CHAPTER – VII

EFFECT OF VERY STRONG UNIFORM MAGNETIC FIELD ON MAGNETO-FLUID DYNAMIC TURBULENCE
CHAPTER VII

EFFECT OF VERY STRONG UNIFORM MAGNETIC FIELD ON MAGNETO-FLUID DYNAMIC TURBULENCE

7.1 Introduction

In a variety of astrophysical and geophysical problems, however, it is often the case that a certain magnetic field such as the cosmic magnetic field, the geomagnetic field etc is imposed on a turbulent motion of a conducting fluid. The essential effect of the presence of an imposed magnetic field is that the mechanical and magnetic modes of turbulence interact not only with each other through the self adjusting processes but also with the external magnetic field. If the external field is very strong, the effect of the latter interaction will predominate that of the self adjusting of processes. Ohji (1964) presented a first order theory for turbulence of an electrically conducting fluid in the presence of a uniform magnetic field which is so strong that the non linear mechanism as well as the dissipation terms when compared with the external coupling terms are of minor importance. In this chapter we have discussed the effect of a uniform magnetic field on velocity and magnetic field covariances. Here we have assumed that the magnetic field is strong enough for non-linear term and the reactions from the turbulence to be negligible.

7.2 Basic Equations

Although we are interested in the effect of a mean magnetic field, let us begin with the more general formulation where the presence of a mean velocity
field is also taken into consideration. If \( \mathbf{U} \) denotes the velocity, \( \mathbf{B} \) the magnetic induction and \( \mathbf{P} \) the pressure, the MFD equations for a conducting fluid of the density \( \rho \), the kinematic viscosity \( \nu \), the conductivity \( \sigma \) and the permeability \( \mu \) are written, in MKS units, as

\[
\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \text{ grad}) \mathbf{U} - \frac{1}{\rho \mu} (\mathbf{B} \text{ grad}) \mathbf{B} = - \text{grad} \mathbf{P} + \frac{1}{2 \mu} \mathbf{B}^2 + \nu \nabla^2 \mathbf{U}, \tag{7.2.1}
\]

for the momentum, and

\[
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{U} \text{ grad}) \mathbf{B} - (\mathbf{B} \text{ grad}) \mathbf{U} = - \frac{1}{\rho \sigma} \nabla^2 \mathbf{B}, \tag{7.2.2}
\]

for the induction, respectively, together with the supplementary equation

\[
\text{div} \mathbf{U} = 0, \quad \text{and} \quad \text{div} \mathbf{B} = 0. \tag{7.2.3}
\]

where \( \rho, \nu, \sigma \) and \( \mu \) are assumed constant. Further, it is convenient to introduce the Alfven velocity

\[
\mathbf{H} = \mathbf{B} \sqrt{\mu \rho},
\]

and the magnetic viscosity

\[
\lambda = 1/\mu \sigma.
\]

For a turbulent flow we can put

\[
\mathbf{U} = \mathbf{U} + \mathbf{u}, \quad \mathbf{H} = \mathbf{H} + \mathbf{h}, \quad \mathbf{P} = \mathbf{P} + \rho,
\]
where $U$, $H$ and $P$ are the mean values and $u$, $h$ and $p$ represent the fluctuating components. Then, taking the statistical average (expressed by an overbar), we have, into the usual index notation.

$$\begin{align*}
\frac{\partial U_i}{\partial t} + \frac{\partial}{\partial x_k} & \left[ U_i U_k - H_i H_k + u_i u_k - h_i h_k \right] \\
\frac{-1}{P} \frac{\partial}{\partial x_i} & \left[ P + \frac{P}{2} (H^2 + h^2) \right] + v \nabla^2 U_i,
\end{align*}$$

(7.2.7)

$$\begin{align*}
\frac{\partial H_i}{\partial t} + \frac{\partial}{\partial x_k} & \left[ H_i U_k - U_i H_k + h_i u_k - u_i h_k \right] + \lambda \nabla^2 H_i,
\frac{\partial U_i}{\partial x_i} = \frac{\partial H_i}{\partial x_i} = 0,
\end{align*}$$

for the mean fields, and subtracting these from Eqs. (1)-(3), we obtain
for the fluctuating fields. Especially, if both \( U \) and \( H \) are steady, uniform and moreover the turbulence is spatially homogeneous, the averaged equation (7.2.7) is satisfied identically and it is readily seen that in Eq. (7.2.8) the terms in the curly brackets vanish.
Now the sets of thus simplified equations at two independent points \( x \) and \( x' = x + r \) are combined to get the two-point correlations equations.

Following the similar procedure to that used in the theory of ordinary homogeneous turbulence, we arrive at

\[
\frac{\partial u_i u_j}{\partial t} + H_k \frac{\partial}{\partial \tau_k} (h_i u_j - u_i h_j) = \frac{\partial}{\partial \tau_k} (u_i u_k h_j - u_i u_k u_j + u_i h_k h_j - h_i h_k u_j) \\
+ \frac{1}{\rho} \left( \frac{\partial P_{u_j}}{\partial \tau_i} - \frac{\partial P_{u_j^*}}{\partial \tau_j} \right) + 2\nu V^2 u_i u_j, \tag{7.2.10}
\]

\[
\frac{\partial h_i h_j}{\partial t} + H_k \frac{\partial}{\partial \tau_k} (h_i h_j - h_i u_j) = \frac{\partial}{\partial \tau_k} (h_i u_k h_j - h_i h_k h_j + h_i h_k u_j) \\
- u_i h_k h_j + 2\lambda V^2 h_i h_j + \frac{\partial}{\partial \tau_k} h_i h_j + H_k \frac{\partial}{\partial \tau_k} (u_i u_k h_j - u_i u_k u_j) \\ 
\tag{7.2.11}
\]

\[
\frac{\partial u_i h_j}{\partial t} + H_k \frac{\partial}{\partial \tau_k} (h_i h_j - u_i h_j) = \frac{\partial}{\partial \tau_k} (u_i u_k h_j - u_i u_k h_j + u_i h_k h_j - h_i h_k u_j) \\
+ \frac{1}{\rho} \frac{\partial}{\partial \tau_i} + (\nu + \lambda) V^2 u_i h_j, \tag{7.2.12}
\]

\[
\frac{\partial h_i u_j}{\partial t} + H_k \frac{\partial}{\partial \tau_k} (u_i u_j - h_i h_j) = \frac{\partial}{\partial \tau_k} (h_i u_k u_j - u_i u_k u_j + h_i h_k h_j - h_i h_k u_j) \\
- \frac{1}{\rho} \frac{\partial}{\partial \tau_i} + (\nu + \lambda) V^2 h_i u_j, \tag{7.2.13}
\]
where

\[ P_0 + \rho + \frac{\nu}{2}(h^2 + 2H_k h_k). \]  

(7.2.14)

and a prime indicates the value at \( X' \). Eqs (7.2.10 – 7.2.13) agree with those derived by Deissler except for the Hall current terms with which we are not concerned here.

7.3 The Case of a Very Strong Imposed Field:

In order to estimate the order of magnitude of various terms in the correlation equations obtained above, we introduce the characteristic length \( L \) and the level of the turbulence \( a = [u^2 + h^2 / 3]^{1/2} \). We have then

\[
\begin{align*}
\text{[triple correlation terms]} & \sim \frac{a^3}{|1|} = \frac{a}{H} (= \text{say}), \\
\text{[external coupling terms]} & \sim \frac{Ha^2}{|1|} = \frac{H}{R_H}, \\
\text{[viscous dissipation terms]} & \sim \frac{va^2}{|1|} = \frac{v}{H}, \\
\text{[magnetic dissipation terms]} & \sim \frac{\lambda a^2}{|1|} = \frac{\lambda}{R_{mH}},
\end{align*}
\]

(7.3.1)

where \( R_n \) and \( R_{mH} \) stand for the Reynolds number and the magnetic Reynolds number with respect to the mean Alfven speed \( H \). If, therefore, the imposed magnetic fields is sufficiently strong, \( \epsilon, 1/R_n \) and \( 1/R_{mH} \) are small in comparison with 1, and hence Equation 7.3.10 becomes

\[
\frac{\partial u_i u_j'}{\partial t} + H_k \frac{\partial}{\partial x_k} (h_i u_j' - u_i h_j') = \frac{1}{\rho} \left( \frac{\partial P u_i'}{\partial x_i} - \frac{\partial u_i P_u}{\partial x_j} \right)
\]

(7.3.2)
to the first approximation. But, by virtue of the solenoidal relations (7.2.9), the divergence of Equation 7.3.2 gives

$$\nabla^2 P^*_u = \nabla^2 u^*_j P^* = 0.$$ 

whose analytic solutions in an unbounded domain are the form

$$P^*_u u^*_j = H_j(t) \quad \text{and} \quad u^*_j P^* = \tilde{H}_j(t),$$

respectively. Accordingly, the right-hand side of Equation 7.3.2 is identically zero. By the similar reason, we can neglect the right-hand sides of Equations 7.2.11 – 7.2.13 as well in the present approximation.

The Fourier transform of these simplified correlation equations are expressed in terms of the spectrum tensors such that

$$\begin{aligned}
\tilde{u}_i u_j &= \int \Phi_{ij}(x,t) \ell^{2r} dk, \\
\tilde{h}_i u_j &= \int \Psi_{ij}(x,t) \ell^{2r} dk, \\
\tilde{u}_i u_j &= \int \Gamma_{ij}(x,t) \ell^{2r} dk, \\
\tilde{h}_i u_j &= \int \Gamma_{ij}(x,t) \ell^{2r} dk.
\end{aligned}$$

(7.3.3)

It should be noted that $\phi_{ij}$ and $\psi_{ij}$ are true tensors but $\Gamma_{ij}$ and $\overline{\Gamma}_{ij}$ are skew tensors, and that

$$\begin{aligned}
\Phi_{ij}(k) &= \Phi_{ji}(-k), \\
\Psi_{ij}(-k) &= \overline{\Gamma}_{ij}(-k) \\
\overline{\Gamma}_{ij}(k) &= \overline{\Gamma}_{ji}(-k) \\
\Gamma_{ij}(k) &= \Gamma_{ji}(-k)
\end{aligned}$$

(7.3.4)

from homogeneity,

$$\begin{aligned}
k_i \Phi_{ij}(k) &= k_i \Psi_{ij}(k) = k_i \Gamma_{ij}(k) = k_i \overline{\Gamma}_{ij}(k) = 0, \\
k_j \Phi_{ij}(k) &= k_j \Psi_{ij}(k) = k_j \Gamma_{ij}(k) = k_j \overline{\Gamma}_{ij}(k) = 0
\end{aligned}$$

(7.3.5)

from solenoidality.
The spectrum equations in the present context then become

\begin{align*}
\phi_{ij}(k, t) + i k \mu H [\Gamma_{ij}(k, t) - \Gamma_{ij}(k, t)] &= 0, \\
\psi_{ij}(k, t) - i k \mu H [\Gamma_{ij}(k, t) - \Gamma_{ij}(k, t)] &= 0,
\end{align*}

(7.3.6)

where a dot means \( \partial / \partial t \) and \( \mu \) denotes hereafter the cosine of the angle between \( z \) and \( H \), i.e., \( k \mu H = k \cdot H_k \). From Equation 7.3.6 it follows immediately that

\begin{align*}
\Phi_{ij}(k, t) + \Psi_{ij}(k, t) &= \text{invariant,} \\
\text{and } \Gamma_{ij}(k, t) + \Gamma^0_{ij}(k, t) &= \text{invariant,}
\end{align*}

(7.3.7)

which are the natural consequences of the linear conservative property of our present system.

Now, let us suppose that an external uniform field \( H \) is suddenly imposed at \( t=0 \). Then appropriate solutions of Eqs. 7.3.6 are readily found to be

\begin{align*}
\phi_{ij}(k, t) &= \frac{1}{2} [i \Phi_{ij}^{(0)} (1 + \cos 2 k \mu H t) + \Psi_{ij}^{(0)} (1 - \cos 2 k \mu H t) + i (\Gamma_{ij}^{(0)} + \Gamma^0_{ij}) \sin 2 k v H t], \\
\psi_{ij}(k, t) &= \frac{1}{2} [i \Phi_{ij}^{(0)} (1 - \cos 2 k \mu H t) + \Psi_{ij}^{(0)} (1 + \cos 2 k \mu H t) - i (\Gamma_{ij}^{(0)} - \Gamma^0_{ij}) \sin 2 k v H t],
\end{align*}

(7.3.8)

where \( \Phi_{ij}^{(0)} = \Phi_{ij}(k, 0) \), etc. A remarkable feature of these solutions is their oscillatory nature. Such oscillations are caused and maintained by the imposed magnetic field which plays an analogous role to the primary field of a
conventional electric dynamo or motor. Eqs 7.3.8 give the velocity and magnetic covariances in spectral form.

### 7.4 Axisymmetric case

The solutions (7.3.4) imply that if the turbulence is initially axisymmetric about the direction of \( H \) it keeps axisymmetry for \( t>0 \). Then, according to the invariant theory of axisymmetric turbulence 7.2.11, we can put

\[
\begin{align*}
\Phi_{ij}(k,t) &= \begin{pmatrix} \psi^{(1)} & \phi^{(2)} \end{pmatrix} \\
\psi_{ij}(k,t) &= \begin{pmatrix} \psi^{(1)} & \phi^{(2)} \end{pmatrix} \\
\Gamma_{ij}(k,t) &= \begin{pmatrix} \gamma^{(1)} & \gamma^{(2)} \end{pmatrix} \\
\bar{\Gamma}_{ij}(k,t) &= \begin{pmatrix} -\gamma^{(1)} & -\gamma^{(2)} \end{pmatrix}
\end{align*}
\]

where \( s \) is a unit vector in the direction of \( H \), and

\[
\begin{align*}
\nabla_{ij}(k) &= (k^2 \delta_{ij} - k_i k_j) / k^2 : \delta_{ij} \text{ is Kronecker's symbol,} \\
\Theta_{ij}(s : k) &= [k^2 (1 - \mu^2) \delta_{ij} - k_i k_j - k^2 s_i s_j + k \mu (s_i k_j + k_i s_j)] / k^2
\end{align*}
\]

while the defining scalars \( \phi^{(2)} \), \( \gamma^{(2)} \) are functions of \( k \) and \( k \mu \) as well as time \( t \). It follows from the homogeneity conditions (7.3.4) that

\[
\phi^{(1,2)}(k, k \mu) = \phi^{(1,2)}(k, -k \mu), \quad \phi^{(1,2)}(k, k \mu) = \phi^{(1,2)}(k, -k \mu),
\]

for true tensors, and

\[
\gamma^{(1,2)}(k, k \mu) = \gamma^{(1,2)}(k, -k \mu),
\]

for skew tensors. Under these conditions, the solutions (7.3.6) are easily transformed into a scalar form. The result is simply to replace \( \Phi_{ij}(k,t) \) by the corresponding scalars \( \phi^{(1,2)}(k,k \mu,t) \) and so on.
Now, we are particularly interested in the amounts of turbulent energy associated with the mechanical and magnetic modes. In this connection we introduce, as in the theory of ordinary axisymmetric turbulence, the energy spectrum functions such that

\[
\begin{align*}
\frac{1}{2} u_{11}^2(t) &= \int_0^\infty E_{11}(k, t) dk, \\
\frac{1}{2} u_{1}^2(t) &= \int_0^\infty E_{1}(k, t) dk, \\
\frac{1}{2} h_{11}^2(t) &= \int_0^\infty F_{11}(k, t) dk, \\
\frac{1}{2} h_{1}^2(t) &= \int_0^\infty F_{1}(k, t) dk,
\end{align*}
\]

(7.4.5)

for which

\[
\begin{align*}
E_{11}(k, t) &= 2\pi k^2 <(1-\mu^2)\phi^{(1)} >_\mu, \\
E_{1}(k, t) &= \pi k^2 <(1+\mu^2)\phi^{(1)} +(1-\mu^2)\phi^{(2)} >_\mu, \\
F_{11}(k, t) &= 2\pi k^2 <(1-\mu^2)\phi^{(1)} >_\mu, \\
F_{1}(k, t) &= \pi k^2 <(1+\mu^2)\phi^{(1)} +(1-\mu^2)\phi^{(2)} >_\mu,
\end{align*}
\]

(7.4.6)

where symbols \( \perp \perp \) and \( \perp \) denote the direction parallel and perpendicular to the imposed magnetic field, and

\[
<...>_\mu = \frac{1}{2} \int_{-1}^1 (...) d\mu.
\]

(7.4.7)

If the initial profiles of \( \phi(u),...,\gamma^{(2)} \) are given, the subsequent changes in the amount of energy are calculated by 7.4.8, 7.4.6 and 7.4.7.

As a simple special case we assume that turbulence is initially isotropic. This condition means that
\[ \begin{align*}
\psi_0^{(1)} &= \frac{3}{4\pi k^2} E_0(k), \\
\phi_0^{(1)} &= \frac{3}{4\pi k^2} F_0(k), \\
\gamma_0^{(1)} &= \frac{3}{4\pi k^2} G_0(k), \\
\tilde{\gamma}_0^{(1)} &= \frac{3}{4\pi k^2} \tilde{G}_0(k),
\end{align*} \]

with
\[ \psi_0^{(2)} = \phi_0^{(2)} = \gamma_0^{(2)} = \tilde{\gamma}_0^{(2)} = 0. \]

(7.4.8)

where subscript 0 denotes the initial data, and \( E_0(k) = E_{\perp}(k,0) = E_{\perp}(k,0), \)
\( F_0(k) = F_{\perp}(k) = F_{\perp}(k,0). \) Using 7.4.8 in 7.3.8 and 7.4.6.

\[ \begin{align*}
E_{\perp}(k,t) &= \frac{1}{2} (E_0 + F_0) + \frac{3}{4} (E_0 - F_0) Q_{\perp}, \\
E_{\perp}(k,t) &= \frac{1}{2} (E_0 + F_0) + \frac{3}{4} (E_0 - F_0) Q_{\perp}, \\
E_{\perp}(k,t) &= \frac{1}{2} (E_0 + F_0) - \frac{3}{4} (E_0 - F_0) Q_{\perp}, \\
E_{\perp}(k,t) &= \frac{1}{2} (E_0 + F_0) - \frac{3}{4} (E_0 - F_0) Q_{\perp}.
\end{align*} \]

(7.4.9)

where \( Q_{\perp} \) and \( Q_{\perp} \) given by

\[ \begin{align*}
Q_{\perp} &= Q_{\perp} (k\tau) = \frac{\sin 2k\tau}{2k^2 H^2 t^2} + \frac{\sin 2k\tau}{4k^3 H^3 t^3}, \\
Q_{\perp} &= Q_{\perp} (k\tau) = \frac{\sin 2k\tau}{2kH^2 t^2} + \frac{\cos 2k\tau}{4k^2 H^2 t^2} - \frac{\sin 2k\tau}{8k^3 H^3 t^3}.
\end{align*} \]

(7.4.10)

7.5 Result:-

The equations 7.3.8 give the expressions for velocity and magnetic field covariances. The spectral equations are discussed for axisymmetric case. At this stage numerical solutions of the equations are required. But due to great complexity involved at this state it is not possible.