# CHAPTER - III

## CONSTANT DECELERATION PARAMETER IN MORDTVEDT'S THEORY

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### 3.1 Introduction:

The success of the inflationary scenario has made it an important milestone in the evolution of the early universe. The phenomenon of cosmological inflation requires a finite period of accelerated expansion in the early universe. Several variants of the original inflationary scenario given by Guth (1981) now exist.

La and Steinhardt (1989) proposed a new type of inflationary scenario based on BD theory of gravity known as extended inflationary scenario. Unlike previous inflationary models based on general theory of relativity, here the inflationary phase transition can be completed via bubble hucleation. It provides a new solution to the graceful exit problem. Hence, the fine tuning of an effective potential to obtain a slow-roll over transition is not required. But the original idea of extended inflation has its own problems.

It the BD parameter w is chosen to satisfy constraints from light-deflection and time-delay experiments ($w > 500$) Reasenberg el.al. (1989), the original model leads to excessive distortions in the cosmic microwave background radiation. Weinberg (1989) obtained upper bounds on the BD parameter ($w < 76$) by requiring that the recovery from the supercooled regime be such that the presently observed universe might emerge. La et.al. (1989) discussed a number of astrophysical and cosmological constraints ($w < 25$) that significantly limit the types of workable theories and the allowed range of physical parameters. A successful inflationary scenario would require a value of $w < 25$ in the early universe. Several means have been devised in order to circumvent this problem.

In order to reconcile the notion of extended inflaiton with limits obtained from measurements of the microwave background anisotropies, several authors have considered extending the gravitational action beyond the BD actiion functional used in the original extended inflation model. The first method of evading the microwave bounds was to introduce a potential term V(0) for the BD field La et.al., Accetta et.al. (1989). This model, however, has a drawback since it requires the BD potential to be very flat so tht it does not dominate over false-vacuum energy and disrupt the first-order inflation Adams et.al. (1991). Another alternative invokes different couplings of the BD field to visible and invisible matter Holman et al. (1990) though these have recently been more tightly constrained Damour, Gundlach (1991). The most popular alternative is to make further modifications to the gravitational action leading to models refererred to as hyperextended inflation Steinhardt, Accetta, (1990), Littl, Wands (1992). Barrow and Maeda (1990) have found examples of a new type of inflation intermediate between the power-law and exponential types. They assumed w as a function of the scalar field Ø which increases during the evolution of the universe. The functional dependence of w was taken to be

$$w(\emptyset) = w_m \emptyset^m + w_o ,$$

where $w_m$ and $w_o$ are dimensionless positive constants and the power index m is a real number. Bellido and Quiros

(1990) proposed a solution to extended inflation by choosing w as

$$2w(\emptyset) + 3 = 2\beta(1-\emptyset/\emptyset_c)^{-\alpha}$$

where $\emptyset_c \simeq M_c^2$ with $M_c$ a mass scale close to the Planck mass $M_{pl}$ today.

The most general example of scalar-tensor theories is the Bergmann-Wagoner theory (1968, 1970), Nordtvedt (1970) in which the coupling of the scalar field $\emptyset$ to the curvature and the cosmological term both depend upon $\emptyset$. This class of theories encompasses other scalar-tensor theories such as Brans-Dicke theory and Nordtvedt's, theory and can be shown to be a low-energy limit of certain superstring theories Green et.al. (1988). Here since the coupling parameter $w = w(\emptyset)$ evolves with time, w could be small initially ($w < 25$) and it could increase with the evolution of the universe to a large value ($w > 500$) at the present epoch. Hence, general scalar-tensor theories like the Nordtvedt's theory, which overcome the problem inherent in working with the BD theory have caught the fascination of cosmologists Burd and Coley, (1991), Banerjee et.al. (1993).

It is worthwhile to observe that most of the well-known models of Einstein's theory and Brans-Dicke theory with curvature parameter k = 0, including inflationary models, are models with constant deceleration parameter. The models with constant deceleration parameter form an interesting class of models in cosmology. This has provided us the motivation to study models with constant deceleration parameter in Nordtvedt's theory. Moreover, the number of equations in Nordtvedt's theory is less than the number of unknowns. Hence, an additional assumption in the form of a constant deceleration parameter would suffice to make the system of equations well defined and obtain a unique solution.

The field equations of Nordtvedt's theory are given. Section 3 deals with the general solution of Nordtvedt's theory with constant deceleration parameter.

## 3.2 Field equations of Nordtvedt's theory:

The gravitational field equations of Nordtvedt's theory are given by

$$(3.1) \qquad G_{ab} = -\frac{8\pi}{\phi} T_{ab} - \frac{w}{\phi^2}\left[\phi_{;a}\phi_{;b} - \frac{1}{2} g_{ab}\phi_{;c}\phi^{;c}\right] - \frac{1}{\phi}\left[\phi_{;a;b} - g_{ab}\,\Box^2\phi\right]$$

$$(3.2) \qquad \Box^2 \phi = \frac{1}{3+2w} \; [8\pi T^a{}_{,a} - w' \phi_{;c}\phi^{;c}] ,$$

where $w' \equiv dw/d\phi$ and $T_{ab}$, the energy-momentum tensor for a perfect fluid, is given by

$$(3.3) \qquad T_{ab} = (\rho + p)\, u_a u_b - p g_{ab}.$$

We now restrict our attention to space-times which are homogeneous, isotropic and flat, for which the line element is just the FRW line element ( with k = 0)

$$(3.4) \qquad ds^2 = dt^2 - R^2(t)\, [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]$$

where R(t) is the scale factor.

The field equations (3.1) and (3.2) with the above metric and the barotropic equation of state

$$(3.5) \qquad p = \gamma\rho , \qquad 0 \le \gamma \le 1$$

now become

$$(3.6) \qquad 3\left(\frac{\dot{R}}{R}\right)^2 + 3\,\frac{\dot{R}}{R}\,\frac{\dot{\phi}}{\phi} - \frac{w}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 = \frac{8\pi\rho}{\phi} ,$$

$$(3.7) \qquad 2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{\ddot{\phi}}{\phi} + \frac{w}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + 2\,\frac{\dot{R}}{R}\,\frac{\dot{\phi}}{\phi} = -\frac{8\pi\gamma\rho}{\phi},$$

$$(3.8) \qquad \ddot{\phi} + \frac{\dot{R}}{R}\dot{\phi} = \frac{8}{3+2w}\,(1-3\gamma)\,\rho - \frac{w}{3+2w}\,\dot{\phi}^2 .$$

Equations (3.6-3.8) lead to the continuity equation,

$$(3.9) \qquad \dot{\rho} + 3\,(1+\gamma)\,\rho H = 0$$

where $H = (\dot{R}/R)$ is the Hubble's function.

Here there are only three independent equations in four unknowns viz., R, $\phi$, $\rho$ and w. Therefore, one more relation is necessary to obtain a unique solution, for which we assume the deceleration parameter to be constant.

Equation (3.9) on integration yields

$$(3.10) \qquad \rho = \rho_0\, R^{-3(1+\gamma)},$$

## 3.3 **Exact solutions with constant deceleration Parameter:**

Consider a model with constant deceleration parameter, i.e., (3.11) $q = -(R\ddot{R})/(\dot{R})^2 = \beta$ (constant).

Equation (3.11) can be wrewritten as

(3.12) $$\frac{\ddot{R}}{R} + \beta\left(\frac{\dot{R}}{R}\right)^2 = 0 \quad .$$

On integration the above equation gives the exact solution

(3.13a), (3.13b)
$$R(t) = \begin{cases} (A+Bt)^{1/(1+\beta)}, & \beta \neq -1 \\ C \exp(Xt), & \beta = -1 \end{cases} \quad ,$$

where A, B, C and X are constants of integration. Also, when $\beta = -1$, H = X = constant.

The above solutions may be relevant during different eras of evolution in Nordtvedt's theory. For example, we can assume the power-law expansion following the big bang, culminating in exponential expansion during inflationary era and reverting to power-law expansion again after exit from inflation.

For singular models (big bang cosmology with $R(0) = 0$), the expression (3.13a) for the scale factor may be rewritten as

$$(3.14) \qquad R(t) \sim t^{1/(1+\beta)}, \qquad \beta \neq -1$$

but during the course of evolution, for instance during inflation,

$$(3.15) \qquad R(t) \sim \exp(Xt) .$$

Now equation (3.10) together with (3.14) yields

$$(3.16) \qquad \rho = \rho_o t^{-3(1+\gamma)/(1+\beta)} ,$$

Equation (3.6) coupled with (3.7) gives

$$(3.17) \qquad 2\frac{\ddot{R}}{R}+4\left(\frac{\dot{R}}{R}\right)^2 + \frac{\ddot{\phi}}{\phi} + 5\frac{\dot{R}}{R}\frac{\dot{\phi}}{\phi} = \frac{8\pi(1-\gamma)\rho}{\phi} .$$

Case (i), $\beta \neq -1$

Using (3.14) and (3.16) in (3.17) we have

$$(3.18) \qquad t^2\ddot{\phi} + \frac{5t}{(1+\beta)}\dot{\phi} + \frac{2(2-\beta)}{(1+\beta)^2}\phi = 8\pi\rho_o t^{2-[3(1+\gamma)]/(1+\beta)}$$

Equation (3.18) can be readily integrated to give

$$(3.19) \qquad \phi = \frac{8\pi(1-\gamma)\,\rho_o(1+\beta)^2}{[2\beta+1-3\gamma][\beta+1-3\gamma]}\,t^{2-[3(1+\gamma)]/(1+\beta)} + a_2 t^{(\beta-2)/(1+\beta)}$$

$a_1$ and $a_2$ being constants of integration.

Using the expressions for $R(t)$, $\rho(t)$ and $\phi(t)$ from (3.14), (3.16) and (3.19) in (3.6) we get $w(t)$ as

$$(3.20) \qquad \frac{w}{6}(1+\beta)^2[\dot{\phi}]^2 = [A_1(t)A_2(t) + A_3(t)A_4(t) - A_5(t)A_6(t)] ,$$

where

$$[\dot{\phi}]^2(1+\beta)^2 = \{(2\beta-1-3\gamma)Kt^{1-[3(1+\gamma)/(1+\beta)} - 2a_1 t^{-2(2+\beta)/(1+\beta)} + (\beta-2)a_2 t^{-3/(1+\beta)}\}^2$$

$$A_1(t) = \{2Kt^{1-[3(1+\gamma)/(1+\beta)]} + a_1 t^{-(3+\beta)/(1+\beta)}\}$$

$$A_2(t) = \{K\,[(2\beta-3\gamma) - \frac{(2\beta+1-3\gamma)(1+\beta-3\gamma)}{3(1-\gamma)}[$$

$$t^{1-[3(1+\gamma)/(1+\beta)]} - \frac{a_1}{2}t^{-(3+\beta)/(1+\beta)}\}$$

$$A_3\ (t) = \{\frac{a_1}{2}\ t^{-(3+\beta)/(1+\beta)} + a_2 t^{-3/(1+\beta)]}\}$$

$$A_4(t) = \{2\ (\beta-1)a_2 t^{-[3/(1+\beta)]} - a_1 t^{-(3+\beta)/(1+\beta)}\}$$

$$A_5(t) = \{K\,[(2\beta-3\gamma) - \frac{(2\beta+1-3\gamma)(1+\beta-3\gamma)}{3(1-\gamma)}]$$

$$t^{1-[3(1+\gamma)/(1+\beta)]} - (\beta-1)a_2 t^{-[3/(1+\beta)]}\}$$

$$A_6(t) = \{Kt^{1-[3(1+\gamma)/(1+\beta)]} - a_2 t^{-[3/(1+\beta)]}\}$$

with

$$K = \frac{8\pi\ (1-\gamma)\ \rho_0 (1+\beta)^2}{(2\beta+1-3\gamma)\ (1+\beta-3\gamma)} . \tag{3.21}$$

When $a_1 = a_2 = 0$, the above solution for $\phi$ reduces to that in BD theory and the coupling parameter w become invariant with respect to time, also (3.20) reduces to Johri, Kalyani (1993)

(3.22) $$w = \frac{2(1-\beta)}{(1-\gamma)(2\beta-1-3\gamma)}$$

From expression (3.20), it might look formidable to study the behaviour of w. But as shown below the expression for w reduces to simpler forms for two different ranges of the deceleration parameter $\beta$ . In the following analysis, we consider the relative contributions of the second term with co-efficient $a_1$ and the third term with co-efficient $a_2$ in (3.19). The term with dominant contribution is retained while the other term is dropped.

Case (i) (a) $-1 < \beta < 1$ .

It is evident, by comparing the exponents of the second and third terms in equation (3.19), that for values of lying in the range $-1 < \beta < 0$ (accelerated expansion), the third term in (3.19) decreases faster with time than the second term and can therefore be neglected for large times. Further, for values of in the range $0 < \beta < 1$ (decelerated expansion), the exponent of the third term in (3.19) is comparable to that of the second term. Therefore, any one of the two terms would suffice. Hence, without loss of generality, we have retained the second term in (3.19). Therefore, for $-1 < \beta < 1$ we have

$$(3.23) \qquad \phi \sim Kt^{(2\beta-1-3\gamma)/(1+\beta)} + a_1 t^{-2/(1+\beta)}.$$

Neglecting $a_2$ in (3.20), we find

$$(3.24) \qquad \frac{w}{6} = \frac{B_1(t)B_2(t)}{[B_3(t)]^2} ,$$

where

$$B_1(t) = \{K + a_1 t^{-(2\beta+1-3\gamma)/(1+\beta)}\}$$

$$B_2(t) = \{K\,[(2\beta-3\gamma) - \frac{(2\beta+1-3\gamma)(1+\beta-3\gamma)}{3(1-\gamma)}$$

$$-a_1 t^{-(2\beta+1-3\gamma)/(1+\beta)}\} ,$$

$$B_3(t) = \{K(2\beta-1-3\gamma) - 2a_1 t^{-(2\beta+1-3\gamma)/(1+\beta)}\}$$

The first derivative of w with respect to time is given by

$$\dot{w} = \frac{C_1(t)C_2(t)}{[C_3(t)]^3} , \tag{3.25}$$

where

$$C_1(t) = 4 \rho_o a_1 (1+\beta)(2\beta+1-3\gamma)$$

$$t^{-(3\beta+2-3\gamma)/(1+\beta)} ,$$

$$C_2(t) = \{K - \frac{(2\beta-1-3\gamma)}{2(1+\beta-3\gamma)} + a_1 t^{-(2\beta+1-3\gamma)/(1+\beta)}\}.$$

$$C_3(t) = \{\frac{K}{2}(2\beta-1-3\gamma) - a_1 t^{-(2\beta+1-3\gamma)/(1+\beta)}\}.$$

The solutions of Banerjee et al (1993) can be obtained as a particular case by substituting $\beta = -(1/2)$ and $\gamma = -1$ in (3.23) and (3.24).

For small values of time a similar analysis can be done. It can be readily seen that for small times the third term in (3.19) is the dominant contributor. Hence, in the analysis the third is to be retained.

Case (i) (b) $\beta > 1$

Here, the second term decreases faster with time than the third term in (3.19) and hence can be neglected for large times. Therefore, we ahve

(3.26) $$\emptyset \quad Kt^{(2\beta-1-3\gamma)/(1+\beta)} + a_2 t^{(\beta-2)/(1+\beta)}.$$

Putting $a_1 = 0$ in (3.20), we get

(3.27) $$\frac{w}{6} = \frac{D_1(t)D_2(t)}{[D_3(t)]^2}$$

where

$$D_1(t) = \{K + a_2 t^{-(\beta+1-3\gamma)/(1+\beta)} \quad ,$$

$$D_2(t) = \{K \quad [(2\beta-3\gamma) - \frac{(2\beta+1-3\gamma)(1+\beta-3\gamma)}{3(1-\gamma)}$$

$$\times (\beta-1)a_2 t^{-(\beta+1-3\gamma)/(1+\beta)}\}$$

$$D_3(t) = \{K(2\beta-1-3\gamma) + (\beta-2)a_2 t^{(\beta+1-3\gamma)/(1+\beta)}\}.$$

Here $\dot{w}$ is given by

$$(3.28) \qquad \dot{w} = \frac{E_1(t)}{[E_2(t)]^3}$$

where

$$E_1(t) = -4\pi \rho_o a_2^2 \frac{(1+\beta)(1+\beta-3\gamma)(2\beta-1-3\gamma)}{(2\beta+1-3\gamma)} t^{-1},$$

$$E_2(t) = \{\frac{K}{2}(2\beta-1-3\gamma) + (\beta-2)\frac{a_2}{2} t^{-(\beta+1-3\gamma)/(1+\beta)}\}.$$

It can be easily observed that for small times the second term of (3.19) is to be retained and a similar analysis can be carried out.

It can be seen from the above expressions that the behaviour of $\phi$ and w are determined by various parameters such as $\gamma$, $\beta$, $a_1$ and $a_2$. The signature of various quantities like $\beta$, $a_1$, $a_2$, $(2\beta+1-3\gamma)$, $(1+\beta-3\gamma)$, $[3(1-\gamma)(2\beta-3\gamma)-(2\beta+1-3\gamma)(1+\beta-3\gamma)]$ should be considered while studying the evolution of $\phi$ and w. Also the signature of the exponents of t in the above expressions plays a crucial role in determining the behaviour of these quantities. As such, there is a variety of models wherein w increases or decreases with time. There also exists models in which w decreases (or increases) up to a certain time and thereafter increases (or decreases). Moreover, for w to be positive, the products of the terms in the numerator of (3.24) or (3.27) should be positive. The first derivative of w with respect to time is useful in determining the behaviour of w. As such we restrict our attention to that particular class of models for which w increases with time.

Consider $a_1 < 0$ and $a_2 = 0$ with

$$(2\beta+1-3\gamma) > 0 \quad ,$$

$$(3.29) \qquad (\beta+1-3\gamma) < 0 \quad ,$$

$$(2\beta-1-3\gamma) < 0 \quad ,$$

then $\beta$ would lie in the range

$$(3.30) \qquad \frac{3\gamma-1}{2} < \beta < 3\gamma-1$$

with $\frac{1}{3} < \gamma < 1$. Therefore, from (3.25) it can be seen that $C_1(t) < 0$, $C_2(t) < 0$ and $C_3(t) > 0$, $\forall t$. Hence, $\dot{w} > 0$ i.e. w increases with time.

Case ii: $\beta = -1$

From equation (3.13) we see that the scale factor is given by

$$(3.31) \qquad R(t) = Ce^{xt}.$$

Using this in (3.10) we obtain

$$(3.32) \qquad \rho = \rho_o \, e^{-3(1+\gamma)xt}$$

Now using (3.31) and (3.32) in (3.17) we have

$$(3.33) \qquad \ddot{\phi} + 5X\dot{\phi} + 6X^2 \phi = 8\pi(1-\gamma) \rho_o e^{-3(1+\gamma)xt}.$$

The above equation on integration yields

$$(3.34) \qquad \phi = \frac{8\pi(1-\gamma)\rho_o}{3X^2\gamma(1+3\gamma)} e^{-3(1+\gamma)xt} + Ae^{-2xt} + Be^{-3xt},$$

where A and B are constants of integration.

Using (3.31), (3.32) and (3.34) in (3.6), w is given by

$$\frac{w}{6}[\dot{\phi}]^2 = -X^2[X_1(t)X_2(t) + X_3(t)+X_5(t)X_6(t)]^2, \tag{3.35}$$

where

$$[\dot{\phi}]^2 = X^2\{3(1+\gamma)K'e^{-3(1+\gamma)xt} + 2Aa^{-2xt} + 3Be^{-3xt}\}$$

$$X_1(t) = \{2K'e^{-3(1+\gamma)xt} + Ae^{-2xt}\} ,$$

$$X_2(t) = \frac{K'(1+\gamma)(3\gamma-1)}{(1-\gamma)} e^{-3(1+\gamma)xt} + \frac{A}{2}e^{-2xt} ,$$

$$X_3(t) = \frac{A}{2}e^{-2xt} + Be^{-3xt} ,$$

$$X_4(t) = \{Ae^{-2xt} + 4Be^{-3xt}\} ,$$

$$X_5(t) = \{2Be^{-3xt} - \frac{K'(1+\gamma)(3\gamma-1)}{(1-\gamma)}e^{-3(1+\gamma)xt}\},$$

$$X_6(t) = \{K'e^{-3(1+\gamma)xt} + Be^{-3xt}\} ,$$

with

$$K' = \frac{8\pi(1-\gamma)\rho_o}{3X^3\gamma(1+3\gamma)} .$$

It would be very difficult to analyse the general behaviour of w from the above expression. But a simpler form for w can be obtained as shown below.

The third term in (3.34) decreases faster than the second term and therefore it can be neglected. Hence,

$$\phi \sim K' e^{-3(1+\gamma)xt} + Ae^{-2xt} \tag{3.36}$$

Therefore, the expression for w simplifies to

$$\frac{w}{6} = - \frac{Y_1(t) Y_2(t)}{[Y_3(t)]^2} \tag{3.37}$$

where

$$Y_1(t) = \{K' + Ae^{(1+3\gamma)xt}\}$$

$$Y_2(t) = \frac{K'(1+\gamma)(3\gamma-1)}{(1-\gamma)} + Ae^{(1+3\gamma)xt}$$

$$Y_3(t) = \{3K'(1+\gamma) + 2Ae^{(1+3\gamma)xt}\}.$$

When $\gamma = -1$, the above solutions (3.36) and (3.37) reduce to the solutions obtained by Banerjeet et al (1993).

## 3.4 Concluding Remarks:

In this chapter, we have obtained exact solutions of the field equations of Nordtvedt's theory for constant deceleration parameter and $k = 0$. We have considered only singular solutions with (i) power-law, (ii) exponential expansion and have examined the particular class of models in which the BD parameter w might increase with time. It is found that there exists a variety of models in which w might increase initially and decrease subsequently and vice versa.

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