CHAPTER – 9

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ON THE BOREL SUMMABILITY OF LEGENDRE SERIES

9.1 The legendre series associated with a Lebesgue integrable function $f(x)$ in the linear interval [-1, +1] is defined by:-

(9.1.1) $$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)\, dx$$

where

(9.1.2) $$a_n = \left(n+\frac{1}{2}\right)\int_{-1}^{+1} f(x)\, P_n(x)\, dx$$

and $P_n(x)$ denotes the n th Legendre polynomial, defined by the following expression

$$\frac{1}{\left(1-2xz+z^2\right)} = \sum_{n=0}^{\infty} z^n\, P_n(x)$$

9.2 A series $\sum_{n=0}^{\infty} u_n$ the sequence of partial sum $\{S_n\}$ is said to be summable by Borel method or simply summable (B) to the sum S if

$$\lim_{r\to\infty} e^{-r} \sum_{n=0}^{\infty} \frac{r^n}{\underline{|n}} S_n$$

exists and equal S.

9.3 Pati (1961) established the following result for the (N,p_n) summability of Fourier series.

THEOREM (1)

If (N,p_n) be a regular Nörlund method defined by a real, non-negative, monotonic, non-increasing sequence of coefficients $\{p_n\}$ such that

(9.3.1) $$P_n = \sum_{v=0}^{n} P_v \rightarrow \infty, \quad \text{as} \quad n \rightarrow \infty$$

and

$$\log n = O(P_n), \quad \text{as} \quad n \rightarrow \infty$$

Then, if

(9.3.2) $$\Phi(t) = \int_0^t |\phi(u)|\, du = 0\left[\frac{t}{P_\tau}\right] \quad \text{as } t \rightarrow 0^+$$

the Fourier series of a function $f(t)$ is summable (N, p_n) to $f(x)$ at the point $t = x$.

In this chapter we have generalised the theorem (1) for Legendre series in following from:

THEOREM:

Let $\{p_n\}$ be a non-negative monotonic, non-increasing sequence of real constant and let $\xi(t)$ be a positive integral, monotonic, decreasing function of t such that

(9.3.3) $$\int_1^n \frac{\xi(u)\, du}{u} = O(P_n) \text{ as } n \rightarrow \infty$$

(9.3.4) $$\text{If } \int_0^t |f(x \pm u) - f(x)|\, du = 0\left[\frac{t\xi\left(\frac{1}{t}\right)}{P_\tau}\right] \text{ as } t \rightarrow +0^+$$

then legendre series is summable (N, p_n) to $f(x)$ at an internal point x of the interval $(-1+\in, 1+\in)$, $o < \in < 1$.

9.4 For the proof of the theorem, we require following lemmas:

Lemma 1.

(9.4.1) $$\sum_{v=0}^{n} (z_v+1) P_v(x) P_v(y) = (n+1) \frac{P_{n+1}(y) P_n(x) - P_n(y) P_{n+1}(x)}{y-x}$$

Lemma 2.

Under the condition of theorem, we have

$$\int_0^t \left| f\{\cos(\theta - v)\} - f(\cos\theta) \right| dv = 0\left(\frac{t}{\log\frac{1}{t}}\right) \text{ as } t \to +0$$

where $x = \cos\theta$, $x+u = \cos\phi$ and $\theta - \phi = v$

The proof of this lemma follows on the line of Foa [1943].

9.5 PROOF OF THE THEOREM :

The nth partial sum of the legendre series (9.1.1) is given by

$$S_n(x) = \sum_{v=0}^{n} a_v P_v(x)$$

$$= \frac{n+1}{2} \int_{-1}^{+1} f(y) \frac{P_{n+1}(y) P_n(x) - P_n(y) P_{n-1}(x)}{y-x} dy$$

by lemma (1)

putting $f(y)=1$ it is easy to see that

$$1 = \frac{n+1}{2} \int_{-1}^{+1} \frac{P_{n+1}(y) P_n(x) - P_n(y) P_{n+1}(x)}{y-x} dy$$

Hence

$$S_n(x) - f(x) = \frac{n+1}{2} \int_{-1}^{+1} [f(y) - f(x)] \frac{P_{n-1}(y) P_n(x) - P_n(y) P_{n-1}(x)}{y-x} dy$$

Let us take a positive number δ less than unity and consider it as the sum of two other numbers μ and r. Let d be another positive number such that $0 < d < \mu$, and let μx and $\mu x'$ are two continuous functions of x with in (-1, +1), which lie within the limits $d \leq \mu x \leq \mu$, $d \leq \mu x' \leq \mu$.

Thus for $-1+S \leq x \leq 1-S$, we have

$$S_n(x) - f(x) = \frac{n+1}{2}\left[\int_{-1}^{x-\mu x} + \int_{x-\mu x}^{x+\mu x'} + \int_{x+\mu x'}^{+1}\right]$$

$$[f(y) - f(x)]\ \frac{P_{n+1}(y)\,P_n(x) - P_n(y)\,P_{n+1}(x)}{y-x}\,dy + 0(1)$$

(9.5.1) $$= A_n(x) + B_n(x) + C_n(x) + 0(1) \text{ say,}$$

Hobson [1909] show that uniformly for $-1 + S \leq x \leq 1 - S$

(9.5.2) $$\lim_{n\to\infty} A_n(x) = 0$$

and $\lim_{n\to\infty} C_n(x) = 0$

Now let us suppose $x = \cos\theta$, $y = \cos\phi$, $0 < \theta < \pi$

$0 < \phi < \pi$, $1-\delta = \cos\rho$, $1-(\mu+\delta) = 1-S = \cos(\rho+\sigma)$

$0 < \rho + \frac{\pi}{2}$, $0 < \delta$; $\sigma + p < \frac{\pi}{2}$

Thus if $\frac{1}{r^\alpha}\left(0 < \alpha < \frac{1}{2}\right)$

denotes the minimum of $[\arccos u - \arccos(u+\mu)]$ for u in $(-1, 1-\mu)$, we have on the lines of Sansone (1959).

$$B_n(\cos\theta) = \frac{n+1}{2}\int_{\theta-\frac{1}{r^\alpha}}^{\theta+\frac{1}{r^\alpha}} [f(\cos\phi) - f(\cos\theta)]$$

$$\frac{P_{n+1}(\cos\phi)\,P_n(\cos\theta) - P_n(\cos\phi)\,P_{n+1}(\cos\theta)}{\cos\phi - \cos\theta}\sin\phi\,d\phi$$

in which

$$\rho + \sigma \le \theta \le \pi - (\rho + \sigma);\ 0 < \frac{1}{r^{\alpha}} < \sigma$$

With successive transformation, we obtain

(9.5.3) $$B_n(\cos\theta) = D_n\theta + E_n\theta$$

where

$$D_n\theta = \frac{1}{2\pi\,|\sin\theta} \int_{\theta-\frac{1}{r^{\alpha}}}^{\theta+\frac{1}{r^{\alpha}}} \left[f(\cos\phi) - f(\cos\theta)\right] \frac{\sin\ (n+1)\ (\theta-\phi)}{\sin\ \frac{1}{2}\ (\theta-\phi)} d\phi$$

and obviously on the line of Sansone (1959).

(9.5.4) $$E_n(\theta) = 0\ (1),\ \text{as } n \to \infty$$

uniformly when x lies within (-1+s, 1-s)

uptting $\theta - \phi = t$, we get

$$S_n(x) - f(x) = \frac{1}{\pi|\sin\theta} \int_0^{\frac{1}{r^{\alpha}}} \left[f\{\cos(\theta - t)\} - f(\cos\theta)\right]$$

$$\sqrt{\sin\ (\theta - t)}\, \frac{1}{P_n} \sum_{k=0}^{n-1} p_k \frac{\sin\ (n+1)}{\sin\ \frac{1}{2}\ t} dt$$

$$= O\left[\int_0^{\frac{1}{r^{\alpha}}} |\psi(t)|\ |N_n(t)| dt\right]$$

$$= 0\left[\left\{\int_0^{\frac{1}{r}} + \int_{\frac{1}{r}}^{\frac{1}{r^{\alpha}}}\right\} |\psi(t)|\ |N_n(t)| dt\right]$$

(9.5.5) $= I + J$, say.

Now uniformly in $0 < t \le \frac{1}{r}$

$$N_n(t) = \frac{1}{P_n} \sum_{k=0}^{n} p_k \sin(n+1)t$$

$$= O\left[\frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{2(n+1)\left|\sin \frac{1}{2}t\right|}{\left|\sin \frac{1}{2}t\right|}\right]$$

$N_n(t) = O(1)$, as $r \to \infty$

Again for $\frac{1}{r} \le t \le \frac{1}{r^\alpha}$

$$N_n(t) = O\left[\frac{1}{P_n \sin\left(\frac{1}{2}t\right)} \left|\sum_{k=0}^{n-1} p_k \sin(n+1)t\right|\right]$$

$$= O\left[\frac{P_\tau}{tP_n}\right].$$

Now considering I, we have

$$I = \int_0^{\frac{1}{r}} |\psi(t)|\, |N_n(t)| dt$$

$$= O\left[n \int_0^{\frac{1}{r}} |\psi(t)|\, dt\right]$$

$$= O\left[n\, 0\left(\frac{\xi(r)}{rP_r}\right)\right]$$

$$= 0\left(\frac{\xi(r)}{rP_r}\right)$$

(9.5.6) $$= 0(1), \text{ as } r \to \infty$$

$$J = O\left[\frac{1}{P_n}\int_{\frac{1}{r}}^{\frac{1}{r^\alpha}} |\psi(t)|\ \frac{P_\tau}{t}dt\right].$$

Integrating by parts, we have

$$J = O\left[\frac{1}{P_n}\left\{\Psi(t)\frac{P_\tau}{t}\right\}_{\frac{1}{r}}^{\frac{1}{r^\alpha}}\right] + O\left[\frac{1}{P_n}\int_{\frac{1}{r}}^{\frac{1}{r^\alpha}}\Psi(t)\frac{P_\tau}{t^2}dt\right] + O\left[\frac{1}{P_n}\int_{\frac{1}{r}}^{\frac{1}{r^\alpha}}\Psi(t)\frac{dP_\tau}{t}\right]$$

$$= J_1 + J_2 + J_3 \quad \text{(say)}$$

$$J_1 = O\left[\frac{1}{P_n}\left\{\Psi(t)\frac{P_\tau}{t}\right\}_{\frac{1}{r}}^{\frac{1}{r^\alpha}}\right]$$

$$= 0\left[\frac{1}{P_n}\left\{\frac{\xi\left(\frac{1}{t}\right)t}{P_\tau}\ \frac{P_\tau}{t}\right\}_{\frac{1}{r}}^{\frac{1}{r^\alpha}}\right]$$

$$= 0\left[\left\{\frac{\xi\left(\frac{1}{t}\right)}{P_n}\right\}_{\frac{1}{r}}^{\frac{1}{r^\alpha}}\right]$$

$$= 0\left[\frac{\xi(n)}{P_n}\right]$$

$$= 0(1) \quad \text{as } n \to \infty.$$

$$J_2 = O\left[\frac{1}{P_n}\int_{\frac{1}{r}}^{\frac{1}{r^\alpha}}\Psi(t)\ \frac{p_\tau}{t^2}dt\right]$$

$$= 0'\left[\frac{1}{P_n}\int_{\frac{1}{r}}^{\frac{1}{r^\alpha}}\frac{\xi\left(\frac{1}{t}\right)t}{P_\tau}\ \frac{p_\tau}{t^2}dt\right]$$

$$= 0\left[\frac{1}{P_n}\int_{\frac{1}{r}}^{\frac{1}{r^\alpha}} \frac{\xi\left(\frac{1}{t}\right)}{t}\, dt\right]$$

$$= 0\left[\frac{1}{P_n}\int_{r^\alpha}^{r} \frac{\xi(u)}{u}\, du\right]$$

$$= 0\,(1) \quad \text{as } n \to \infty.$$

Lastly,

$$J_3 = O\left[\frac{1}{P_n}\int_{\frac{1}{r}}^{\frac{1}{r^\alpha}} \Psi\,(t)\; \frac{dp_\tau}{t}\right]$$

$$= O\left[\frac{1}{P_n}\int_{r}^{r^\alpha} \Psi\left(\frac{1}{u}\right) u\, d\, p_u\right]$$

$$= 0\,(1)\; + O\left[\frac{1}{P_n}\sum_{k=1}^{n-1} kp_k \Psi\left(\frac{1}{k}\right)\right]$$

$$= 0\,(1)\; + O\left[\frac{1}{P_n}\sum_{k=1}^{n-1} P_k \Psi\left(\frac{1}{k}\right)\right]$$

$$= 0\,(1)\; + O\left[\frac{1}{P_n}\sum_{k=1}^{n-1} 0\left(\frac{\lambda(k)}{k}\right)\right]$$

$$= 0\,(1)\; + O\left[\frac{1}{P_n} 0\left(\int_{1}^{n} \lambda \frac{(u)}{u} du\right)\right]$$

$$= 0\,(1)\; + 0\,(1) \quad \text{as } n \to \infty$$

$$= 0\,(1) \qquad \text{as } n \to \infty$$

$$J = 0\,(1)\ + 0\,(1)\ +\ 0\,(1)$$

(9.5.7) $$= 0\,(1)\ \to \infty .$$

The combining (9.5.1), (9.5.5), (9.5.6) and (9.5.7) we get

$$e^{-r}\sum_{n=0}^{\infty}\frac{r^{n}}{\underline{|n}}\{S_{n}(x)-f(x)\}=0\,(1),\ \text{as } r\to\infty .$$

This completes the proof of the theorem.