CHAPTER – 4

On Almost (N,p,q) Summability Of Derived Fourier Series

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ON ALMOST (N, p, q) SUMMABILITY OF DERIVED FOURIER SERIES

4.1 Let $\sum a_n$ be an infinite series with $\{S_n\}$ as the sequence of its n^{th} partial sums. Lorentz (1948) has given the following definition :

A bounded sequence $\{S_n\}$ is said to be almost convergent to a limit S, if

(4.1.1) $$\lim_{n\to\infty} \frac{1}{n+1} \sum_{v=m}^{n+m} S_v = S$$

uniformly with respect to m.

Let $\{p_n\}$ and $\{q_n\}$ be the two sequence of non- zero real constants and

(4.1.2) $$P_n = p_0 + p_1 + p_2 + \dots\dots\dots\dots + p_n, P_{-1} = p_{-1} = 0$$

(4.1.3) $$Q_n = q_0 + q_1 + q_2 + \dots\dots\dots\dots + q_n, Q_{-1} = q_{-1} = 0$$

Given two sequence $\{p_n\},\{q_n\}$, convolution p*q is defined by

$$R_n = (p*q)_n = \sum_{k=0}^{n} p_k q_{n-k}.$$

It is familiar and can be easily verified that the operation of convolution is commutative and associative and

$$(p*1)_n = \sum_{k=0}^{n} p_k$$

We define that the series $\sum a_n$ is said to be almost generalized Nörlund (N,p,q) summable to S (Qureshi [36] (1981) if

(4.1.4) $$t_{n,m} = \frac{1}{R_n} = \sum_{v=0}^{n} p_{n-v}\, q_v\, S_{v,m}$$

tends to S, as $n \to \infty$, uniformly with m where

(4.1.5) $$S_{v,m} = \frac{1}{v+1} = \sum_{k=m}^{v+m} S_k .$$

PARTICULAR CASES :

Four important particular cases of almost (N,p,q) means are.

1. Almost (N,p,q) method reduces to almost Norlund method (N, p_n) if $q_n = 1$ for all n.

2. Almost (N, p, q) method reduces to almost Riesz method $(\overline{N}, p_n)$ if $p_n = 1$ for all n.

3. In the special case when $p_n = \binom{n+\alpha+1}{\alpha-1}$, $\alpha < 0$, the method (N, p_n) reduces to the well known method of summability (C, α).

4. $P_n = (n+1)^{-1}$ of the Nörlund mean is known as harmonic mean and is written as $\left(N, \frac{1}{n+1}\right)$.

Let f (t) be a periodic function with period 2π and integrable in the sense of Lebesgue over an interval $(-\pi, \pi)$.

Let its Fourier series be given by

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

(4.1.6) $$= \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t)$$

and then the conjugate series of (4.1.6) is given by

(4.1.7) $$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t).$$

The series

(4.1.8) $f'(t) = \sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt)$

which is obtained by differentiating term by term is called the first derived series or the derived Fourier series of f (t).

We shall use the following notations:

(4.1.9) $\phi(t) = f(x+t) + f(x-t) - 2f(x)$

(4.1.10) $\psi(t) = f(x+t) + f(x-t)$

(4.1.11) $g(t) = f(x+t) - f(x-t) - 2tf'(x)$

(4.1.12) $G(t) = \int_0^t |dg(u)|$

(4.1.13) $$N_{n,m}(t) = \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v} q_v \frac{\cos mt - \cos(v+m+1)t}{2(v+1)\sin^2 \frac{t}{2}}$$

$\tau = \left[\frac{1}{t}\right]$ = the integral part of $\frac{1}{t}$.

4.2 KNOWN RESULT :

Pandey (1983) established a result on Nörlund summability of fourier series of the following form.

THERÖREM A : $\Phi(t) = 0[t\,\alpha(t)]$, as $t \to +0$, where $\alpha(t)$ is a monotonic non-decreasing function of t such that

(4.2.1) $\alpha\left(\frac{1}{n}\right) = 0(1)$, as $n \to \infty$

(4.2.2) $\int_{\frac{1}{n}}^{s} \alpha(t)\, p_\tau \frac{dt}{\tau} = O(P_n)$, as $n \to \infty$

then the Fourier series (4.1.6) of f (t) at t = x is summable (N, p_n) to f (x), where $\{p_n\}$ is a real non - negative and non- decreasing sequence such that

$p_n \to \infty$, as $n \to \infty$

He also proved:

THEOREM B : If the sequence $\{p_n\}$ and $\alpha(t)$ be the same as in the theorem A, then if

(4.2.3) $\psi(t) = 0[t\,\alpha(t)]$ as $t \to +O$

then the conjugate series of Fourier series (4.1.7) is summable

(N, P_n) to $\frac{1}{2\pi}\int_0^{\pi} \psi(t)\cot\frac{t}{2}\,dt$,at every point where this integral exists.

4.3 MAIN THEOREM :

The object of this chapter is to improve theorem B for almost (N, p, q) summability of derived Fourier series of Fourier series in the following form :

THEOREM :

Let $\{p_n\}$ and $\{q_n\}$ be the monotonic non-increasing sequence of real constants such that $R_n = \sum_{v=0}^{n} p_v\, q_{n-v} \to \infty$, as $n \to \infty$

(4.3.1) If $G(t) = \int_0^t |dg(u)| = 0\left[\frac{\alpha\left(\frac{1}{t}\right)t}{R_{\left(\frac{1}{t}\right)}}\right]$ as $t \to +O$

(4.3.2) $\int_{\frac{1}{(n+m)}}^{\frac{1}{(n+m)^{\delta}}} \frac{|dg(t)|}{t^2} = 0(n)$, as $n \to \infty$

where $0 < \delta < \frac{1}{2}$ uniformly with respect to m and $\alpha(t)$ is a positive function of t such that

(4.3.3) $\alpha(n) \log n = O[R_n]$ as $n \to \infty$

and

(4.3.4) $$\sum_{v=0}^{n} \left(\frac{p_{n-v}\, q_v}{(v+1)} \right) = O\left(\frac{R_n}{n}\right)$$

then the derived Fourier series (4.1.8) is almost (N, p, q) summable o f' (x) at the point t = x

4.4 LEMMAS :

For the proof of our theorem following lemmas are required :

Lemma (4.4.1) :

For $O < t < \frac{1}{n+m}$

we have

$$N_{n,m}(t) = \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v}\, q_v \frac{\cos mt - \cos(v+m+1)t}{2(v+1)\sin^2\left(\frac{t}{2}\right)}$$

$$= \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v}\, q_v \frac{\sin\left(m+\frac{v+1}{2}\right)t \sin\left(\frac{v+1}{2}\right)t}{(v+1)\sin^2\left(\frac{t}{2}\right)}$$

$$N_{n,m}(t) \le \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v} q_v \frac{(2m+v+1)\left|\sin\frac{t}{2}\right|(v+1)\left|\sin\frac{t}{2}\right|}{(v+1)\sin^2\left(\frac{t}{2}\right)}$$

$$= \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v}\, q_v\,(2m+v+1)$$

$$\leq \frac{1}{2\pi R_n}\left|\sum_{v=0}^{n} p_{n-v}\, q_v\right|(2m+n+1)$$

$$= O(n+m)\,\frac{1}{R_n}\sum_{v=0}^{n} p_{n-v}\, q_v$$

or $\left|N_{n,m}(t)\right| = O(n+m)$.

LEMMA (4.4.2) :

For $\frac{1}{n+m} < t < \pi$

$$N_{n,m}(t) = \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_{n-v} q_v \frac{\sin\left(m+\frac{v+1}{2}\right)t \sin\left(\frac{v+1}{2}\right)t}{(v+1)\sin^2\frac{t}{2}}$$

$$\left|N_{n,m}(t)\right| \leq \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_{n-v}\, q_v \frac{\sin\left(m+\frac{v+1}{2}\right)t \sin\left(\frac{v+1}{2}\right)t}{(v+1)\sin^2\left(\frac{t}{2}\right)}$$

$$\leq \frac{1}{2\pi R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{(v+1)}\frac{1}{\sin^2\left(\frac{t}{2}\right)}$$

$$= O\left(\frac{1}{t^2}\right)\frac{1}{R_n}\sum_{v=0}^{n}\left(\frac{p_{n-v}q_v}{(v+1)}\right)$$

$$\left|N_{n,m}(t)\right| = O\left(\frac{1}{t^2 n}\right). \qquad (\text{by } 4.3.4).$$

4.5 PROOF OF THE THEOREM :

Denoting by $S_n(x)$ the sum of the first n terms of the series (4.1.8) at the point $t = x$ we get

(4.5.1) $$S_n(x) = -\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{d}{dx}\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}f(u)\,du$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} f(u)\left\{\frac{d}{dx}\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}du$$

$$= -\frac{1}{2\pi}\int_0^{2\pi}\{f(x+t) - f(x-t)\}\left\{\frac{d}{dt}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right\}dt.$$

Now integrating by parts the right hand side (4.5.1), we get

$$S_n(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,d\{f(x+t) - f(x-t)\}$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dg(t) + f'(x).$$

Therefore

$$S_n(x) - f'(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dg(t)$$

So that

$$S_{v,m} - f'(x) = \frac{1}{v+1}\sum_{n=m}^{v+m}\{S_n - f'(x)\}$$

$$= \frac{1}{2\pi(v+1)} \sum_{n=m}^{v+m} \left\{ \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t} \, dg(t) \right\}$$

$$= \frac{1}{2\pi(v+1)} \int_0^{\pi} \sum_{n=m}^{v+m} \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t} \, dg(t)$$

$$S_{v,m} - f'(x) = \frac{1}{2\pi\,(v+1)} \int_0^{\pi} \frac{\cos mt - \cos(v+m+1)}{2\sin^2\left(\frac{1}{2}\right)} \, dg(t).$$

Now by (4.1.4), we have

$$t_{n,m} - f'(x) = \frac{1}{2\pi} \int_0^{\pi} \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v \frac{\cos mt - \cos(v+m+1)t}{2(v+1)\sin^2\frac{t}{2}} dg(t)$$

$$= \int_0^{\pi} N_{n,m}(t)\, dg(t).$$

In order to prove the theorem, we have to show that under our assumpticn

$$\int_0^{\pi} N_{n,m}(t)\, dg(t) = 0\,(1), \text{ as } n \to \infty$$

uniformly with respect to m.

Now we have

$$\int_0^{\pi} N_{n,m}(t)\, dg(t) = \left\{ \int_0^{\frac{1}{n+m}} + \int_{\frac{1}{n+m}}^{\frac{1}{(n+m)^{\delta}}} + \int_{\frac{1}{(n+m)^{\delta}}}^{\pi} \right\} N_{n,m}(t)\, dg(t)$$

(4.5.2) $$= I_1 + I_2 + I_3.$$

Now, we take

$$I_1 = \int_0^{\frac{1}{n+m}} N_{n,m}(t)\, dg(t)$$

$$|I_1| = O\left[\int_0^{\frac{1}{n+m}} |N_{n,m}(t)|\,|dg(t)|\right]$$

$$= O(n+m)\left[\int_0^{\frac{1}{n+m}} |dg(t)|\right] \text{ by Lemma (4.4.1)}$$

$$= O(n+m)\ 0\left[\frac{\alpha(n+m)}{(n+m)\,R_{(n+m)}}\right] \text{ by (4.3.1)}$$

$$= 0\left[\frac{\alpha(n+m)}{R_{(n+m)}}\right]$$

(4.5.3) $|I_1| = 0(1)$, as $n \to \infty$

uniformly with respect to m.

Nextly we consider

$$I_2 = \int_{\frac{1}{n+m}}^{\frac{1}{(n+m)^\delta}} N_{n,m}(t)\, dg(t)$$

$$|I_2| = O\left[\int_{\frac{1}{n+m}}^{\frac{1}{(n+m)^\delta}} |N_{n,m}(t)|\cdot|dg(t)|\right]$$

$$= O\left[\int_{\frac{1}{n+m}}^{\frac{1}{(n+m)^\delta}} \frac{|dg(t)|}{t^2 n}\right] \text{ by lemma (4.4.2)}$$

$$= O\left[\frac{1}{n}\int_{\frac{1}{n+m}}^{\frac{1}{(n+m)^\delta}} \frac{|dg(t)|}{t^2}\right]$$

$$= O\left(\frac{1}{n}\right) 0(n) \text{ by (4.3.2)}$$

(4.5.4) $|I_2| = 0(1)$, as $n \to \infty$

uniformly with respect to m.

Lastly, we have

$$|I_3| = \left[\int_{\frac{1}{(n+m)^\delta}}^{\pi} |N_{n,m}(t)|\,|dg(t)|\right]$$

$$= \int_{\frac{1}{(n+m)^\delta}}^{\pi} \frac{1}{2\pi}\frac{1}{R_n}\sum_{v=0}^{n} p_{n-v}q_v \frac{\cos mt - \cos(v+m+1)t}{2(v+1)\sin^2\frac{t}{2}}|dg(t)|$$

$$= \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_{n-v}q_v \left[\int_{\frac{1}{(n+m)^\delta}}^{\pi} \frac{\cos mt}{2(v+1)\sin^2\frac{t}{2}}|dg(t)|\right.$$

$$\left. - \int_{\frac{1}{(n+m)^\delta}}^{\pi} \frac{\cos(v+m+1)t}{2(v+1)\sin^2\frac{t}{2}}dg(t)\right]$$

(4.5.5) $|I_3| = I_{3.1} - I_{3.2}$ (say).

Now by using second value theorem, we have

$$I_{3.1} \le \frac{1}{2\pi R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{2(v+1)}\frac{1}{2\,\mathrm{Sin}^2\left(\frac{1}{2(n+m)^\delta}\right)}$$

$$\int_{\frac{1}{(n+m)^\delta}}^{\varepsilon} \cos mt\, dg(t)$$

Where $\frac{1}{(n+m)^\delta} < \varepsilon < \pi$ and $0 < \delta < \frac{1}{2}$

$$= O\left(\frac{1}{n}\right)(n+m)^{2\delta}\left[\frac{\frac{1}{2(n+m)^\delta}}{\sin\frac{1}{2(n+m)^\delta}}\right]^2 \int_{\frac{1}{(n+m)^\delta}}^{\varepsilon} dg(t)$$

(4.5.6) $|I_{3.1}| = 0\,(1)$, as $n \to \infty$

uniformly with respect to m.

Now,

$$I_{3.2} = \frac{1}{2\pi R_n} \int_{\frac{1}{(n+m)^{\delta}}}^{\pi} \sum_{v=0}^{n} p_{n-v} q_v \frac{\cos (v+m+1)t}{2(v+1) \sin^2 \frac{t}{2}} dg\,(t)$$

$$I = \frac{1}{2 \sin^2\left(\frac{1}{(n+m)^{\delta}}\right)} \int_{\frac{1}{(n+m)^{\delta}}}^{\varepsilon} dg\,(t)$$

(4.5.7) $I_{3.2} = 0\,(1)$, as $n \to \infty$

uniformly with respect to m.

Hence

(4.5.8) $I_3 = 0\,(1)$, as $n \to \infty$

Now by combining (4.5.2), (4.5.3), (4.5.4) and (4.5.8) we have

$$\int_0^{\pi} N_{n,m}(t)\, dg\,(t) = 0\,(1), \text{ as } n \to \infty$$

uniformly with respect to m.

Thus the theorem is established.