPROXIMATION OF THE CONJUGATE OF A FUNCTIONS BELONGING TO THE CLASS LIP (α, p) BY (N, p_n) MEANS OF A CONJUGATE SERIES

10.1 DEFINITIONS AND NOTATIONS

Let $f(x)$ be a function with period 2π and integrable in the sense of lebesgue over $(-\pi,\pi)$ let its Fourier series be given by

(10.1.1) $$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=1}^{\infty} A_n(x)$$

then

(10.1.2) $$f\overline{(x)} \sim \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x)$$

is called the conjugate series of $f(x)$ we shall use the notations

(10.1.3) $$\psi(t) = f(x+t) - f(x-t)$$

we define the norm $\text{II}.\,\text{II}_p$ as

(10.1.4) $$\text{II}\ f\ \text{II}_p = \left\{ \int_0^{2\pi} |f(x)|^p\, dx \right\}^{\frac{1}{p}},\ p \geq 1$$

and let the degree of approximation $E_n(f)$ be given by (see zygmund 1959)

(10.1.5) $$E_n(f) = \min_{T_n} \| f - T_n \|_p$$

where $T_n(x)$ is a trignometrical polynomial of degree n. We say that

$f(x) \in \text{Lip}(\alpha, q)$ for $a \leq x \leq b$

(10.1.6) $$\text{If} \left\{ \int_a^b |f(x+h) - f(x)| \leq A|h|^{\alpha},\ 0 < \alpha \leq 1 \right\}$$

[See Def. 5.38 of Mc. Fadden (1942)]

10.2 Let $\{p_n\}$ be a non-negative, non-increasing generating sequence for the (N, p_n) method such that

(10.2.1) $P_n \equiv P(n) = p_0 + p_1 \,........ + P_n \rightarrow \infty$ as $n \rightarrow \infty$

we write $p(y) = p[y]$ and $P_{(y)} = P_{[y]}$

where [y] as usual denotes the greatest integer less than y.

In 1981 Qureshi proved the following theorem

Theorem A-

If the sequence $\{p_n\}$ satisfies the following conditions.

$n \mid p_n \mid < C \mid P_n \mid$

and $\sum_{k=1}^{n} k \mid p_k - p_{k-1} \mid < C \mid P_n \mid$

then the degree of approximation of f (x), conjugate to a periodic function f with period 2π and belonging to the class of Lip α, $0 < \alpha \leq 1$, λ by (N, p_n) means of its conjugate series, is given by

$$\left| \overline{f}(x) - \overline{t}_n(x) \right| = O\left(\frac{1}{P_n} \sum_{k=1}^{n} \frac{P_k}{k^{\alpha+1}} \right)$$

where $t_n(x)$ are the (N, p_n) - means of the conjugate series (10.1.2)

10.2 MAIN THEOREM

In this chapter we have generalised the above theorem (A) in the following form :

THEOREM

If (N, p_n) is a regular Nörlund summability method defined by real non-negative monotonic, non-increasing sequence of cocfficients $\{p_n\}$ such that $P_n = \sum_{v=0}^{n} p_v \neq 0$, then the degree of approximation of a function $\overline{f(x)}$, conjugate to a periodic function f with period 2π and belonging to the class Lip (α,p), $\left(\frac{1}{p}<\alpha\leq 1\right)$, by Nörlund means (N, p_n) of its conjugate series satisfies

$$\|\bar{t}_n(x)-\bar{f}(x)\|_p = O\left(\frac{1}{(n+1)^{\alpha-1/p}}\right), \quad n=1,\ 2,\ 3\ \ldots\ldots$$

where $\bar{t}_n(x)=\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\bar{s}_k(x)$

i.e. (N, p_n) means of the conjugate series of the fourier series (10.1.2),

10.3 LEMMAS

For the proof of the theorem, we require the following lemma :

LEMMA 1

(Mc Fadden 1942, Lemmas 5.11) -

If $\{p_n\}$ is non-negative and non-increasing then for

$0 \leq a \leq b < \infty;\ \ 0 \leq t \leq \pi$ and any n, we have

$$\left|\sum_{k=a}^{q} p_k\, e^{(n-k)t}\right| \leq P_{\left(\frac{1}{t}\right)}.$$

LEMMA 2

(Mc Fadden 1942, Lemma 5.40)-

If f (x) belongs to Lip (α,q) on $[0,\pi]$, then $\psi(t)$ also belongs to Lip (α,q) on $[0,\pi]$

10.3 PROOF OF THE THEOREM :

After Qureshi (1981), we write

$$\bar{f}(x) - \bar{t}_n(x) = \frac{1}{\pi}\int_0^{\pi} \psi(t)\,\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\,\frac{\cos\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\,dt$$

where $\psi(t) = f(x+t) - f(x-t)$

we have

$$\bar{f}(x) - \bar{t}_n(x) = \frac{1}{2\pi P_n}\left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n+1}}^{\pi}\right)\frac{\psi(t)}{\sin\frac{t}{2}}\sum_{k=0}^{n} p_{n-k}\cos\left(k+\frac{1}{2}\right)t\,dt$$

$= I_1 + I_2$ say.

Now

$$I_1 = \frac{1}{2\pi P_n}\int_0^{\frac{1}{n+1}}\frac{\psi(t)}{\sin\frac{t}{2}}\sum_{k=0}^{n} p_{n-k}\cos\left(k+\frac{1}{2}\right)t\,dt$$

By Holders inequality and Lemma 2, we have

$$I_1 \le \frac{1}{2\pi P_n}\left(\int_0^{\frac{1}{n+1}}\left(\frac{t\,|\,\psi(t)\,|}{t^{\alpha}}\right)^p dt\right)^{\frac{1}{p}}$$

$$\left(\int_0^{\frac{1}{n+1}}\left(\frac{1}{\sin\left(\frac{t}{2}\right)t^{1-\alpha}}\left|\sum_{k=0}^{n} p_{n-k}\cos\left(k+\frac{1}{2}\right)t\right|\right)^q dt\right)^{\frac{1}{q}}$$

$$= O\left(\frac{1}{P_n}\right)O\left(\frac{1}{n}\right)O\left\{\left(\int_0^{\frac{1}{n-1}}\left(\frac{1}{t^{2-\alpha}}\sum_{k=0}^{n} p_{n-k}\right)^q dt\right)^{\frac{1}{q}}\right\}$$

$$= O\left(\frac{1}{n+1}\right)O\left\{\left(\int_0^{\frac{1}{n+1}} t^{\alpha q-2q}\,dt\right)^{\frac{1}{q}}\right\}$$

$$= O\left(\frac{1}{n+1}\right)O\left\{\left(\frac{1}{n+1}\right)^{\alpha-2+\frac{1}{q}}\right\}$$

$$= O\left\{\left(\frac{1}{n+1}\right)^{\alpha-\frac{1}{p}}\right\}$$

Also

$$I_2 = \frac{1}{2\pi P_n q}\int_{\frac{1}{n+1}}^{\pi}\frac{\psi(t)}{\sin\left(\frac{t}{2}\right)}\sum_{k=0}^{n} p_{n-k}\cos\left(k+\frac{1}{2}\right)t\,dt$$

Similarly, as above, we have

$$I_2 = O\left(\frac{1}{P_n}\right)\left\{\left(\int_{\frac{1}{n+1}}^{\pi}\left(t^{-\delta}\frac{|\psi(t)|}{t^{\alpha}}\right)^{p}dt\right)^{\frac{1}{p}}\right\}$$

$$\left\{\left(\int_{\frac{1}{n+1}}^{\pi}\left|\frac{P\left(\frac{1}{t}\right)}{t^{1-\delta-\sigma}}\right|^{q}dt\right)^{\frac{1}{q}}\right\} \text{ (by lemma 1)}$$

$$= 0\left(\frac{1}{P(n)}\right)\left\{\left(\int_{\frac{1}{n+1}}^{\pi}\left(t^{-\delta}\frac{t^{\alpha-\left(\frac{1}{p}\right)}}{t^{\alpha}}dt\right)^{\frac{1}{p}}\right)\right\}$$

$$\left\{\left(\int_{1}^{n+1}\left(\frac{P(y)}{y^{\alpha+\delta-1}}\right)^{q}\frac{dy}{y^{2}}\right)^{\frac{1}{q}}\right\}$$

$$= O\left(\frac{1}{P(n)}\right)\left\{\left(\int_{\frac{1}{n+1}}^{\pi}t^{-\delta p-1}\,dt\right)^{\frac{1}{p}}\right\}$$

$$\left\{\left(\int_1^{n+1} \frac{(P(y))^q}{y^{q\alpha+q\delta-q+2}}\, dy\right)^{\frac{1}{q}}\right\}$$

$$= O\left(\frac{1}{P(n)}\right) O\left\{\left(\frac{1}{n+1}\right)^{-\delta}\right\}\left\{O\left(\frac{P(n)}{(n+1)^{\alpha+\delta-1+\frac{1}{q}}}\right)\right\}$$

$$= O\left(\frac{1}{(n+1)^{\alpha-1+\frac{1}{q}}}\right)$$

$$= O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right) \qquad \left(\because \frac{1}{p}+\frac{1}{q}=1\right)$$

Hence $\left|\, \overline{t_n}(x) - \overline{f}(x) \,\right| = O\left\{\left(\frac{1}{n+1}\right)^{\alpha-\left(\frac{1}{p}\right)}\right\}$.

Therefore

$$\left\| \overline{t_n}(x) - \overline{f}(x) \right\|_p = O\left[\left\{\int_0^{2\pi}\left(\left(\frac{1}{n+1}\right)^{\alpha-\frac{1}{p}}\right)^p dx\right\}^{\frac{1}{p}}\right]$$

$$= O\left[\left(\frac{1}{n+1}\right)^{\alpha-\frac{1}{p}}\left(\int_0^{2\pi} dx\right)^{\frac{1}{p}}\right]$$

$$= O\left[\frac{1}{(n+1)^{\alpha-1/p}}\right]$$

This completes the proof of the theorem.