CHAPTER – 5

On Uniform Matrix Summability Of A Fourier Series

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ON UNIFORM MATRIX SUMMABILITY OF A FOURIER SERIES

5.1 Let $T = (a_{n,k})$ be an infinite triangular matrix satisfying the Silverman-Töpelitz (1913) conditions of regularity,

i.e., $\sum_{k=0}^{n} a_{n,k} \to 1$ as $n \to \infty$,

$a_{n,k} = 0$ for $k > n$

and $\sum_{k=0}^{n} |a_{n,k}| \leq M,$ a finite constant.

Let $f \in L(-\pi, \pi)$ and be periodic with period 2π and let

(5.1.1) $$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be considered Fourier series of the function f.

Let $\sum_{m=0}^{\infty} u_m(x)$ be an infinite series such that

(5.1.2) $$S_k(x) = u_0(x) + u_1(x) + \ldots\ldots\ldots\ldots + u_k(x) = \sum_{v=0}^{k} u_v(x)$$

If there exists a function

$s = s(x)$ such that

$$t_n(x) = \sum_{k=0}^{n} a_{n,k} \{S_k(x) - s\}$$

This chapter in the from of a reseach paper has been published in a "Acta ciencia Indica" vol. XXXI M. No. 3, 815 (2005).

$$= \sum_{k=0}^{n} a_{n,n-k}\left(S_{n-k}(x) - S\right)$$

$$= 0\ (1) \text{ as } n \to \infty$$

uniformly in a set E in which s = s (x) is bounded, then we say that the series $\sum_{m=0}^{\infty} u_m(x)$ is summable (T) uniformly in set E to the sum s.

We shall use following notations:

$$\phi(t) = f(x+t) - f(x-t) - 2f(x)$$

$$\Phi(t) = \int_0^t |\phi(u)|\,du$$

$$A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k}, \text{ Where } \tau = \text{Integral Part of } \frac{1}{t} = \left[\frac{1}{t}\right],$$

$$M_n = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}$$

5.2 MAIN THEOREM.

Saxena (1965) has applied the concept of uniform summability for Fourier series of harmonic means. The object of present paper to prove a theorem on uniform Matrix (T) summability.

We shall prove the following:

THEOREM:

Let $(a_{n,k})$ be an infinite triangular matrix such that elements $(a_{n.k})$ are non-negative, and non- decreasing with $k \le n$ and if

(5.2.1) $$\Phi(t) = \int_0^t |\phi(u)|\,du = 0\left(\frac{\in\left(\frac{1}{t}\right).t}{P_t}\right) \text{ as } t \to +0$$

uniformly in set E, where $\in(t)$ is a positive function of t such that

$\in(n).\ \log n = O\ (P_n)$ as $n \to \infty$

then Fourier series is summable (T) uniformly in E to sum f (x) .

$$\tau = \left[\frac{1}{t}\right]$$

5.3 LEMMAS :

We shall require following lemmas for the proof of our theorem.

Lemma (5.3.1) :

For $0 \leq t \leq \frac{1}{n}$ and under the condition of our theorem on $(a_{n,k})$

$$M_n(t) = 0(n)$$

Proof:

$$M_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\sin\left(n - k + \frac{1}{2}\right)t}{\sin \frac{t}{2}}$$

$$= \frac{1}{2\pi} \left| \sum_{k=0}^{n} \frac{a_{n,n-k} (2n - 2k + 1) \left| \sin \frac{t}{2} \right|}{\left| \sin \frac{t}{2} \right|} \right|$$

$$= 0\ (2n+1) \left| \sum_{k=0}^{n} a_{n,n-k} \right|$$

$$= 0\ (2n+1) \left| A_{n,n} \right|$$

$$= 0\ (n)\ 0\ (1)$$

$$= 0\ (n).$$

Lemma (5.3.2) : (Mc Fadden (1942) : If $\{p_n\}$ be a non - negative, non- increasing sequence, then for $0 \le t \le \pi,\ 0 \le a \le b \le \infty$ and any n,

$$\left| \sum_{k=a}^{b} p_k\, e^{i(n-k)t} \right| = 0\,(p_\tau)$$

Lemma (5.3.3) : For $\frac{1}{n} < t \le \delta < \pi$

$$M_n(t) = 0\left(\frac{A_{n,\tau}}{t}\right)$$

Proof: Since for $\frac{1}{n} < t \le \delta < \pi$

$$\sin\left(\frac{t}{2}\right) > \frac{t}{\pi}.$$

Therefore,

$$|M_n(t)| = \left| \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)} \right|$$

$$= 0\left(\frac{1}{t} \text{ Imaginary part of} \left| \sum_{k=0}^{n} a_{n,n-k}\, e^{i\left(n-k+\frac{1}{2}\right)t} \right| \right)$$

$$= 0\left(\frac{1}{t} \left| \sum_{k=0}^{n} a_{n,n-k}\, e^{i(n-k)t} \right| \left| e^{i\frac{t}{2}} \right| \right)$$

$$= 0\left(\frac{1}{t} \left| \sum_{k=0}^{n} a_{n,n-k}\, .e^{i(n-k)t} \right| \right)$$

$$= 0\left(\frac{A_{n,\tau}}{t}\right) \quad \text{by lemma (5.3.2).}$$

Which proves the lemma.

5.4 PROOF OF THE THEOREM :

Following Titchmarsh (1939) we have

$$S_k(x) - f(x) = \frac{1}{2\pi}\int_0^\delta \phi(t)\,\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt + 0(1),$$

uniformly in E.

Then

$$t_n(x) = \sum_{k=0}^{n} a_{n,n-k}\left\{ s_{n-k}^{(x)} - f(x) \right\}$$

$$= \frac{1}{2\pi}\int_0^\delta \left(\sum_{k=0}^{n} a_{n,n-k}\,\frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t} \right) \phi(t)\,dt + 0(1)$$

$$= \int_0^\delta M_n(t)\,\phi(t)\,dt + 0(1), \text{ uniformly in E}$$

$$= \left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} \right) M_n(t)\,\phi(t) + 0(1) \text{ uniformly in E}$$

(5.4.1) $$= I_1 + I_2 + 0(1) \text{ uniformly in E}$$

Now,

$$I_1 = \int_0^{\frac{1}{n}} \phi(t)\,M_n(t)\,dt \text{ uniformly in E}$$

$$|I_1| \le \int_0^{\frac{1}{n}} |\phi(t)|\,|M_n(t)|\,dt \text{ uniformly in E}$$

$$|I_1| = O(n)\int_0^{\frac{1}{t}} |\phi(t)|\,dt \quad \text{uniformly in E by Lemma (5.3.1)}$$

$$= O(n)\, 0\left(\frac{\varepsilon(n)}{nP_n}\right)$$

$$= 0\left(\frac{n\,\varepsilon(n)}{n\,P_n}\right)$$

$$= 0\left(\frac{1}{\log n}\right)$$

(5.4.2) $= 0(1)$ as $n \to \infty$ (by hypothesis of theorem)

$$|I_2| = 0(1)\int_{\frac{1}{n}}^{\delta} |\phi(t)|\, \frac{A_{n,\tau}}{t}\, dt$$

$$= \left\{ \frac{A_{n,\tau}}{t}\, \Phi(t) \right\}_{\frac{1}{n}}^{\delta} + \int_{\frac{1}{n}}^{\delta} \frac{A_{n,\tau}}{t^2}\, \Phi(t)\, dt + \int_{\frac{1}{n}}^{\delta} \frac{\Phi(t)}{t}\, d\left(A_{n,\tau}\right)$$

$$= 0\left\{ \frac{A_{n,\tau}}{t}\, \frac{\varepsilon\left(\frac{1}{t}\right) t}{P_\tau} \right\}_{\frac{1}{n}}^{\delta} + \int_{\frac{1}{n}}^{\delta} \frac{A_{n,\tau}}{t^2}\, 0\left(\frac{\varepsilon\left(\frac{1}{t}\right) t}{P_\tau}\right) dt$$

$$+ \int_{\frac{1}{n}}^{\delta} \frac{a_{n,n-\tau}}{t}\, 0\left(\frac{\varepsilon\left(\frac{1}{t}\right) t}{P_\tau}\right) dt$$

$$= 0\left(\frac{A_{n,\left(\frac{1}{\delta}\right)}\, \varepsilon\left(\frac{1}{\delta}\right)}{P_{\left(\frac{1}{\delta}\right)}}\right) + 0\left(\frac{A_{n,n}\varepsilon(n)}{P_n}\right)$$

$$+ \int_{\frac{1}{\delta}}^{u} \frac{A_{n,u}.\varepsilon(u)}{u.P_u}\, du + \int_{\frac{1}{\delta}}^{n} \frac{a_{n-y}}{P_y}\, \frac{\varepsilon(y)}{y^2}\, dy$$

$$= 0(1) + 0(1) + 0\left(\frac{A_{n,n}}{\log n}\right) + 0\left[\sum_{k=1}^{n} \frac{a_{n,n-k}\,\varepsilon(k)}{P_k . k^2}\right],$$

by mean value theorem

$$= 0(1) + 0(1) + 0(1) + 0\left(\frac{\varepsilon(n)}{P_n}\right) O\left(\frac{A_{n,n}}{n^2}\right)$$

by hypothesis of theorem

$$= 0(1) + 0(1) + 0(1) + 0\left(\frac{1}{\log n}\right) O\left(\frac{1}{n^2}\right)$$

$= 0(1) + 0(1)$, as $n \to \infty$

(5.4.3) $= 0(1)$, uniformly in E as $n \to \infty$.

Collecting (5.4.1), (5.4.2) and (5.4.3) we have

$$t_n(x) = \sum_{k=0}^{n} a_{n,n-k} \{s_{n-k}(x) - f(x)\}$$

$= 0(1)$, as $n \to \infty$, uniformly in E.

Thus the theorem is completely etablished.

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