# CHAPTER – 6

# *On Uniform Triangular Matrix Summability Of Legendre Series*

# CHAPTER – 6

## UNIFORM TRIANGULAR MATRIX SUMMABILITY OF LEGENDRE SERIES

6.1 Let $\sum_{n=0}^{\infty} u_n(x)$ be an infinite series with $\{ S_n(x) \}$ as the sequence of its nth partial sums. Let $( \lambda_{n,k} )$ $(n = 0,1,2,........., k = 0, 1,.........., n; \lambda_{n,0} = 1)$ be a triangular matrix of real or complex numbers.

Let

(6.1.1) $$\sigma_n(x) = \sum_{k=0}^{n} \lambda_{n,k} u_k(x) = \sum_{k=0}^{n} \Delta \lambda_{n,k} S_k(x)$$

Where

$$\Delta \lambda_{n,k} = \lambda_{n,k} - \lambda_{n,k+1}$$

and

$$\Delta^2 \lambda_{n,k} = \Delta \lambda_{n,k} - \Delta \lambda_{n,k+1}$$

If

(6.1.2) $$\sigma_n(x) \to s(x) \quad \text{as } n \to \infty$$

then we say that the series $\sum u_n(x)$ is summable $(\wedge)$ to s (x) at a point x.

(6.1.3) If $\sigma_n(x) - S(x) = 0(1)$

as $n \to \infty$ uniformly in a set E,

then we say that the series $\sum u_n(x)$ is summable $(\wedge)$ uniformly in set E to the sum S (x).

In particular, if

(6.1.4) $\Delta \lambda_{n,k} = \begin{cases} [(n+1-k) \log n]^{-1}, & k \le n \\ 0 & k > n \end{cases}$

then $\sigma_n(x)$ defined by (6.1.1) is same as the harmonic mean $\left(N, \frac{1}{n+1}\right)$ of the sequence $\{S_n(x)\}$.

6.2 The Legendre series associated with a Lebesgue integrable function f (x) in the range (-1,1) is given by

(6.2.1) $$f(x) \sim \sum_{n=0}^{\infty} a_n \, p_n(x)$$

where

(6.2.2) $$a_n = \left(n+\frac{1}{2}\right) \int_{-1}^{1} f(x) \, p_n(x) \, dx$$

and the $n^{th}$ Legendre polynomial $p_n(x)$ is defined by the generating function

(6.2.3) $$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} p_n(x) \, z^n .$$

We use the following notation

(6.2.4) $$\psi(t) = \psi_0(t) = f\{\cos(\theta - t)\} - f(\cos\theta)$$

and

(6.2.5) $$N_n(t) = \sum_{k=0}^{n} \Delta \lambda_{n,k} \frac{\sin(k+1)t}{\sin\frac{t}{2}}$$

6.3 Dwivedi [1970] has established the following theorem on uniform harmonic summability of Legendre series:

**THEOREM A :**

If

$$(6.3.1) \qquad \int_0^t \left| f(x \pm u) - f(x) \right| du = 0 \left[ \frac{t}{\log\left(\frac{1}{t}\right)} \right] \text{as } t \to +0$$

uniformly in a set E defined in the interval (-1,1) in wich f (x) is bounded then the series (6.2.1) is summable by harmonic means uniformly in E to the sum f (x).In the present chapter, we propose to extend the above result for uniform triangular matrix summability of Legendre series by proving following:

**THEOREM :**

$$(6.3.2) \qquad \int_t^{\eta} \frac{|\psi(u)|}{u^2} du = 0 \left[ \frac{1}{tP_{\left(\frac{1}{t}\right)}} \right] \quad \text{as } t \to +0$$

as $t \to +0$, where $0 \leq t < \eta < \pi$ is fixed, uniformly in a set E in the interval (-1,1) in which f(x) is bounded such that $P_n \to \infty$ and $\log n = O(P_n)$ as $n \to \infty$ then the series (6.2.1) is summable (^) uniformly in a set E to the sum f (x).

Note.

It is worth noticing that our condition (6.3.2) is less stronger than the condition (6.3.1) in Theorem A in the following sense.

Following on the lines Foa' [1943], it may be easily proved that under condition (6.3.1).

we have

(6.3.3) $$\int_0^t \left| f\{\cos(\theta - y)\} - f(\cos\theta) \right| dy = 0\left[\frac{t}{P_{\left(\frac{1}{t}\right)}}\right] \text{ as } t \to +0$$

where $x = \cos\theta,\ x + u = \cos\phi,\ \theta - \phi = y$;

so that in view of (6.2.4)

(6.3.4) $$\int_0^t |\psi(y)|\, dy = 0\left[\frac{t}{P_{\left(\frac{1}{t}\right)}}\right] \text{ as } t \to +0.$$

Now it can be easily seen, as under, that (6.3.2) implies (6.3.4) on integrating by parts. we have

$$\int_0^t |\psi(y)|\, dy = \int_0^t y^2 \frac{|\psi(y)|}{y^2} dy$$

$$= \left[ y^2\, 0\left(\frac{1}{yP_{\left(\frac{1}{y}\right)}}\right) \right]_0^t - 2\int_0^t y\, 0\left(\frac{1}{yP_{\left(\frac{1}{y}\right)}}\right) dy$$

$$= 0\left[\frac{t}{P_{\left(\frac{1}{t}\right)}}\right], \text{ as } t \to 0$$

6.4 In due course of the proof of our theorem, we shall use the following lemmas:

**Lemma 1**

$$\sum_{v=0}^{n} (2v+1)\, p_v(x)\, p_v(y)$$

(6.4.1) $$= \frac{(n+1)\, p_{n+1}(y)\, p_n(x) - p_{n+1}(x)\, p_n(y)}{y - x}.$$

This identity is known as Christoffel's formula of summation.

**Lemma 2**

If $\{\Delta\lambda_{n,k}\}_{k=0}^{n}$ is a non- negative and non-decreasing sequence with respect to k, then for $0 \le a < b < \infty, 0 < t \le \pi$ for any n,

(6.4.2) $$\left| \sum_{k=0}^{b} \Delta \lambda_{n,n-k} e^{i(n-k)t} \right| = O\left[ \frac{1}{t} \Delta \lambda_{n,n-\tau} \right]$$

where $\tau$ is the integral part of $\frac{1}{t}$.

**Lemma 3**

If $\{\Delta\lambda_{n,k}\}_{k=0}^{n}$ is a non- negative and non-decreasing with respect to k, such that

$$\sum_{k=0}^{n} \Delta \lambda_{n,k} = 1$$

then

as $n \to \infty$

$$\Delta\lambda_{n,k} = O\left(\frac{1}{n-k+1}\right)$$

uniformly for all $k \le n$, so that we get

(6.4.3) $$\Delta \lambda_{n,0} = O\left(\frac{1}{n}\right).$$

**Lemma 4**

If $N_n(t)$ be as given (6.2.5) then

$$|N_n(t)| = O(n)$$

as $n \to \infty$

uniformly in $0 < t \leq \frac{1}{n}$.

**PROOF OF LEMMA 4**

We have,

$$|N_n(t)| = \left| \sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{\sin(k+1)t}{\sin\frac{t}{2}} \right|$$

$$= O\left[\sum_{k=0}^{n} |\Delta\lambda_{n,k}| (k+1)\right].$$

Now, using Abel's transformation, we get

$$|N_n(t)| = O\left[\sum_{k=0}^{n-1} |k+1-k-2| \sum_{v=0}^{k} |\Delta\lambda_{n,v}| + (n+1) \sum_{k=0}^{n} |\Delta\lambda_{n,k}|\right]$$

$$= O\left[\sum_{k=0}^{n-1} \left\{ \sum_{v=0}^{k} |\Delta\lambda_{n,v}| \right\} + (n+1) \sum_{k=0}^{n} |\Delta\lambda_{n,k}|\right].$$

By the regularity condition of (^) summability, there exists a constant M such that

$$\sum_{k=0}^{\infty} |\Delta\lambda_{n,k}| < M$$

for any n.

Therefore, we have

$$|N_n(t)| = O[Mn + (n+1)M]$$

$$= O(n) \text{ as } n \to \infty$$

uniformly in $0 < t \leq \frac{1}{n}$.

## 6.5 PROOF OF THE THEOREM

The $n^{th}$ partial sum of the series (6.2.1) is

$$S_n(x)=\sum_{v=0}^{n} a_v p_v(x)$$

$$=\sum_{v=0}^{n} \frac{(2v+1)}{2}\int_{-1}^{1} f(y)\, p_v(y)\, p_v(x)\, dy \quad \text{by (6.2.2)}$$

$$=\sum_{v=0}^{n} \frac{(2v+1)}{2}\int_{-1}^{1} f(y)\, p_v(y)\, p_v(x)\, dy$$

$$=\frac{(n+1)}{2}\int_{-1}^{1}\frac{p_{n+1}(y)\, p_n(x)-p_{n+1}(x)\, p_n(y)}{y-x} f(y)\, dy \quad \text{by (6.4.1)}$$

putting $f(y)=1$ it can be easily seen that

$$1=\frac{(n+)}{2}\int_{-1}^{1}\frac{p_{n+1}(y)\, p_n(x)-p_{n+1}(x)\, p_n(y)}{y-x} dy$$

therefore

$$S_n(x)-f(x)=\frac{n+1}{2}\int_{-1}^{1}[\, f(y)-f(x)\,]$$

$$\frac{p_{n+1}(y)\, p_n(x)-\, p_{n+1}(x)\, p_n(y)}{y-x} dy$$

Let us take a positive number S<1 and consider it as the sum of two other positive numbers $\alpha$ and $\beta$. Let $\delta$ be another positive number such that $0<\delta\ <\ \alpha$, $\alpha x$ and $\alpha x$ be two continuous functions of x within (-1,1) which lie with in the limits $\delta\ \le\ \alpha x\ \le\ \alpha,\ \delta\ \le\ \alpha x'\ \le \alpha$

Therefore

$$S_n(x)-f(x)=\frac{n+1}{2}\left[\int_{-1}^{x-\alpha x} + \int_{x-\alpha x}^{x+\alpha x'} + \int_{x+\alpha x'}^{1}\right]$$

(6.5.1) $= A_n(x) + B_n(x) + C_n(x)$ say.

Hobson [1909] has shown that uniformly for

$-1+s \le x \le 1-s$

(6.5.2) $$\begin{cases} \lim\limits_{n\to\infty} A_n(x) = 0 \\ \lim\limits_{n\to\infty} C_n(x) = 0 \end{cases}$$

Now we suppose that

$x = \cos\theta,\ y = \cos\phi, 0 < \theta < \pi,\ 0 < \phi < \pi,\ 1-\beta = \cos\rho,$
$1-(\alpha+\beta) = 1-s = \cos(\rho+\sigma)$

$1 < \rho < \frac{\pi}{2},\ 0 < \sigma,\ \rho + \sigma < \frac{\pi}{2}$

Thus if $\eta$ denotes the minimum of

[arc cos u - arc cos $(u+\alpha)$ ]

for u in $(-1, 1-\alpha)$, we have on the lines of Sansone [1959] .

$$B_n(\cos\theta) = \frac{n+1}{2} \int_{\theta-\eta}^{\theta+\eta} \left[ f(\cos\phi) - f(\cos\theta) \right]$$

$$\frac{p_{n+1}(\cos\phi)\, p_n(\cos\theta) - p_{n+1}(\cos\theta) p_n(\cos\phi)}{\cos\phi - \cos\theta} \times \sin\phi\, dy$$

in which $\rho + \sigma \le \theta \le \pi - (\rho+\sigma),\ 0 < \eta \le \bar{\sigma}$. With successive transformation, we get

(6.5.3) $B_n(\cos\theta) = D_n(\theta) + E_n(\theta)$

where

$$D_n(\theta) = \frac{1}{2\pi\sqrt{\sin\theta}} \int_{\theta-\eta}^{\theta+\eta} \frac{f(\cos\phi) - f(\cos\theta)}{\sin\frac{1}{2}(\theta-\phi)} \times \sin\{(n+1)(\theta-\phi)\}\sqrt{\sin\phi}\, d\phi$$

and obviously on the lines of Sansone [1959]

$E_n(\theta) = 0\,(1)$, as $n \to \infty$

uniformly where x lies within $(-1+s,\ 1-s)$ i.e. in the set E.

Putting $\theta - \phi = t$, we get

(6.5.4)

$$D_n(\theta) = \frac{1}{\pi\sqrt{\sin\theta}} \int_0^{\eta} [f\{\cos(\theta-t)\} - f(\cos\theta)] \times \frac{\sin(n+1)}{\sin\frac{t}{2}} t\sqrt{\sin(\theta-t)}\, dt$$

So, we get from (6.5.1) to (6.5.4)

$$S_n(x) - f(x) = \frac{1}{\pi\sqrt{\sin\theta}} \int_0^{\eta} [f\{\cos(\theta-t)\} - f(\cos\theta)]$$

$$\times \frac{\sin(n+1)t}{\sin\frac{t}{2}} \sqrt{\sin(\theta-t)}\, dt + 0\,(1)$$

$$= O\left[\int_0^{\eta} \{f(\cos(\theta-t)) - f(\cos\theta)\} \frac{\sin(n+1)t}{\sin\frac{t}{2}}\, dt\right]$$

(6.5.5) $$= O\left[\int_0^{\eta} \psi(t) \frac{\sin(n+1)t}{\sin\frac{t}{2}}\, dt\right] + 0\,(1)$$

uniformly in E.

Now, if $\sigma_n(x)$ be the (^)-mean of the sequence $\{S_n(x)\}$ of partial sums of the series (6.2.1) then by the application of (6.1.1), we have

$$\sigma_n(x)-f(x)=\sum_{k=0}^{n}\Delta\lambda_{n,k}\left[S_k(x)-f(x)\right].$$

$$=O\left[\int_0^{\pi}\left\{\sum_{k=0}^{n}\Delta\lambda_{n,k}\frac{\sin\ (k+1)t}{\sin\ \frac{t}{2}}\right\}.\ \psi\ (t)\ dt\right]+0\ (1)$$

$$=O\left[\int_0^{\pi}|\psi\ (t)|\ |N_n(t)|\ dt\right]+0\ (1),$$

$$=O\ (n)\left[\left\{\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\pi}\right\}|\psi\ (t)|\ |N_n(t)|\ dt\right]+0\ (1)$$

(6.5.6) $$=O\ (I_1)+O\ (I_2)+0\ (1),$$

uniformly in E.

Now, the theorem will be established, if we show that

(6.5.7) $$I_1=0\ (1)$$

and $I_2=0\ (1)$ as $n\to\infty$

uniformly in E.

Let us first consider $I_1$.

We have by (6.4.4)

$$I_1=O\ (n)\left[\int_0^{\frac{1}{n}}|\psi\ (t)|\ dt\right]$$

$$=O\ (n)\left[\frac{1}{n}.\frac{1}{P_n}\right]$$

by (6.3.4) uniformly in E,

$$= O\left[n.\frac{1}{n}.\frac{1}{P_n}\right]$$

$$= 0\left[\frac{1}{P_n}\right]$$

(6.5.8) $$= 0\,(1) \text{ as } n \to \infty$$

by (6.3.2) (iii), as $n \to \infty$, uniformly in E.

Next, considering $I_2$ we have

$$I_2 = \int_{\frac{1}{n}}^{\eta} |\psi(t)|\,|N_n(t)|\,dt$$

$$= \int_{\frac{1}{n}}^{\eta} |\psi(t)| \left|\sum_{k=0}^{n} \Delta\lambda_{n,k} \frac{\sin\,(k+1)\,t}{\sin\frac{t}{2}}\right| dt$$

$$= \int_{\frac{1}{n}}^{\eta} |\psi(t)| \left|\sum_{k=0}^{n} \Delta\lambda_{n,n-k} \frac{\sin\,(n-k+1)\,t}{\sin\frac{t}{2}}\right| dt$$

$$\leq \int_{\frac{1}{n}}^{\eta} \frac{|\psi(t)|}{t} \left|\text{Im} \sum_{k=0}^{n} \Delta\lambda_{n,n-k} e^{i\,(n-k+1)t}\right| dt$$

$$= O\left[\int_{\frac{1}{n}}^{\eta} \frac{|\psi(t)|}{t^2} \sum_{k=0}^{n} \Delta\lambda_{n,n-\tau} dt\right] \text{ by (6.4.2)}$$

$$= O\left[\int_{\frac{1}{n}}^{\eta} \frac{|\psi(t)|}{t^2} \Delta\lambda_{n,0} dt\right]$$

$$= O\left(\frac{1}{n}\right) \int_{\frac{1}{n}}^{\eta} \frac{|\psi(t)|}{t^2} dt \text{ by (6.4.3)}$$

$$= O\left(\frac{1}{n}\right) 0\left(\frac{n}{P_n}\right)$$

$$= 0\left(\frac{1}{P_n}\right)$$

(6.5.9) $$= 0\,(1) \text{ as } n \to \infty$$

uniformly in E.

Now combining (6.5.6), (6.5.7), 6.5.8), (6.5.9) we get the required result.

This completes the proof of the theorem.