CHAPTER – VI

RATE OF CHANGE OF VORTICITY COVARIANCE OF DUSTY FLUID TURBULENCE IN THE PRESENCE OF MAGNETIC FIELD IN A ROTATING SYSTEM
6.1 INTRODUCTION

The main feature of turbulent flow is that turbulent fluctuations are random in nature. Taylor (1935) studied the correlation coefficients between two fluctuating quantities in turbulent flow. With the help of the correlation function, the statistical property of the random variable may be described. Batchelor (1951), Jain (1962), Kishore and Sinha (1987), Kishore and Sarkar (1989) considered acceleration covariance and vorticity covariance in hydrodynamic turbulence and hydrodynamic dusty turbulence. Dixit (1989) discussed the rate of change of vorticity covariance considering the Beltrami flow field in the presence of a magnetic field. In this chapter, we consider the turbulent flow in a rotating system. When the motion is referred to axes which rotate steadily with the bulk of the fluid, the coriolis and centrifugal force must be supposed to act on the fluid. The coriolis force due to rotation plays an important role in a rotating system of turbulent flow, while the centrifugal force with the potential is incorporated into pressure. The main aim of the present chapter is to derive an equation for the rate of change of vorticity covariance in a rotating system.
6.2 Discussion of the Problem

The vorticity equation for dusty fluid turbulence is given by (cf. Dixit, 1990)

\[
\frac{\partial \omega_i}{\partial t} = \nu \frac{\partial^2 \omega_i}{\partial x_k \partial x_k} + \frac{\partial h_i}{\partial t} + f (W_i - \omega_i) + 2(u_k \frac{\partial \Omega_i}{\partial x_k} - \Omega_k \frac{\partial u_i}{\partial x_k})
\]  

(2.1)

where

\[ u_i = \text{ith component of velocity fluctuation} \]
\[ v_i = \text{ith component of velocity of dust particles} \]
\[ \omega_i = \text{ith component of vorticity fluctuation of vorticity vector} \]
\[ W_i = \text{component of vorticity of dust particles} \]
\[ \Omega_i = \text{ith component of angular velocity vector}. \]

Here we have assumed that the dust particles be non-conducting i.e.

\[ h_i' v_j = h_j' v_i = 0 \] and instantaneous velocities at one point remain unaffected by dust particles of the other point, i.e. \( u_i u_j = u_j' v_i = 0 \). A similar equation for the ith component becomes

\[
\frac{\partial w_i'}{\partial t} = \nu \frac{\partial^2 w_i'}{\partial x_k \partial x_k} + \frac{\partial h_i'}{\partial t} + f (W_i' - \omega_i') + 2(u_k \frac{\partial \Omega_i'}{\partial x_k} - \Omega_k \frac{\partial u_i'}{\partial x_k})
\]  

(6.2.2)

If we multiply equation (6.2.1) by \( \omega_j' \) and (6.2.2) by \( \omega_i \), then adding and after taking ensemble average, we have

\[
\frac{\partial}{\partial t} \langle \omega_i \omega_j' \rangle = 2\nu \frac{\partial^2}{\partial x_k \partial x_k} \langle \omega_i \omega_j' \rangle + \frac{\partial}{\partial t} \langle h_i w_j' \rangle + \frac{\partial}{\partial t} \langle \omega_i h_j' \rangle
\]
\begin{align*}
+ f \left( \frac{\partial u_k}{\partial x_k} \Omega_k \omega_j - \frac{\partial}{\partial x_k} u_i \Omega_k \omega_j + \frac{\partial}{\partial x_k} \omega_i u_k \Omega_j \right) & \\
+ 2 \left[ \frac{\partial u_k}{\partial x_k} \Omega_k \omega_j - \frac{\partial}{\partial x_k} u_k \Omega_j \omega_j + \frac{\partial}{\partial x_k} (u_k \Omega_j \omega_i) - u_j \Omega_k \omega_i \right] & \\
(6.2.3)
\end{align*}

Let $\xi_k = x_k' - x_k$ and

$$\frac{\partial}{\partial \xi_k} = -\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_k}$$

If we use the condition of homogeneity, equation (6.2.3) becomes

$$\frac{\partial}{\partial t} \omega_i \omega_j = 2\nu \frac{\partial^2}{\partial \xi_k \partial \xi_k} \omega_i \omega_j + \frac{\partial}{\partial t} h_i \omega_j + \frac{\partial}{\partial t} \omega_i h_j$$

$$+ f \left( \frac{\partial}{\partial \xi_k} \Omega_j \omega_j - 2 \frac{\partial}{\partial \xi_k} \omega_i \omega_j + \frac{\partial}{\partial \xi_k} \omega_i \Omega_j \right)$$

$$+ 2 \left[ \frac{\partial}{\partial \xi_k} u_k \Omega_k \omega_j - \frac{\partial}{\partial \xi_k} u_k \Omega_j \omega_i + \frac{\partial}{\partial \xi_k} (u_k \Omega_j \omega_i) - u_j \Omega_k \omega_i \right]$$

(6.2.4)

Following Chandrasekhar (1955), we put

$$L_{ij} = \omega_i \omega_j, \quad h_i \omega_j = Q_{ij}, \quad \omega_i h_j = T_{ij},$$

$$W_i \omega_j = P_{ij}, \quad \omega_i \omega_j = R_{ij}$$

$$(u_i \Omega_k - \Omega_i u_k) \omega_j = S_{ik,j}$$

$$(u_k \Omega_j - \Omega_k u_j) \omega_i = F_{k,j,i}$$

(6.2.5)

If we substitute the above relations in equation (6.2.4), we get
\[
\frac{\partial}{\partial t} \omega_i \omega_j = 2v \frac{\partial^2}{\partial \xi_k \partial \xi_k} L_{ij} + \frac{\partial}{\partial t} Q_{ij} + \frac{\partial}{\partial t} T_{ij} + f(P_{ij} - 2L_{ij} + R_{ij}) + 2 \left[ \frac{\partial}{\partial \xi_k} S_{ik,j} + \frac{\partial}{\partial \xi_k} F_{k,i,j} \right]
\]

(6.2.6)

The linear \( L_{ij} \) is solenoidal in its indices; therefore, it can be expressed as

\[
L_{ij} = \frac{L'}{\gamma} \xi_i \xi_j - (rL' + 2L) \delta_{ij}
\]

(6.2.7)

where \( r = |\xi_k| \) and \( L(r,t) \) is the defining scalar of \( L_{ij} \). The primes attached to the scalar function \( L \) denote differentiation with respect to \( r \).

\[
\frac{\partial^2 L_{ij}}{\partial \xi_k \partial \xi_k} = \left( \frac{L'''}{r} + \frac{4L''}{r^2} - \frac{4L'}{r^3} \right) \xi_i \xi_j - (rL'' + 6L' + \frac{4L'}{r} \delta_{ij}
\]

(6.2.8)

\[
P_{ij} = \frac{P'}{r} \xi_i \xi_j - (rP' + 2P) \delta_{ij}
\]

(6.2.9)

\[
R_{ij} = \frac{R'}{r} \xi_i \xi_j - (rR' + 2R) \delta_{ij}
\]

(6.2.10)

Let

\[
\omega_i \omega_j = \alpha(r,t) \xi_i \xi_j + \beta(r,t) \delta_{ij} + \eta(r,t) \epsilon_{ij \ell} \xi_{\ell}
\]

(6.2.10a)

\[
\omega_i h_j = \gamma(r,t) \xi_i \xi_j + \delta(r,t) \delta_{ij}
\]

\[
h_i \omega_j = \phi(r,t) \xi_i \xi_j + \psi(r,t) \delta_{ij}
\]

where \( \alpha(r,t), \beta(r,t), \delta(r,t), \phi(r,t), \psi(r,t) \) are defining scalars.

Also
\[ \frac{\partial}{\partial \xi_k} S_{i k, j} = \frac{s'}{r} \xi_i \xi_j - (rs' + 2s) \delta_{ij} \quad (6.2.11) \]

Let

\[ F_{k, j, i} = (2F + rF') \xi_{i j k} - \frac{F'}{r} \xi_i \xi_j \xi_k \]

and

\[ \frac{\partial}{\partial \xi_k} F_{k, j, i} = \left( F'' + \frac{4F'}{r} \right) \xi_{i j \ell} \xi_{\ell} \quad (6.2.11) \]

If we put these expressions in (6.2.6), we get

\[ \frac{\partial}{\partial t} \omega_i \omega_j = \left[ 2v \left( \frac{L'''}{r} + \frac{4}{r^2} L'' - \frac{4}{r^3} L' \right) \right. \]
\[ + \frac{f}{r} \left( P' - 2L' + R' \right) + \frac{2s'}{r} \xi_i \xi_j \]
\[ - \left[ 2v \left( rL'''' + 6L'' + \frac{4L'}{r} \right) + f \left( P' + 2L' - R' \right) \right. \]
\[ - 2f \left( P + 2L + R \right) + 2(rs' + 2s) \right] \delta_{ij} \]
\[ + \frac{\partial}{\partial t} Q_{i j} + \frac{\partial}{\partial t} T_{i j} + \left[ F'' + \frac{4}{r} P' \right] \xi_{i j \ell} \xi_{\ell} \quad (6.2.12) \]

Thus

\[ \frac{\partial}{\partial t} \alpha(r, t) = 2v \left( \frac{L'''}{r} + \frac{4}{r^2} L'' - \frac{4L'}{r^3} \right) \]
\[ + f(P' - 2L' + R') + \frac{2s'}{r} + \frac{\partial}{\partial t} \gamma(r, t) + \frac{\partial}{\partial t} \phi(r, t) \quad (6.2.13) \]
\[ \frac{\partial}{\partial t} \beta(r, t) = -2v \left( rL'''' + 6L'' + \frac{4}{r} L' \right) \]
Thus, with the help of the above three independent scalars, equations (6.2.13), (6.2.14) and (6.2.15) the rate of change of vorticity covariance of dusty turbulence in a rotating system can be determined by equation (6.2.10a).

6.3 Concluding Remarks

The dynamical effects of the rotation are contained in the coriolis force and this is not related to the location of the rotation axis, only to the magnitude and direction of the angular velocity vector. When the Rossby number \( R = \frac{U}{\Omega L} \) is small, the fluid is almost in a state of rigid rotation with the container. Again when the Rossby number is very large, there may be similarities with the case of the non rotating container.

If the system is non rotating, the coriolis force will be neglected and, we get

\[
\frac{\partial}{\partial t} \alpha (r, t) = 2 \nu \left( \frac{L'''}{r^3} + \frac{4}{r^2} L'' - \frac{4L'}{r^3} \right) + f (P' - 2L' + R')
\]
\[ + \frac{\partial}{\partial t} \gamma(r, t) \] 

\[ \frac{\partial}{\partial t} \beta(r, t) = -2 \gamma(r L'' + 6 L'' + \frac{4}{r} L') \]

\[ + f r (P' + 2 L' - R') \]

\[ + 2 f (P + 2 L + R) \]

\[ + \frac{\partial}{\partial t} \delta(r, t) + \frac{\partial}{\partial t} \psi(r, t) \] 

\[ \frac{\partial}{\partial t} \eta(r, t) = 0 \]

which gives the same results given by Dixit (1990).