CHAPTER 1

Certain Investigation in the field generalized hypergeometric series, continued fraction and partitions
1. In this chapter, we give a brief background of the hypergeometric series, and certain related functions which form the subject matter of the present thesis. Basic hypergeometric series have assumed great importance during the last four decades or so because of their applications in diverse fields, like additive number theory, combinatorial analysis, statistical and quantum mechanics, vector spaces etc. They have provided the analysts with a very handy tool to unify and sub-sum numerous isolated results in theory of numbers, under a single umbrella. A fresh interest in these functions was aroused by the discovery of Ramanujan's "Lost" Note book by G.E. Andrews in 1976. A beautiful account of the discovery of the 'Lost' Notebook and its contents has been given by him [3] in 1976 in the American Mathematical monthly. The enormous mass of literature on basic hypergeometric series (or-q-hypergeometric series as we often call it) has become so significant and important that their study has acquired an independent respectable status of its own rather merely being treated as a generalization of the ordinary hypergeometric series.

As a generalization of Gauss hypergeometric series, Heine[1,2,3] introduced the series
\[
1 + \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)} z + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})} z^2 + ..., \tag{1}
\]

where it is assumed that \( c \neq 0, -1, -2, \ldots \) and \(|q| < 1, |z| < 1\) to ensure convergence of the series when it does not terminate. Since

\[\lim_{q \to 1} \frac{1-q^a}{1-q} = a,\] \tag{2}

The series (1.1.1) converges to the series

\[1 + \frac{a.b}{1.c} z + \frac{a(a+1)b(b+1)}{1.2.c(c+1)} z^2 + ... . \tag{3}\]

In the limit \( q \to 1 \).

A basic hypergeometric series is generally defined to be a series of the type \( \sum_{n=0}^{\infty} a_n z^n \), where \( a_{n+1}/a_n \) is a rational function of \( q^n \), \( q \) being a fixed complex parameter, called the base of the series, usually with modulus less than 1. An explicit representation of such series is given by

\[r \Phi_s \left[ \begin{array}{c} a_1, \ a_2, \ldots, a_r; \ q; \ z \\ b_1, \ b_2, \ldots, b_s; \ q^j \end{array} \right] = \sum_{n=0}^{\infty} q^n \left( \begin{array}{c} n \\ 2 \end{array} \right) z^n, \tag{4}\]

where \( \left( \begin{array}{c} n \\ 2 \end{array} \right) = n(n-1)/2 \) and

\[(a_1, a_2, \ldots, a_r; q)_n = (a_1, q)_n (a_2, q)_n \ldots (a_r, q)_n \]

with the q-shifted factorials defined by

\[(a; q)_n = \begin{cases} 1 & \text{if } n = 0 \\ (1-a)(1-aq)\ldots(1-aq^{n-1}) & \text{if } n = 1, 2, \ldots \tag{5} \end{cases} \]

For convergence of the series (1.1.4) we need \(|q| < 1 \) and \(|z| < \infty\), when \( i = 1, 2, \ldots \) or max. \((|q|, |z|) < 1\) when \( i = 0 \), provided that no zeros appear in
the denominator. Some elementary properties of the q-shifted factorials that we shall need in the subsequent chapters of the present thesis are:

\[
\begin{align*}
(a;q)_{m+n} &= (a;q)_m (aq^m;q)_n, \\
(a;q)_n &= a^{\binom{n}{2}} q^{\binom{n}{2}}, \\
(aq^{-n};q)_k &= \frac{(a;q)_k (q/a;q)_n q^{-nk}}{(q^{1-k}/a;q)_n}, \\
(a^{1-n};q)_n &= (a^{-1};q)_n (-a)^n q^{\binom{n}{2}}, \\
(a;q)_{2n} &= (a,aq;q^2)_n, \\
(a^2;q^2)_n &= (a,-a;q)_n, \\
\end{align*}
\]

We shall also use the notations

\[
\begin{align*}
(a;q)_\alpha &= (a;q)_\infty / (aq^\alpha;q)_\infty, \\
(a;q)_{\infty} &= \prod_{n=0}^{\infty} (1-aq^n),
\end{align*}
\]

(7)

The basic hypergeometric series (1.1.4), for \( r=s+1 \) and \( z=q \), if the product of the denominator parameters is \( q^k \) times the product of the numerator parameters (\( k \) being a positive integer), i.e.,

\[
b_1 b_2 \ldots b_s = q^k a_1 a_2 \ldots a_{s+1},
\]

is called a \( k \)-balanced basic hypergeometric series and for \( k=1 \), it is called a Saalschützian basic hypergeometric series.

The basic hypergeometric series (1.1.4) for \( r=s+1 \), is called well-poised if \( qa_1 = a_2 b_1 = \ldots = a_{s+1} b_s \). (8)

It is also called very well-poised (a name coined by R. Askey [1])

If it is well-poised and \( a_2 = q \sqrt{a_1}, a_3 = -q \sqrt{a_1} \).
The series (1.1.4), for \( r = s + 1 \) is called a nearly-poised series of the first kind if
\[
q a_1 \neq a_2 b_1 = a_3 b_2 = \ldots = a_{s+1} b_s
\]
and a nearly-poised series of the second kind if
\[
q a_1 = a_2 b_1 = a_3 b_2 = \ldots = a_s b_{s-1} \neq a_{s+1} b_s.
\]
A basic bilateral hypergeometric series is defined, for \(|q| < 1\), as
\[
\psi_A \left[ \begin{array}{c} a_1, a_2, \ldots, a_A; q; z \\ b_1, b_2, \ldots, b_A \end{array} \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_A; q)_n z^n}{(b_1, b_2, \ldots, b_A; q)_n},
\]
where \(|b_1 b_2 \ldots b_A/a_1 a_2 \ldots a_A| < |z| < 1\), for convergence and
\[
(a; q)_n = \frac{(-1)^n q^{n(n+1)/2}}{(q/a; q)_n a^n}
\]
(10)
The basic bilateral series (1.1.9) is said to be \( k \)-balanced if,
\[
b_1 b_2 = b_A = q^k a_1 a_2 \ldots a_A,
\]
and Saalschützian if \( k = 1 \) in (1.1.11). It is called well poised if
\[
a_1 b_1 = a_2 b_2 = \ldots = a_A b_A,
\]
nearly poised of first kind if
\[
a_1 b_1 \neq a_2 b_2 = \ldots = a_{A-1} b_{A-1} = a_A b_A;
\]
nearly poised of second kind if
\[
a_1 b_1 = a_2 b_2 = \ldots \neq a_A b_A;
\]
and very well-poised if it is well-poised and
\[
a_1 = -a_2 = q b_1 = -q b_2.
\]
For \( i, j, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), a generalized basic hypergeometric series if two variables is defined as
\[
\phi_{A+B; C+D} \left[ \begin{array}{c} (a); (b); (b'); q; x, y \\ (c); (d); (d'); q^i, q^j, q^k \end{array} \right] = \sum_{m, n=0}^{\infty} \frac{\binom{m+j}{n+j} q^n}{q^{mn+i+j}} \times
\]
The double series converges absolutely for all bounded values of the complex arguments \(x\) and \(y\) when \(i, j, k \in \mathbb{N}\) and \(|q| < 1\), and also when \(i=j=k=0\), provided further that \(\max(\{|q|,|x|,|y|\}) < 1\).

The generalized 'bibasic' hypergeometric series of one variable is defined as:

\[
\Phi^{A+B}_{C+D} \left( \begin{array}{l} (a); \; (b); \; q; \; q_1; \; z \\ (c); \; (d); \; q^i; \; q^j \\ \end{array} \right) = \sum_{n=0}^{\infty} \frac{[(a);q^n][(b);q_1^n]z^n q^{in} q^{jn}}{(q;q)_n[(c);q^n][(d);q_1^n]},
\]

where \(i,j>0\), \(|z|<\infty\) and \(|q|<1\), \(|q_1|<1\); \(i,j=0\), max. \((|z|,|q|,|q_1|)<1\) for convergence. In the definitions (1.1.12) and (1.1.13) \((a)\) stands for the sequence of \(A\)-parameters of the form \(a_1,a_2,...,a_A\) and 

\[
[(a);q^n] = (a_1,a_2,...,a_A;q)_n.
\]

In the special case, when \(i=j=k=0\), the first member of (1.1.12) will be written simply as:

\[
\Phi^{A+B}_{C+D} \left( \begin{array}{l} (a); \; (b); \; (b'); \; q; \; x, \; y \\ (c); \; (d); \; (d'); \\ \end{array} \right),
\]

and a similar notational simplifications will be made in writing first members of (1.1.4) and (1.1.3) for \((i=0\) and \(i=j=0\) respectively.

A basic hypergeometric series of \(n\) variables is defined as:
where \( (a_p) \) stands for the \( p \)-parameters \( a_1, a_2, \ldots, a_p \). In what follows the notations carry their usual meanings. For the convergence of this series we require \( \max(|q|, |x|, \ldots, |x_n|) < 1 \).

2(A) THE q-BINOMIAL THEOREM, THE q-GAMMA AND q-BETA FUNCTIONS.

The most fundamental summation formula in the theory of basic hypergeometric series is the q-binomial formula:

\[
\Phi \left[ \begin{array}{c}
(a_p); \quad (b^{(1)}_M) ; \quad (b^{(2)}_M) ; \quad \ldots ; \quad (b^{(n)}_M) ; \\
(c_t); \quad (d^{(1)}_N) ; \quad (d^{(2)}_N) ; \quad \ldots; \quad (d^{(n)}_N);
\end{array} \right]_{x_1, x_2, \ldots, x_n} = \\
\sum_{m_1, \ldots, m_n \geq 0} \frac{[\!(a_p)\!]_{m_1} \cdots \cdots [\!(b^{(1)}_M)\!]_{m_1} \cdots \cdots [\!(b^{(n)}_M)\!]_{m_n} x_1^{m_1} \cdots x_n^{m_n}}{[\!(c_t)\!]_{m_1} \cdots \cdots [\!(d^{(1)}_N)\!]_{m_1} \cdots \cdots [\!(d^{(n)}_N)\!]_{m_n}}
\]

was first introduced by Thomae[1] and later by Jackson [2].

For \( 0 < q < 1 \), the q-gamma function is defined by

\[
\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad \text{Re } x > 0,
\]

was first introduced by Thomae[1] and later by Jackson [2].
Also \( \lim_{q \to 1} \Gamma_q[x] = \Gamma(x) \). \hspace{2cm} (4)

In view of [1.2(A).3] it is natural to define the q-beta function by

\[
B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}, \quad 0 < q < 1, \, \text{Re} \, x, y > 0.
\]

By [1.2(A).3] and [1.2(A).5] we have

\[
B_q(x,y) = (1-q) \left( \frac{q}{q^x, q^y; q} \right)_\infty
\]

\[
= (1-q) \left( \frac{q;q}{q^y;q} \right)_\infty \sum_{n=0}^{\infty} \left( \frac{q^y;q}{q;q} \right)_n q^n x^n
\]

\[
= (1-q) \sum_{n=0}^{\infty} \left( \frac{q^{n+1};q}{q^n+y;q} \right)_\infty q^n x^n
\]

Thomae [1,2] and Jackson [4] introduced the q-integral

\[
\int_a^b f(t) \, dq_t = (1-q) \sum_{n=0}^{\infty} f(q^n) q^n,
\]

which was generalized by Jackson to

\[
\int_a^b f(t) \, dq_t = \int_a^b f(t) \, dq_t - \int_a^0 f(t) \, dq_t, \hspace{2cm} (7)
\]

with

\[
\int_a^0 f(t) \, dq_t = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n,
\]

and

\[
\int_0^\infty f(t) \, dq_t = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n
\]

2(B) SUMMATION AND TRANSFORMATION FORMULAE FOR BASIC HYPERGEOMETRIC SERIES.

Hiene [1.2] established the transformation formula
\[ 2\Phi_1\left[\begin{array}{c} a, \\ b, \\ c \end{array} \mid q; z \right] = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} 2\Phi_1\left[\begin{array}{c} c/b, \\ z, \\ q; \\ z \end{array} \mid az \right] \]

where \( \max(|q|, |z|, |b|) < 1 \). By iterating the transformation [1.2(B).1] we obtain two more formulae

\[ 2\Phi_1\left[\begin{array}{c} a, \\ b; \\ q; \\ z \\
\end{array} \mid c \right] = \frac{(c/b, bz; q)_\infty}{(c, z; q)_\infty} 2\Phi_1\left[\begin{array}{c} abz/c, \\ b; \\ q; \\ c/b \end{array} \mid bz \right] \]

\[ = \frac{(abz/c; q)_\infty}{(z; q)_\infty} 2\Phi_1\left[\begin{array}{c} \frac{c}{a}, \\ \frac{c}{b}; \\ q; \\ \frac{abz}{c} \end{array} \mid \right] \]

By putting \( abz/c=1 \) in [1.2(B).2] we get q-Gauss summation formula

\[ 2\Phi_1\left[\begin{array}{c} a, \\ b; \\ q; \\ c/ab \\
\end{array} \mid c \right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \quad |c/ab| < 1. \]

When \( a = q^{-n}, \ n=0,1,2,... \) [1.2(B).2] reduces to the q- Vandermonde summation formula

\[ 2\Phi_1\left[\begin{array}{c} q^{-n}, \\ b; \\ q; \\ cq^n/b \\
\end{array} \mid c \right] = \frac{(c/b; q)_n}{(c; q)_n}. \]

By reversing the series in [1.2(B).5] we also find that

\[ 2\Phi_1\left[\begin{array}{c} q^{-n}, \\ b; \\ q; \\ q \\
\end{array} \mid c \right] = \frac{(c/b; q)_n b^n}{(c; q)_n}. \]

Equating the coefficients of \( z^n \) of both sides in [1.2(B).3] we find:

\[ \frac{(a,b;q)_n}{(q,c;b)_n} = \frac{(ab/c; q)_n}{(q,c; q)_n} \sum_{k=0}^{\infty} \left( \frac{q^{-n}, c/a, c/b; q}{q, c, cq^{1-n}/ab; q} \right)_k. \]

which can be rewritten as

\[ 3\Phi_2\left[\begin{array}{c} q^{-n}, \\ c/a, \\ c/b; \\ q; \\ q \\
\end{array} \mid c, \\ cq^{1-n}/ab \right] = \frac{(a,b;q)_n}{(c, ab/c; q)_n}. \]
This is known as the q-Saalschütz formula. We have also a different type of summation formula

\[ 2\Phi_1 \left[ \begin{array}{c} a, b; q, q \\ c \end{array} \right] + \frac{(q/c, a, b; q)_\infty}{(c/q, aq/c, bq/c; q)_\infty} 2\Phi_1 \left[ \begin{array}{c} aq/c, bq/c; q, q \\ q^2/c \end{array} \right] = \frac{(q/c, abq/c; q)_\infty}{(aq/c, bq/c; q)_\infty}. \] (9)

which can be expresses as a q-integral

\[ \frac{d}{qd/c} \frac{(qt/d, ct/d; q)_\infty}{(at/d, bt/d; q)_\infty} d_q t = \frac{d(1-q)(q, c, q/c, abq/c; q)_\infty}{(a, b, aq/c, bq/c; q)_\infty}. \] (10)

where \( d \) is an arbitrary non-zero parameter. This integral also yields Andrews and Askey's [1] q-beta integral.

\[ \frac{b}{-a} \frac{(-qt/a, t/b; q)_\infty}{(-ct/a, dt/b; q)_\infty} d_q t = \frac{b(1-q)(q, -a/b, bq/a)_\infty}{(c, d, -bc/a, -ad/b; q)_\infty}. \] (11)

Since, by [1.2(B).1]

\[ 2\Phi_1 \left[ \begin{array}{c} aq/c, bq/c; q, q \\ q^2/c \end{array} \right] = \frac{(aq/c, bq^2/c; q)_\infty}{(q^2/c, q/q)_\infty} \sum_{r=0}^{\infty} \frac{(q/a, q)_r}{(bq^2/c; q)_r} (aq/c)^r, \]

\(|aq/c| < 1, \]

\[ = \frac{(1-c/q)(aq/c, bq/c; q)_\infty}{(1-a)(q/c, q; q)_\infty} \sum_{r=1}^{\infty} \frac{(1/a, q)_r}{(bq/c; q)_r} (aq/c)^r \] (12)

We find by the use of the identity (1.1.10) that the formula [1.2(B).9] also contains Ramanujan's celebrated summation formula

\[ \sum_{n=-\infty}^{\infty} (a; q)_n z^n = (q, b/a, az, q/az; q)_\infty, \] (13)

where \(|b/a| < |z| < 1\). Replacing \( a, b, z \) by \(-q^a, -q^b\) and \( q^z \) respectively, we may express [1.2(B).13] in the form of q-integral
The transformation formula

\[
2 \Phi_1 \left[ \begin{array}{c} a, b; q; z \\ c 
\end{array} \right] = \frac{(az;q)_\infty}{(z;q)_\infty} 2 \Phi_2 \left[ \begin{array}{c} a, c/b; q; -bz \\ c, az; q^1
\end{array} \right].
\]  

(15)

was given by Jackson [3] as a q-analogue of Pfaff-Kummer transformation formula. If \( a = q^{-n} \), \( n=0,1,... \), then we may reverse the Sears of summation in [1.2(B).15] to get Sear’s [1] transformation formula.

\[
2 \Phi_1 \left[ \begin{array}{c} q^{-n}, b; q; z \\ c 
\end{array} \right] = \frac{(c/b;q)_\infty}{(c;q)_\infty} (bz/q)^n 3 \Phi_1 \left[ \begin{array}{c} q^{-n}, q^{1-n}/c, q/z; q; q \\ \right] b^q q^{-n}/c 
\]  

(16)

If we take the identity

\[
(x;q)_\alpha (-x;q)_\alpha = \left( x^2; q^2 \right)_\alpha.
\]  

(17)

and make use of the results

\[
(x;q)_\alpha = \frac{(x;q)_\infty}{(xq^\alpha; q)_\infty}
\]

and

\[
1 \Phi_0 [a; -; q; z] = \frac{(az;q)_\infty}{(z;q)_\infty}.
\]  

(18)

we obtain the formula

\[
\sum_{n=0}^\infty \frac{(q^{-\alpha}; q)_n}{(q;q)_n} (xq^\alpha)^n 2 \Phi_1 \left[ \begin{array}{c} q^{-n}, q^{-\alpha}; q; -q^{\alpha+1} \\ q^{\alpha+1-n} \end{array} \right]
\]

\[
\infty \frac{t^{z-1}}{-tq^a q^-a} = \left(-\frac{q^{1-a-z}}{q^a_1} q^{z} \right) \ldots B_q (z, b - a - z). \]

(14)
This immediately leads to the summation formula

\[
2 _2\Phi_1 \left[ \begin{array}{c} q^{-n}, \quad q^{-\alpha}, \quad q; \quad -q^{\alpha+1} \\ q^{\alpha+1-n} \end{array} \right] = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{(q;q)_n (q^{-2\alpha}; q^2)^{n/2}}{(q^{-\alpha}; q)_n (q^2; q^2)^{n/2}}, & \text{if } n \text{ is even} \end{cases}
\] (20)

For the sake of brevity, we shall use the following notation for a very well-poised basic hypergeometric series.

\[
\frac{r+1}{W_r} \left[ a; a_1, a_2, \ldots, a_{r-2}; q; z \right] = r+1 \Phi_r \left[ a, q\sqrt{a}, -q\sqrt{a}, a_1, \ldots, a_{r-2}; q; z \right]
\] (21)

By Saalschütz summation formula, we have

\[
\frac{(b, c; q)_k}{(aq/b, aq/c; q)_k} = \left( \frac{bc}{aq} \right)^k _3\Phi_2 \left[ \begin{array}{c} q^{-k}, \quad aq^k, \quad aq/bc; q; q \\ aq/b, \quad aq/c \end{array} \right]
\] (22)

k=0,1,2,\ldots. If \( \Omega_k \) is an arbitrary bounded complex function then we have the expansion formula

\[
\sum_{k=0}^{n} \frac{(b, c, q^{-n}; q)_k}{(q, aq/b, aq/c; q)_k} \Omega_k
\]

\[
= \sum_{j=0}^{\infty} \left( \frac{aq}{bc}, q^{-n}; q \right)_j \left( a; q \right)_j \frac{(-)^j q^{-2j} \binom{j}{2}}{\binom{j}{2} \times (q, a, aq/b, aq/c; q)_j}
\]
\[
\sum_{i=0}^{n-j} \frac{aq^{2j}q^{j-n}q_i}{(aqq^j;q)_i}q^{-ij}(bc/aq)^i q^{ij}\Omega_{i+j}, \quad n=0,1,2,\ldots
\] (23)

Now choosing
\[
\Omega_k = \frac{(a,a_1,a_2,\ldots,a_r;q)_k}{(b_1,b_2,\ldots,b_{r+1};q)_k}
\]
we find that
\[
\frac{r+4\Phi_{r+3}}{r+2\Phi_{r+1}} \left[ \frac{a,b,c,a_1,a_2,\ldots,a_{r-1},a_r,q^{-n};q;z}{aq/b,aq/c,b_1,b_2,\ldots,b_r,b_{r+1}} \right]
\]
\[
= \frac{n}{r+2\Phi_{r+1}} \left[ \frac{(aq/bc,a_1,a_2,\ldots,a_r,q^{-n};q)_j}{(aq/b,aq/c,b_1,b_2,\ldots,b_r,b_{r+1};q)_j} \right] \left( \frac{bczq^{-j-1}}{aq} \right) \frac{j}{2} \times
\]
\[
\sum_{j=0}^{r+2\Phi_{r+1}} \left[ \frac{aq^{2j},a_1q^j,\ldots,a_rq^j,q^{-n};q,bcq^{-j-1}}{b_1q^j,\ldots,b_rq^j,b_{r+1}q^j} \right].
\] (24)

Setting \( a_1 = q\sqrt{a} = -a_2, \; b_1 = \sqrt{a} = -b_2, \; b_k = a_k, k=3,\ldots,r, \; b_{r+1} = aq^{n+1} \)
and \( z = aq^{n+1}/bc \) and then making use of the summation formula
\[
4\Phi_3 \left[ \frac{a, \; q\sqrt{a}, \; -q\sqrt{a}, \; q^{-n}, \; q, \; q^n}{\sqrt{a}, \; -\sqrt{a}, \; aq^{n+1}} \right] = \delta_{m,n},
\]
where \( \delta_{m,n} \) is the kronecker delta, [1.2(B).24] gives the sum of a terminating very well-poised \( 6\Phi_5 \) series:
\[
6W_5 \left[ a;b,c,q^{-n};q;aq^{n+1};bc \right] = (aq,aq/bc;q)_n
\]
(25)

Again setting \( a_1 = q\sqrt{a} = -a_2, \; b_1 = \sqrt{a} = -b_2, \; a_3 = d, a_4 = e, \; b_3 = aq/d, \; b_4 = aq/e, \; b_k = a_k, k=5,\ldots,r, \; b_{r+1} = aq^{n+1} \) and
\[ z = a^{2q^{n+2}/bcde} \text{ in } [1.2(B).24] \text{ and using } [1.2(B).25] \text{ on the right side one can obtain Watson's}[1] \text{ Transformation formula and then making use of the summation formula} \]

\[ gW_7\left[ a; b, c, d, e, q^{-n}, q; a^{2q^{n+2}/bcde} \right] \]

\[ = \frac{(aq, aq/de; q)_n}{(aq/b, aq/c, 2q^{n+2}/bcde)} 4\Phi_3 \left[ q^{-n}, d, e, aq/bc; q, q \right] \text{. (26)} \]

In the special case \( aq/bc = deq^{-n}/a \), \( 4\Phi_3 \) series on the right side of \([1.2(B).26]\) reduces to a terminating balanced \( 3\Phi_2 \) series which can be summed by Saalschütz summation formula. Thus we obtain Jackson’s[5] summation formula.

\[ gW_7\left[ a; b, c, d, e, q^{-n}, q; q \right] \]

\[ = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}. \text{ (27)} \]

where \( a^{2q^{n+1}/bcde} \). As \( n \to \infty \), \([1.2(B).27]\) yields

\[ 6W_5[a; b, c, d, aq/bcd] \]

\[ = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}, |aq/bcd| < 1. \text{ (28)} \]

Using Jackson’s formula \([1.2(B).27]\) we have

\[ \frac{(a, b, c; q)_k}{(q, aq/b, aq/c; q)_k} = \frac{(\lambda bc/a; q)_k}{(qa^2/\lambda bc; q)_k} \sum_{j=0}^{k} \frac{1 - \lambda^2 q^{2j}}{1 - \lambda} \times \]

\[ \frac{(\lambda, \lambda b/a, \lambda c/a, aq/bc; q)_j}{(q, aq/b, aq/c, \lambda bc/a; q)_j} \frac{(a; q)_k^j}{(\lambda q; q)_k^j} \frac{(a/\lambda; q)_k^j}{(q; q)_k^j} (a/\lambda^j) \text{ (29)} \]

where \( \lambda \) is an arbitrary parameter.

Thus, for an arbitrary sequence \( (\alpha_k) \).
\[
\sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(q, aq/b, aq/c; q)_k} \alpha_k
\]
\[
= \sum_{j=0}^{k} \frac{(1 - \lambda q^2)^j (\lambda, \lambda b/a, \lambda c/a, aq/bc; q)_j (a; q)_2j}{(1 - \lambda) (q, aq/b, aq/c, qa^2/\lambda bc; q)_j (\lambda q; q)_2j} \left( \frac{a}{\lambda} \right)^j
\]
\[
\times \sum_{k=0}^{\infty} \frac{(aq^2j, a/\lambda, \lambda bcq^j/a; q)_k}{(q, \lambda q^2j+1, a^2q^j+1/\lambda bc; q)_k} \alpha_{j+k},
\]
(30)

Provided all the series terminate or absolutely convergent.

Choosing \( \lambda = qa^2/bcd \), and

\[
\alpha_k = \frac{(1 - aq^{2k})(d, e, f, g, h; q)_k q^k}{(1 - a)(aq/d, aq/e, aq/f, aq/g, aq/h; q)_k},
\]

where at least one of the parameters e, f, g, h is of the form \( q^{-n} \), \( n=0,1,2,... \) and \( q^2a^3 = bcdefgh \),

We obtain by using [1.2(B).27] for the inner series on the right side of [1.2(B).30], Bailey's [1] transformation formula for terminating very well-poised balanced \( 10 \Phi_9 \)-series,

\[
10W_9 \left[ a; b, c, d, e, f, a^3q^{n+2}/bcdef, q^{-n}; q; q \right]
\]
\[
= \left( aq, aq/ef, a^2q^2/bcde, a^2q^2/bcdf; q \right)_n \times \left( aq/e, aq/f, a^2q^2/bcdef, a^2q^2/bcd; q \right)_n
\]
\[
\times 10W_9 \left[ qa^2/bcd; aq/bc, aq/bd, aq/cd, e, f, a^3q^{n+2}/bcdef, q^{-n}; q; q \right]. \quad (31)
\]

An iteration of this formula gives another transformation formula

\[
10W_9 \left[ a; b, c, d, e, f, a^3q^{n+2}/bcdef, q^{-n}; q; q \right]
\]
If we now take the limit $n \to \infty$ in [1.2(B).31], we immediately get the transformation formula:

$$\begin{align*}
\mathbf{8W}_7[a; b, c, d, e, f; q; aq/ef] \\
= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f; q)}{(aq/e, aq/f, \lambda q/ef, \lambda q; q}_{\infty}} \\
\times 8W_7[\lambda; \lambda b/a, \lambda c/a, \lambda c/a e, f; q; aq/ef].
\end{align*}$$

(33)

where $\lambda = qa^2/bcd$, provided $\max(|aq/ef|, |\lambda q/ef|) < 1$.

Bailey [2] choose $n$ to be an odd integer, say $n = 2m + 1$, and divided the series on the right-side of [1.2(B).32] into two halves, each containing $(m+1)$ terms and then reversed the order of the second series. This procedure followed by the limit $n \to \infty$ yields the transformation formula.

$$\begin{align*}
\mathbf{8W}_7[a; b, c, d, e, f; a^2q^2/bcdef] \\
= \frac{(aq, aq/de, aq/df, aq/ef; q)}{(aq/d, aq/e, aq/f, aq/def; q)}_{n} \\
\times 4\Phi_3\left[\begin{array}{cccc}
d, & e, & f, & aq/bc; q; q \\
aq/b, & aq/c, & def/a
\end{array}\right] \\
+ \frac{(aq, aq/bc, d, e, f, a^2q^2/cdef; q)}{(aq/b, aq/c, aq/d, aq/e, a^2q^2/bcdef, def/qa; q)}_{\infty} \\
\times 4\Phi_3\left[\begin{array}{cccc}
aq/de, & aq/df, & aq/fe, & a^2q^2/bcdef; q; q \\
a^2q^2/bcdef, & a^2q^2/cdef, & a^2q^2/def
\end{array}\right].
\end{align*}$$

(34)
Both $4_3\Phi_3$-series on the right side of [1.2(b).34] are balance and non-terminating and hence [1.2(B).34] is a non terminating extension of Watson’s formula[1.2(B).26].

2(C) BIBASIC SUMMATION AND TRANSFORMATION FORMULAE.

A. Verma and M. Upadhyay [1.2] developed a theory for generalized basic hypergeometric series with two bases $q$ and $q^{1/2}$. Later R.P. Agarwal and A. Verma [1.2] extended the existing theory of the generalized basic hypergeometric series with a single base $q$ and those with two bases $q$ and $q^{1/2}$ to the transformation theory for basic hypergeometric series with two unconnected bases $q$ and $q_1$. Following is the general transformation established by Agarwal and Verma.

\[
\frac{1}{q^{r-1}} \sum_{t=0}^{\infty} \prod_{u=0}^{r-1} \left[ \begin{array}{c} (a) + f_1, \quad (b) - f_1; \quad q \\ 1, \quad (e) + f_1, \quad (f') - f_1 \end{array} \right] \\
\times \prod \left[ \begin{array}{c} (c) + m[f_1] + u_1, \quad (d) - m[f_1] - u_1; \quad q \\ (g) + m[f_1] + u_1, \quad h - m[f_1] - u_1 \end{array} \right] \\
\rho^{\Phi \sigma} \left[ \begin{array}{c} (e) + f_1, \quad 1 - (b) + f_1; \quad 1 - (d) + m[f_1] + u_1, \quad (g) + m[f_1] + u_1; \quad q_1 \quad Q_1 \\
(a) + f_1, \quad 1 - (f) + f_1; \quad 1 - (h) + m[f_1] + u_1, \quad (c) + m[f_1] + u_1; \quad q^{F-B}, q_1^{H-D} \end{array} \right] \\
+ \text{idem} \left( f_1; f_2, f_3, \ldots, f_\ell \right) \\
+ \frac{1}{q^{r-1}} \prod_{t=0}^{\infty} \prod_{u=0}^{r-1} \left[ \begin{array}{c} (a) + M[h_1], \quad (b) - M[h_1]; \quad q \\ (e) + M[h_1], \quad (f) - M[h_1] \end{array} \right] \\
\times \prod \left[ \begin{array}{c} (c) + h_1, \quad (d) - h_1; \quad q_1 \quad Q_2 \\
(g) + h_1, \quad 1 - (d) + h_1; \quad (c) + h_1, \quad 1 - (h) + h_1; \quad q^{F-B}, q_1^{H-D} \end{array} \right] \\
+ \text{idem} \left( h_1; h_2, h_3, \ldots, h_H \right)
\]
\[
\sum_{u=0}^{r-1} \frac{1}{t} \prod \left[ \begin{array}{c}
(a) - e_1, (b) + e_1, q \\
(e) - e_1, f + e_1
\end{array} \right] \times
\prod \left[ \begin{array}{c}
(c) - m[e_1] - u_1, (d) + m[e_1] + u_1; q_1 \\
(g) - m[e_1] - u_1, h + m[e_1] + u_1
\end{array} \right] \\
\rho \Phi_\sigma \left[ \begin{array}{c}
(f) + e_1, 1 - (a) + e_1; (h) + m[e_1] + u_1, 1 - (c) + m(e_1) + u_1; Q_3 \\
(b) + e_1, 1 - (e) + e_1; (d) + m[e_1] + u_1, 1 - (g) + m(e_1) + u_1; q^{E-A, q_1 G-C}
\end{array} \right] \\
+ \text{idem} \left( e_1; e_2, e_3, \ldots, e_E \right)
\]

\[
\sum_{u=0}^{r-1} \frac{1}{t} \prod \left[ \begin{array}{c}
(a) - M[g_1], (b) + M[g_1] \\
(e) + M[g_1], (f) + M[g_1]
\end{array} \right] \prod \left[ \begin{array}{c}
(d) + g_1, (c) - g_1; q_1 \\
(h) + g_1, (g) - g_1, 1
\end{array} \right] \\
\rho \Phi_\sigma \left[ \begin{array}{c}
(d) + M[g_1], 1 - (a) + M[g_1]; (h) + g_1, 1 - (c) + g_1; Q_4 \\
(b) + M[g_1], 1 - (e) + M[g_1]; (d) + g_1, 1 - (g) + g_1; q^{E-A, q_1 G-C}
\end{array} \right] \\
+ \text{idem} \left( g_1; g_2, g_3, \ldots, g_G \right)
\]
where the numbers \((\infty) + (2\pi i/t)m(\alpha)\) have been abbreviated to \(m[(\infty)]\) and the numbers \((\alpha) + (2\pi i/t|)m(\alpha)\) to \(M[(\alpha)]\). The parameters in [1.2(C).1] are subject to the following restrictions:

(i) \(F>B\) or \(F=B\) and \(\text{Re} \left[ \Sigma(b) - \Sigma(f) \right] > 0\),

(ii) \(H>D\) or \(H=D\) and \(\text{Re} \left[ \Sigma(d) - \Sigma(h) \right] > 0\),

(iii) \(E>A\) or \(E=A\) and \(\text{Re} \left[ \Sigma(a) - \Sigma(e) \right] > 0\),

(iv) \(G>C\) or \(G=C\) and \(\text{Re} \left[ \Sigma(c) - \Sigma(g) \right] > 0\),

(3)

Verma [see Agarwal [5; 7.12(2)]] established the sum of a truncated polybasic hypergeometric series

\[
3\Phi_3 \left[ \begin{array}{c} q^a, q_1^a, zq^{a+1}q_1^{b+1} \\ zq^{a+1}, zq_1^{b+1}, zq_1^a q_1^b \end{array} \right]_N
= \frac{(q^a;q)_N(q_1^b;q)_N}{(z-1)(1-zq_1^a q_1^b)} \frac{z^{N+1}}{N!} \frac{(1-zq^a)(1-zq_1^b)}{(z-1)(1-zq^a q_1^b)}. \tag{4}
\]

where the underlined terms are on base \(p = qq_1\). Now, if we further let \(|q| < 1, |q_1| < 1, |z| < 1\), and then let \(N \to \infty\) in [1.2(C) .4] we get:

\[
3\Phi_3 \left[ \begin{array}{c} q^a, q^b, zq^{a+1}q_1^{b+1} \\ zq^{a+1}, zq_1^{b+1}, zq_1^a q_1^b \end{array} \right]_N
= \frac{(1-zq^a)(1-zq_1^b)}{(1-z)(1-zq^a q_1^b)}. \tag{5}
\]

We have the summation formula for a very well-poised \(4\Phi_3\)-series.

\[
4\Phi_3 \left[ \begin{array}{c} a, q\sqrt{a}, -q\sqrt{a}, q^{-n}; q; q^n \\ \sqrt{a}, -\sqrt{a}, aq^{n+1} \end{array} \right] = \sum_{k=0}^{n} \frac{(1-aq^{2k})(a,q^{-n};q)_k}{(1-a)(q,aq^{n+1};q)_k} q^{nk} = \delta_{n,0}. \tag{6}
\]
A bibasic generalization of [1.2(C). 6], namely
\[ \sum_{k=0}^{n} \frac{(1-aq)^k q^k}{(1-a)(q; q)_k (apq^n; p)_n} \left( \frac{q^{-n}; q}{q_n; q} \right)^k q^{nk} = \delta_{n,0}. \]

was found by Carlitz [1] and made extensive use of by Gassel and Stanton [1] and Rahman [4].

Applying the difference operator \( \Delta \) defined by \( \Delta U_k = U_k - U_{k-1} \), Gasper [1] established the following indefinite bibasic summation formula.

\[ \sum_{k=0}^{n} \frac{(1-ap^k q^k)(1-bp^k q^{-k})}{(1-a)(1-b)(q; q)_k (ap/c, bcp; p)_k} (c, a/bc; q)_k \frac{c^{-k}}{c^k} = \frac{(aq, bp;p)_n (cq, aq/bc; q)_n}{(q; q)_n (ap/c, bcp; p)_n}. \tag{7} \]

As \( b \to 0 \) in [1.2(C) .7] we find another bibasic summation formula due to Gasper (See Gasper and Rahman [2;3.6.8], p.71):

\[ \sum_{k=-m}^{n} \frac{(1-adp^k q^k)(1-bdp^k q^{-k})}{(1-a)(1-b/d)(dq, adq/b; q)_k (adp/c, bcp/d; p)_k} q^k \frac{c^{-k}}{c^k} = \frac{(ap; p)_n (cq; q)_n}{(q; q)_n (ap/c; p)_n}. \]

Gasper and Rahman [1] extended [1.2(C) .7] in the following form:

\[ \sum_{k=-m}^{n} \frac{(1-adp^k q^k)(1-bdp^k q^{-k})}{(1-ad)(1-b/d)(dq, adq/b; q)_k (adp/c, bcp/d; p)_k} q^k \frac{c^{-k}}{c^k} = \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-ad)(1-b/d)(1-c/d)(1-ad/bc)} \times \frac{(ap, bp; p)_n (cq, ad^2q/bc; q)_n}{(dq, adq/b; q)_n (adp/c, bcp/d; p)_n}. \]
For \( n, m = 0, \pm 1, \pm 2, \ldots \), for \( d = 1 \), [1.2(C).8] yields [1.2(C).7]. If \(|p| < 1, |q| < 1\), then letting \( m, n \to \infty \) in [1.2(C).9] we find:

\[
\sum_{k=-\infty}^{\infty} \frac{(1 - ad^k q^k) \left( 1 - \frac{b p^k q^{-k}}{d} \right) (a, b; p)_k (c, ad^2/bc; q)_k}{(1 - ad)(1 - b/d)(dq, adq/b; q)_k (adp/c, bcp/d; p)_k} q^k
\]

\[
= \frac{(1 - a)(1 - b)(1 - c)(1 - ad^2/bc)}{d(1 - ad)(1 - b/d)(1 - c/d)(1 - ad/bc)} \times \left\{ \frac{(ap, bp; p)_\infty (cq, ad^2q/bc; q)_\infty}{(dq, adq/b; q)_\infty (adp/c, bcp/d; p)_\infty} \right\}.
\]

Taking \( m = 0 \) in [1.2(C).9] we have

\[
\sum_{k=0}^{n} \frac{(1 - ad^k q^k) \left( 1 - \frac{b p^k q^{-k}}{d} \right) (a, b; p)_k (c, ad^2/bc; q)_k}{(1 - ad)(1 - b/d)(dq, adq/b; q)_k (adp/c, bcp/d; p)_k} q^k
\]

\[
= \frac{(1 - a)(1 - b)(1 - c)(1 - ad^2/bc)}{d(1 - ad)(1 - b/d)(1 - c/d)(1 - ad/bc)} \times \left\{ \frac{(ap, bp; p)_n (cq, ad^2q/bc; q)_n}{(dq, adq/b; q)_n (adp/c, bcp/d; p)_n} \right\}.
\]

\[
\frac{(1 - d)(1 - ad/b)(1 - ad/c)(1 - bc/d)}{d(1 - ad)(1 - b/d)(1 - c/d)(1 - ad/bc)}.
\]
If we put \( c = q^{-n} \) in [1.2 (C). 11], it yields

\[
\sum_{k=0}^{n} \frac{(1-adp^k q^k)(1-bp^k q^k - \frac{d}{d})(a,b;p)_k(q^{-n},ad^2q^n/b;q)^k}{(1-ad)(1-b/d)(dq,adq/b;q)_k(apdq^n, bq^{-n/p}/d;p)_k}q^k
\]

\[
= \frac{(1-d)(1-ad/b)(1-adq^n)(1-dq^n/b)}{(1-ad)(1-d/b)(1-dq^n)(1-adq^n/b)}, \quad n=0,1,2,\ldots \tag{12}
\]

The \( d \to 1 \) limit case of [1.2(C). 12] is

\[
\sum_{k=0}^{n} \frac{(1-adp^k q^k)(1-bp^k q^k - \frac{d}{d})(a,b;p)_k(q^{-n},aq^n/b;q)_k}{(1-a)(1-b)(q,aq/b;q)_k(apq^n,bpq^{-n/p};p)_k}q^k = \delta_{n,0}. \tag{13}
\]

By a little manipulation [1.2(C).13] yields the summation formula:

\[
(1-a/p)(1-c/p) \sum_{k=0}^{n} \frac{(aq^k,bq^{-k};p)_{n-1}(1-aq^{2k}/b)(-)^k q^2}{(q;q)_k(q;q)_{n-k}(aq^k/b;q)_{n+1}}
\]

\[
= \delta_{n,0} \tag{14}
\]

One can employ [1.2(C).14] to find the following bibasic expansion formula:

\[
\sum_{n=0}^{\infty} A_n B_n (xw)^n = \sum_{k=0}^{\infty} \frac{(apq^k,bpq^{-k};p)_{k-1}(-x)^k q^2}{(q,aq^k/b;q)_k} \times
\]

\[
\times \sum_{n=0}^{k} \frac{(1-ap^n q^n)(1-bp^n q^{-n})(q^{-k},aq^k/b;p)_n A_n w^n}{(apq^k,bpq^{-k};p)_k}.
\]
\[
\sum_{j=0}^{\infty} \frac{\left( a p^k q^k, b p^k q^{-k} ; p \right)_j}{\left( q, a q^k + 1/b ; q \right)_j} \cdot B_{j+k}(x q)^j
\]

For bibasic summation, transformation and expansion theorems, one is referred to the unpublished monograph “Generalized Hypergeometric series and its Application to the theory of Combinatorial analysis and Partition Theory” of Prof. R.P. Agarwal[5] and Basic Hypergeometric Series by G. Gaspar and M. Rahman [1].

2.(D) ASKEY-WILSON POLYNOMIALS AND q-BETA INTEGRALS.

The polynomials

\[
P_n(z; a, b, c, d) = _4\Phi_3 \left[ \begin{array}{c} q^{-n}, \ abcdq^{n-1}, \ ae^{i\theta}, \ ae^{-i\theta} \\ ab, \ ac, \ ad \end{array} \right]_{q} \]

which are symmetric in a,b,c,d are called the Askey-Wilson polynomials.

The orthogonality relation of these polynomials is given by

\[
\int_{-1}^{1} P_n(z; a, b, c, d) P_m(z; a, b, c, d) W(z; a, b, c, d) \, dz = \frac{\delta_{m,n}}{h_n},
\]

where \( W(z; a, b, c, d) = \frac{h(z; 1) h(z; -1) h(z; \sqrt{q}) h(z; -\sqrt{q})}{h(z; a) h(z; b) h(z; c) h(z; d) \sqrt{1 - z^2}} \),

\[
h(z; a) = \prod_{n=0}^{\infty} \left( 1 - 2a z^q + a^2 z^{2q} \right),
\]

\[
h_n = K^{-1}(a, b, c, d) \frac{\left( abcdq^{-1}, ab, ac, ad; q \right)_n \left( 1 - abcdq^{2n-1} \right)}{(q, bc, bd, cd; q)_n a^{2n} \left( 1 - abcdq^{-1} \right)}
\]

and
\[ K(a,b,c,d) = \int_{-1}^{1} w(z; a, b, c, d) dz \]
\[ = \frac{2\pi (abcd; q)_{\infty}}{(a, bc, ac, bc, bd, cd; q)_{n}}, \] (3)

Provided \( \max.(|q|, |a|, |b|, |c|, |d|) < 1 \).

In a beautiful piece of work, Askey and Wilson showed that the integral [1.2(D).3] is the Beta integral. Setting \( a = -b = \sqrt{q}, \ c = q^{\alpha + 1/2}, \ d = -q^{\beta + 1/2} \), in [1.2(D).3] one can show that in the limit \( q \to 1 \), it reduces to the beta integral
\[ \frac{1}{1} \int_{-1}^{1} (1 - z)^\alpha (1 + z)^\beta dz = 2^{\alpha + \beta + 1} B(\alpha + 1, \beta + 1). \] (4)

Rahman [1] established the following product formula for \( P_n(z; a, b, c, d) \):
\[ P_n(x; a, b, c, d)P_n(y; a, b, c, d) = \left( \frac{bc, bd; q}{ac, ad; q} \right)_n \frac{a}{b} \times \]
\[ \sum_{k=0}^{n} \left( q^{-n}, abcdq^{n-1}, ae^i\theta, ae^{-i}\theta, ae^{i\phi} \right)_k q^k \times \]
\[ \left( q, ab, ac, ad, a/b; q \right)_k \]
\[ 10W_9 \left[ bq^{-k/a}, q^{-1-k/ac}, q^{-1-k/ad}, q^{-k}, be^i\theta, be^{-i}\theta, be^{i\phi}, be^{-i\phi}; q, qcdq/ab \right], \] (5)

where \( x = \cos\theta \) and \( y = \cos\phi \).

By applying Whipple’s transformation of a terminating balanced \( 4\Phi_3 \)-series, viz.,
\[ 4\Phi_3 \left[ q^{-n}, abcdq^{n-1}, ae, af; q, q \right] \]
\[ \left[ ab, ace, adf \right] \]
\[ = \left( \frac{bd/e, bc/f; q}{ace, adf; q} \right)_n \left( \frac{ae}{b} \right)_n \times \]
One can easily see that
\[ P_n(z;a,b,c,d) = \frac{(bc,bd;q)_n}{(ac,ad;q)_n} P_n(z;a,b,c,d) \]
\[ = \frac{(cd,cd;q)_n}{(ab,ad;q)_n} \frac{a}{c} P_n(z; c, b, a, d) \]
\[ = \frac{(de,db;q)_n}{(ab,ad;q)_n} \frac{a}{d} P_n(z; d, b, c, a). \] (7)

Nassrullah and Rahman [1] obtained the following integral representation of an \( 8 \Phi_7 \)-series:
\[ 8W_7 \left[ g^2/q; g/a, g/b, g/c, g/d, g/f; q; abcdf/g \right] \]
\[ = \frac{\left( q, a, b, c, d, a, f, b, c, d, c, f; g^2, q \right)}{2\pi (ag, bg, cg, dg, fg, abcdf/g; q)_{\infty}} \times \]
\[ \int_{-1}^{1} \frac{h(x; l)h(x; -l)h(x; \sqrt{q})h(x; -\sqrt{q})h(x; g)}{h(x; a)h(x; b)h(x; c)h(x; d)h(x; f)} \frac{dx}{\sqrt{1-x^2}}. \]

when \( |abcdf/g| < 1 \) and \( \max. \ (|a|, |b|, |c|, |d|, |f|, |g|) < 1 \).

**Ramanujan’s theta functions and Modular equations**

Ramanujan recorded hundreds of modular equations in his three notebooks. Chapters 19 – 21 in Ramanujan’s second notebook are almost exclusively devoted to modular equations. Ramanujan used modular equations to evaluate class invariants, certain \( q \) – continued fractions, theta functions and certain other quotients and products of theta functions.
Theta functions are at the focal point in Ramanujan's theories. His general theta function \( f(a,b) \) is defined by
\[
f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |a,b| < 1.
\] (1)

By an appeal of Jacobi's triple product identity, we have
\[
f(a,b) = \left[ -a, -b, ab; ab \right]_\infty.
\] (2)

The three most important special cases of (1) are
\[
\phi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \left[ q; q \right]_\infty,
\] (3)
\[
\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \left[ q^2; q^2 \right]_\infty,
\] (4)
\[
f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-)^n q^{n(3n+1)/2} = [q; q]_\infty.
\] (5)

We also define,
\[
\chi(-q) = \left[ q; q^2 \right]_\infty.
\] (6)

Let \( n \) denote a fixed natural number and assume that
\[
\frac{2F_1 \left[ \frac{1}{r}, \frac{r-1}{r}; 1; 1 - \alpha \right]}{2F_1 \left[ \frac{1}{r}, \frac{r-1}{r}; 1; 1 - \beta \right]} = \frac{2F_1 \left[ \frac{1}{r}, \frac{r-1}{r}; 1; 1 - \beta \right]}{2F_1 \left[ \frac{1}{r}, \frac{r-1}{r}; 1; 1 - \beta \right]}, \quad r = 2, 3, 4 \text{ or } 6.
\] (7)

where \( r = 2, 3, 4 \) or 6. Then a modular equation of degree \( n \) in the signature \( r \) is a relation between \( \alpha \) and \( \beta \) induced by (7). The multiplier \( m(r) \) is defined by
\[
m(r) = \frac{2F_1 \left[ \frac{1}{r}, \frac{r-1}{r}; 1; 1 - \alpha \right]}{2F_1 \left[ \frac{1}{r}, \frac{r-1}{r}; 1; 1 - \beta \right]} \quad \text{for } r = 2, 3, 4 \text{ or } 6.
\]
1.3 BASIC HYPERGEOMETRIC IDENTITIES AND ADDITIVE NUMBER THEORY.

Let us start with the following well-known generating functions which are very useful in the study of the theory of partitions.

\[ \sum_{n=0}^{\infty} P_m(n)q^n = \frac{1}{(q;q)_m}, \]  

(1)

where \( P_m(n) \) stands for the number of partitions of \( n \) in which no part is greater than \( m \).

\[ \sum_{n=0}^{\infty} P(n)q^n = \frac{1}{(q;q)_{\infty}}, \]  

(2)

where \( P(n) \) denotes the number of partitions of \( n \).

\[ \sum_{n=0}^{\infty} d_m(n)q^n = (-q; q)_\infty, \]  

(3)

where \( d_m(n) \) is the number of partitions of \( n \) into distinct parts none greater than \( m \).

\[ \sum_{n=0}^{\infty} d(n)q^n = (-q; q)_\infty, \]  

(4)

where \( d(n) \) denotes the number of partitions of \( n \) into distinct parts.

L.J. Rogers in 1984 established identities which were later discovered by Ramanujan in 1913, namely,

\[ \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_{\infty}}, \]  

(5)

\[ \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_{\infty}}, \]  

(6)

These identities have the following very elegant Combinatorial interpretations:
"The number of partitions of $n$ into parts with least difference 2 is equal to the number of the partitions of $n$ into parts $\equiv 1$ or $4$(mod 5)" and

The number of partitions of $n$ into not less than 2 and with minimal difference 2 is equal to the number of partitions of $n$ into parts $\equiv 2$ or $3$ (mod 5)”, respectively.

The simplest of the several proofs of [1.3.5] and [1.3.6] given by Rogers and then by Ramanujan depends on general formula

$$1 + \sum_{n=0}^{\infty} (-)^n a^{2n} q^{n(5n-1)/2} (1-\frac{aq^{2n}}{1-a})(a;q)_{n} (q;q)_{n}$$

$$= (a;q)_{\infty} \sum_{n=0}^{\infty} \frac{a^n q^n}{(q;q)_n}$$ \hspace{1cm} (7)

and Jacobi’s triple product identity

$$\sum_{n=-\infty}^{\infty} (-)^n q^{n^2} z^n = \left(q^2, zq, q/z; q^2\right)_{\infty}, \hspace{1cm} |q|<1 \hspace{1cm} (8)$$

Identities [1.3.5] and [1.3.6] can be obtained from [1.3.7] by taking $a=1$ and $a=q$, respectively and then using [1.3.8].

In 1929, an interesting proof depending on the transformation [1.2.(B).26] was given by G.N. Watson[1].

Letting $b,c,d,e \rightarrow \infty$ in [1.2(B).26] we get:

$$1 + \sum_{r=1}^{\infty} \frac{(aq;q)_{r-1} \left(1-\frac{aq^{2r}}{1-a}\right) \left(q^{-n}; q\right)_{r} a^{2r} q^{2r^2+nr}}{(q;q)_{r} (aq^{n+1}; q)_{r}}$$

$$= (aq;q)_{n} \left\{ 1 + \sum_{r=1}^{\infty} \frac{(-)^{r} q^{(r+1)/2} \left(q^{-n}; q\right)_{r} a^{r} q^{nr}}{(q;q)_{r}} \right\}. \hspace{1cm} (9)$$

For $n \rightarrow \infty$, [1.3.9] reduces to [1.3.7].

Further, if we take $bc=aq$ and $d,e,n \rightarrow \infty$ in [1.2(B).26], we find
\begin{equation}
1 + \sum_{r=1}^{\infty} (-)^n a^n q^{n(3n-1)/2} \left( 1 - a q^{2n} \right) \frac{(aq;q)_n}{(q;q)_n} = (aq;q)_\infty, \tag{10}
\end{equation}

which for a=1, yields Euler's identity viz.,

\begin{equation}
\sum_{n=-\infty}^{\infty} (-)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{11}
\end{equation}

Again, taking \( bc=aq, \ d=q\sqrt{a}, \ e,n \to \infty \) and then \( a=1 \) in \([1.2(B).26]\), we have;

\begin{equation}
1 + \sum_{r=1}^{\infty} q^{r(r+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{12}
\end{equation}

which is a classical identity due to Gauss.

W.N. Bailey[4,5] combined Rogers method freely with the known summation formulae of basic hypergeometric series to obtain some general transformations leading to Rogers-Ramanujan identities as limiting cases.

The fundamental theorem used by him was

"If \( \beta_n = \sum_{r=0}^{n} U_{n-r} V_{n+r} \alpha_r \)

and \( \gamma_n = \sum_{r=n}^{\infty} U_{r+n} V_{r-n} \delta_r \)

where \( \alpha_r, \delta_r, \gamma_r \) and \( U_r \) are functions of \( r \) only then

\begin{equation}
\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n
\end{equation}

Provided that all the series, involved are either convergent or terminating.

Similarly \( \gamma_n \), by basic analogue of Gauss theorem one obtains the transformation:

\begin{equation}
\sum_{n=0}^{\infty} (y,z;q)_n \beta_n (x/yz)^n
\end{equation}
By giving different values to the sequence $\alpha_n$, Bailey obtained different identities from it. However, a systematic attempt was made not until 1951, to use the above theorem to obtain various identities, when L.J. Slater [1] obtained a number is general transformation by giving different values to $\alpha_r, \delta_r, U_r$ and $V_r$ with the help of known basic bilateral q-series summation theorems. The most widely and fruitfully used summation theorem in this direction is

$$
\psi_6\left[ \frac{q \sqrt{a}, -q \sqrt{a}, b, c, d, e; \frac{a^2 q}{bcde}}{\sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/bcde} \right] = \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, b, c, d, e, qa^2/bcde; q)_\infty}.
$$

Putting $b = q^{-n/3}, c = q^{(1-n)/3}, d = q^{(2-n)/3}$ and replacing $q$ by $q^3$ and then taking $a = q$ in [1.3.14] we get:

$$
\sum_{r=\left[ -n/3 \right]}^{\left[ n/3 \right]} \frac{\left( 1 - q^{6r+1} \right) (-q)^r \left( q^4/e; q^3 \right)_r}{(q; q)_{n+3r+1}(q; q)_{n-3r}(q^4/e; q^3)_r} e^r
$$

$$
= \frac{\left( q^2/e; q^3 \right)_n}{(q; q)_{2n} \left( q^2/e; q \right)_n}.
$$

Letting $e \to \infty$, we find

$$
\sum_{r=\left[ -n/3 \right]}^{\left[ n/3 \right]} \frac{\left( 1 - q^{6r+1} \right) q^{r(6r-1)}}{(q; q)_{n+3r+1}(q; q)_{n-3r}} = \frac{1}{(q; q)_{2n}}.
$$

Since
\[
q^{r(6r-1)}(1-q^{6r+1}) = q^{r(6r-1)}(1-q^{n+3r+1}) - q^{(2r+1)(3r+1)}(1-q^{n-3r})
\]

So we get from [1.3.16], on simplification

\[
\frac{(1-q)}{(q;q)_n (q;q)_{n+1}} = \sum_{r=1}^{\lfloor n/3 \rfloor} \frac{q^{r(6r-1)} + q^{r(6r+1)}}{(q;q)_{n-3r} (q;q)_{n+3r}}
\]

\[
- \sum_{r=1}^{\lfloor (n+1)/3 \rfloor} \frac{q^{(2r+1)(3r+1)}}{(q;q)_{n+3r+1} (q;q)_{n-3r-1}} - \\
- \sum_{r=1}^{\lfloor (n+1)/3 \rfloor} \frac{q^{(2r-1)(3r-1)}}{(q;q)_{n-3r+1} (q;q)_{n+3r-1}} = \frac{1}{(q;q)_{2n}}
\]

(17)

Again, taking

\[
\beta_n = \frac{1}{(q;q)_{2n}}, \quad \alpha_{3n-1} = -q^{(2n-1)(3n-1)}
\]

\[
\alpha_{3n} = q^{n(6n-1)(3n-1)} + q^{(6n+1)}, \quad \alpha_{3n+1} = -q^{(3n+1)(2n+1)}
\]

And \(x=q, \ y=q^{1/2}/u, \ z=q^{1/2}/v\) in (1.3.13) we obtain;

\[
\sum_{n=0}^{\infty} \left( u^{q^{1/2}/u} q^{q^{1/2}/v} ; q \right)_n u^n v^n
\]

\[
= \frac{(u^{q^{1/2}}, v^{q^{1/2}} ; q)_\infty}{(q, uv;q)_\infty} \sum_{n=0}^{\infty} \left( u^{q^{1/2}/u} v^{q^{1/2}/v} ; q \right)_{3n} u^{3n} v^{3n}
\]

\[
\times \left\{ q^{n(6n-1)} - \frac{(1-u^{3n-1/2})(1-v^{3n-1/2}) q^{(2n-1)(3n-1)}}{(u-q^{2n-1/2})(v-q^{3n-1/2})} \right\}
\]

\[
\times q^{n(6n+1)} - \frac{(u-q^{3n+1/2})(v-q^{3n+1/2}) q^{(2n+1)(3n+1)}}{(1-uq^{3n+1/2})(1-vq^{3n+1/2})} \right\}
\]

(18)

This is a classical result proved by Rogers. He expressed it in the form:
\[ a_0 + a_2 + a_4 + \ldots = b_0 - b_2q - b_4q^2 + b_6\left(q^5 + q^7\right) - b_3q^{12} \ldots \] (19)

where

\[ a_{2n} = \frac{(q^{1/2}/u, q^{1/2}/v, q)_n}{(q; q)_n} \cdot u^n v^n \]

and

\[ b_{2n} = \frac{(u^q q^{1/2}, v^q q^{1/2}/u, q^{1/2}/v; q)_n}{(q, uv, q^{1/2}/v; q)_n} \]

Giving different values to \( u \) and \( v \) in [1.3.18], different identities can be obtained. For example for \( u = v = 0 \) we find

\[
(q; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_{2n}} = \left(q^{10}, q^{14}, q^{16}; q^{30}\right)_\infty - q^2 \left(q^{30}, q^4, q^{26}, q^{30}\right)_\infty.
\] (20)

Again, taking \( u = 1, v = 1 \) and then replacing \( q \) by \( q^2 \), [1.3.18] yields,

\[
\frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-)^n q^{n^2}}{(-q; q^2)_n (q^4, q^4)_n} = \left(q^{42}, q^{19}, q^{23}, q^{42}\right)_\infty + q^3 \left(q^{42}, q^5, q^{37}, q^{42}\right)_\infty.
\] (21)

Using similar technique, Slater[1] obtained a list of 130 identities with single and double products. Still there are gaps left out in her list of modular identities.

In 1951, Bailey showed that the identities with double products, deduced in the work of Slater and Rogers, can be reduced to single products identity.

He showed that
When the power of $q$ in the product advances by $n$ and $n$ is a multiple of 3, then two products can be reduced to a single one by using the following results:

\[
(q; q)_{\infty} (-z, -q/z; q)_{\infty} \left( q/z^2, qz^2; q^2 \right)_{\infty}
\]

\[
= \left( q^3, z^3 q, q^2 / z^3; q^3 \right)_{\infty} + z \left( q^3, z^3 q^2, q / z^3; q^3 \right)_{\infty}.
\] (22)

Replacing $q$ by $q^{14}$ and $z = q^3$ we get:

\[
\left( -q^3, -q^{13}, q^{14}; q^{14} \right)_{\infty} \left( q^8, q^{20}; q^{28} \right)
\]

\[
= \left( q^{42}, q^{19}, q^{23}; q^{42} \right)_{\infty} + q^3 \left( q^5, q^{37}, q^{42}; q^{42} \right)_{\infty}
\] (23)

Now, comparing [1.3.21] and [1.3.23], we have

\[
\frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{(-)^n q^n n^2}{(-q; q^2)_{n} (q^4; q^4)_{n}}
\]

\[
= \left( -q^3, -q^{11}, q^{14}; q^{14} \right)_{\infty} + \left( q^8, q^{20}, q^{28}; q^{28} \right)
\] (24)

When the power of $q$ in the product advances by $n$ and $n$ is not a multiple of 3, then we have:

\[
\left( -q z^2, -q^3 / z^2, q^4; q^4 \right)_{\infty} + z \left( -q^3 z^2, -q / z^2, q^4; q^4 \right)
\]

\[
= (-z, -q/z; q)_{\infty}
\] (25)

Recently, V.K. Jain and Verma [1] in 1982, used a quadratic transformation:

\[
\Phi_3 \left[ \begin{array}{cccc}
   a^2 & b^2 & c & d; \\
   ab\sqrt{q}, & -ab\sqrt{q}, & -cd
\end{array} \right] q; q
\]

\[
= \Phi_3 \left[ \begin{array}{cccc}
   a^2 & b^2 & c^2 & d^2; \\
   a^2 & b^2 & -cd & -cdq
\end{array} \right] q^2; q^2
\] (26)
where $a,b,c,$ or $d$ is of the form $q^{-N}$, $N$ being a non-negative integer to obtain a new form of $q$-analogue of whipple’s transformation viz.,

$$
8\Phi_7\left[\begin{array}{l}
\sqrt{a},-\sqrt{a},aq/c,aq/e,-aq/c,-aq^{n+1},aq^n;\\
\frac{a^2q^{2n+2}}{e^2c}
\end{array}\middle| \begin{array}{l}
a,q\sqrt{a},-q\sqrt{a},c,e,-e,-q^{-n},q^{-n};
\end{array}\right]
$$

which yields an identity of modules 13 for $c \to \infty$.

A Verma and V.K. Jain[2] observing that all the transformations used by them or by others are either between two series with the same base or between two series one with base $q$ and other different from $q$, developed transformations for terminating basic hypergeometric series, using the general theory of bibasic hypergeometric series given by R.P. Agarwal and A. Verma[1,2] in 1967-68. From these transformations they obtained a number of new Rogers–Ramanujan type of identities related to the moduli 11,13,17,19,22,23,26 and 38 etc. Later, Verma and Jain[2] extended their own transformations to obtained identities for the moduli 33,39,51 and 57.

In 1984, Prabha Rastogi [1] in her thesis approved for Ph.D. Degree of Lucknow University, Lucknow established some bibasic hypergeometric transformations, with the help of some known summation theorems. She made an attempt to fill up the gaps in the Slater’s list [1951-52].
Recently, M.D. Hirchhorn [1], MV. Subbarao [1] and other have made attempts to give partitions theoretic interpretations of certain identities due to Slater[1]. One of the Hirchhorn’s theorem is

"The number of partitions of \( k \),

\[ k = a_1 + a_2 + a_3 + \ldots \]

with \( a_1 \geq a_2 \geq a_3 \geq \ldots \)

Is equal to the number of partitions of \( k \) into parts congruent to \( 1, 3, 4, 5, 7, 9, 11, 13, 15, 16, 17 \) or \( 19 \)(mod 20)."

He used the following identity due to Slater [1] in order to establish the theorem [1.3.28].

\[
\sum_{r=0}^{\infty} \frac{q^{r^2}}{(q; q)_{2r}^2} = \frac{1}{(q, q^3, q^4, q^5, q^7, q^9, q^{11}, q^{13}, q^{15}, q^{16}, q^{17}, q^{19}; q^{20})_\infty}.
\]

(29)

M.V. Subbarao[1] making use of an identity [Slater[1];(94) ] established an other theorem, viz.,

"the number of partitions of \( n \) such that the parts in the first half of each partition have minimal difference 2, is equal to the number of partitions of \( n \) into parts = \( \pm 1, \pm 2, \pm 5, \pm 6, \pm 8, \pm 9 \)(mod 20)."