C H A P T E R - I V

ACCELERATED EXPANSION OF THE UNIVERSE

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4.1 Introduction:

The stress tensor T^a_b for any source term in a Friedmann universe, described by an expansion factor a(t), must have the form $T^a_b(t) = \text{dia}\,[\rho(t), -p(t), -p(t), -p(t)]$. Given an equation of state which specifies p as a function of ρ, we will be left with two degrees of freedom a(t) and $\rho(t)$ which can be determined by two independent Einstein's equations for the Friedmann model. The situation, however, is slightly different if the source is described by an "adjustable function" as in the case of scalar fields. If, for example, the source is described by a scalar field with a Lagrangian $L = (1/2)\,\partial_a\phi\,\partial^a\phi - V(\phi)$, then it is possible to choose $V(\phi)$ in order to have a specific evolution for the universe. Given any $a(t)$ it is always possible to obtain a $V(\phi)$ such that it results in a consistent cosmological evolution. In fact, this can be achieved even in the presence of other energy densities in the universe (like matter, radiation etc.) in addition to the scalar field. This should not be surprising, since the existence of a free function $V(\phi)$ allows a trade off with another function $a(t)$.

Recently, it has been suggested that the tachyonic condensate in a class of string theories can be described by an effective scalar field with a Lagrangian of the form $L = - V(\phi)[1-\partial_a\phi\,\partial^a\phi]^{\frac{1}{2}}$. The evolution of this condensate can have cosmological significance which may be worth exploring. Since this Lagrangian also has a potential function $V(\phi)$, it seems reasonable to expect that any form of cosmological evolution (that is, any $a(t)$) can be obtained with the tachyonic field as the source by choosing $V(\phi)$ "suitably". It turns out that this is indeed true.

I will outline a recipe for constructing $V(\phi)$ given a particular form of evolution for the universe $a(t)$ in the two cases (normal scalar field and tachyonic field) mentioned above. The first case corresponds to quintessence/dark energy models and has been a favourite pastime of the cosmologists in the last several years Ratra et.al. (1988). In this case, there is very little (independent) constraint on $V(\phi)$ and hence it is not possible to evaluate the relative merits of different choices for $V(\phi)$. In the case of tachyonic scalar field, there are some constraints on the form of $V(\phi)$, especially on the asymptotic behaviour, which could rule out certain class of cosmological models. It is nevertheless possible to construct several interesting models satisfying the asymptotic constraints on $V(\phi)$. In particular, it may be possible to have a rapidly accelerated phase of expansion for the universe at late times which seems to have some observational support.

4.2 Recipe for the Scalar Field Potential:

Consider a k = 0 universe with a normal scalar field having a potential $V(\phi)$ as the source. We assume that the evolution of the universe is already specified so that a(t), $H(t) \equiv (\dot{a}/a)$ .... etc. are known functions of time and we need to determine $V(\phi)$ such that Friedmann equations

$$(4.1) \qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho; \quad \left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3}(\rho + 3p) \quad ,$$

as well as the equation of motion for the scalar field

(4.2) $$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}$$

are satisfied. (Of course, only two of these three equations are independent when the universe is driven by a single source). In a Friedmann universe, $\phi(t,x) = \phi(t)$ and the energy density and pressure of the scalar field is given by

(4.3) $$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi); \quad P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

It is convenient to define a time dependent parameter $w(t)$ by the relation $w(t) \equiv P_\phi(t)/\rho_\phi(t)$. The equation of motion for the scalar field, written in the form $d(\rho a^3) = -w\rho d(a^3)$ can be integrated to give $(\dot{\rho}_\phi/\rho_\phi) = -3H(1+w)$. The Friedmann equation, on the other hand, gives $\rho_\phi \propto H^2$ so that $(\dot{\rho}_\phi/\rho_\phi) = 2(\dot{H}/H)$. Combining the two relations we get

(4.4) $$1+w(t) = -\frac{2}{3}\frac{\dot{H}}{H^2} ,$$

thereby determining $w(t)$. (Note that we have not used the specific form of the source so far; so, this equation will be satisfied by any source in a FRW model.) From the definition of w and (4.3), it follows that $\dot{\phi}^2/2V = (1+w)(1-w)^{-1} \equiv f(t)$, say. Writing this as $\dot{\phi}^2 = 2fV$, differentiating with respect to time and using (4.2) we fined that

$$(4.5) \qquad \frac{\dot{V}}{V} = -\frac{\dot{f}+6H\ f}{1+f} \quad ,$$

Integrating this equation and using the definition of f (t) and equation (4.4) we get

$$(4.6) \qquad V(t) = \frac{3H^2}{8\pi G}\left[1 + \frac{\dot{H}}{3H^2}\right] \quad .$$

Substituting back in the relation $\dot{\phi}^2 = 2fV$, we can determine $\phi(t)$ to be

$$(4.7) \qquad \phi(t) = \int dt \quad \left[-\frac{\dot{H}}{4\pi G}\right]^{\frac{1}{2}} \quad .$$

Equations (4.6) and (4.7) completely solve the problem of finding a potential $V(\phi)$ which will lead to a given $a(t)$. These equations determine $\phi(t)$ and $V(t)$ in terms of $a(t)$ thereby implicitly determining $V(\phi)$.

In fact, the same method works even when matter other than scalar field with some known energy density known(t) present in the universe. In this case, equations (4.6) and (4.7) generalizes to

$$(4.8) \quad V(t) = \frac{1}{16\pi G} H(1-Q) \quad [6H + \frac{2\dot{H}}{H} - \frac{\dot{Q}}{1-Q}] \quad ,$$

$$(4.9) \quad \phi(t) = \int dt \, [\frac{H(1-Q)}{8\pi G}]^{\frac{1}{2}} \quad [\frac{\dot{Q}}{1-Q} - \frac{2\dot{H}}{H}]^{\frac{1}{2}}$$

where $Q(t) \equiv [8\pi G \rho_{known}(t)/3H^2(t)]$.

As an example of using (4.6) and (4.7), let us consider a universe in which $a(t) = a_o t^n$. Elementary algebra now gives the potential to be of the form

$$(4.10) \quad V(\phi) = V_o \exp \quad [- \sqrt{\frac{2}{n}} \frac{\phi}{M_{pl}})$$

where V_o and n are constants and $M_{pl}^2 = 1/8\pi G$. The corresponding evolution of $\phi(t)$ is given by

$$(4.11) \quad \frac{\phi(t)}{M_{pl}} = \sqrt{2n} \, \ln \left(\sqrt{\frac{V_o}{n(3n-1)}} \frac{t}{M_{pl}} \right) .$$

As a second example, consider an evolution of the form

$$(4.12)\quad a(t) \propto \exp(\alpha t^f), \quad f = \frac{\beta}{4+\beta}, \quad 0 < f < 1, \ \alpha > 0$$

In this case, we can determine the potential to be

$$(4.13)\quad V(\phi) \propto \left(\frac{\phi}{M_{pl}}\right)^{-\beta} \left(1 - \frac{\beta^2}{6}\frac{M_{pl}^2}{\phi^2}\right) ,$$

where β is a constant. The two potentials described bove have been used extensively in inflationary models.

In fact, virtually all other potentials used in quintessence/dark energy models for the universe can be obtained by the recipe given above. Since one seldom worries seriously about the microscopic origin of $V(\phi)$ in these models, it may be mathematically more convenient to choose one's favourite cosmological evolution in terms of $a(t)$ and then construct $V(\phi)$ and study its properties.

4.3 Recipe for Tachyonic Potential:

Consider next a universe with a given $a(t)$ and a tachyonic source with the Lagrangian $L = - V(\phi)$

$[1 - \partial_a \phi \, \partial^a \phi]^{1/2}$. When $\phi = \phi(t)$, the energy density and pressure are given by

$$(4.14) \quad \rho = \frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}} \quad ; \quad p = -V\sqrt{1-\dot{\phi}^2}$$

In this case, the "reverse engineering" to determine $V(\phi)$ from $a(t)$ is almost trivial. For any source with a parameter $w(t)$, we must have

$$(4.15) \quad \frac{\dot{\rho}}{\rho} = -3H(t)\,(1+w) = \frac{2\dot{H}}{H} \quad ,$$

leading to (4.4). On the other hand, for the tachyonic model, $p(t)/\rho(t) \equiv w(t) = \dot{\phi}^2 - 1$. Combining these, we can determine $\dot{\phi}^2 = -(2/3)\,(\dot{H}/H^2) = (2/3)(dH^{-1}/dt)$ in terms of H and obtain

$$(4.16) \quad \phi(t) = \int dt \, \left(\frac{2}{3}\,\frac{\dot{H}}{H^2}\right)^{1/2} \quad .$$

Multiplying the two equations in (4.4) and using (4.4) and the Friedman equation, we get

$$(4.17) \quad V = (-w)^{1/2}\,\rho = \frac{3H^2}{8\pi G}\,\left(1 + \frac{2}{3}\,\frac{\dot{H}}{H^2}\right)^{1/2} \quad .$$

Equations (4.16) and (4.17) completely solve the problem. Given any $a(t)$, these equations determine $V(t)$ and $\phi(t)$ and thus the potential $V(\phi)$. Equation (4.16) also implies that $H < 0$ for these models.

Note the similarity between the forms of $V(\phi)$ in (4.17) and (4.6). The fact that both tachyonic and normal scalar field potentials can be used to drive the expansion of the universe, suggests that- as far as cosmological evolution is concerned- there exists a mapping between the two potentials directly. For example, we found that an exponentially decaying potential for the normal scalar field leads to power law growth for $a(t)$. I will now show that a tachyonic potential of the form $V(\phi) \propto \phi^{-2}$ will lead to the same kind of cosmological evolution.

Consider a universe with power law expansion $a = t^n$. In this case, $(\dot{H}/H^2)$ in equation (4.16) is a constant making $\dot{\phi}$ a constant. The complete solutioin is given by

$$\phi(t) = \left(\frac{2}{3n}\right)^{1/2} t + \phi_0; \quad V(t) = \frac{3n^2}{8\pi G}\left(1 - \frac{2}{3n}\right)^{1/2} \frac{1}{t^2}, \tag{4.18}$$

where $n > (2/3)$. [I will comment on the $n = (2/3)$ case later on.] Combining the two, we find the potential to be

(4.19) $$V(\phi) = \frac{n}{4\pi G}\left(1-\frac{2}{3n}\right)^{\frac{1}{2}} (\phi-\phi_0)^{-2} .$$

For such a potential, it is possible to have arbitrarily rapid expansion with large n. [It is also possible to reproduce the evolution in (4.12) by a more complicated choice of the potential, of the form $V \propto \phi^{x}\,(1+c_1\phi^{y})^{\frac{1}{2}}$. The reverse-engineering procedure is exactly the same].]

If $\phi_a\phi^a << 1$ the tachyonic Lagrangian can be approximated by the form

(4.20) $$L = -V(\phi)\,[1-\partial_a\phi\,\partial^a\phi]^{\frac{1}{2}} \approx \frac{1}{2}\psi_a\psi^a - U(\psi) ,$$

with

(4.21) $$\psi = \int\sqrt{V(\phi)}\,d\phi; \quad U(\psi) = V[\phi(\psi)] .$$

In our case, $V(\phi) = A/\phi^2$ (say), giving $\psi = \sqrt{A}\ln\phi$ and $U(\psi) \propto \exp(-2\psi/\sqrt{A})$. Curiously, this has the same form of the potential we found in (4.10) for a normal scalar field to produce the $a = t^n$ evolution though the coefficient of ψ matches with that in (4.10) only when n >> 1. In this limit, the mapping in (4.20) becomes increasingly accurate.

The potential in (4.19) has the reasonable behaviour of V --> 0 as ϕ --> ∞ though its form for small and intermediate values of ϕ is not supported by string theory. It is however possible to show that the asymptotic form of the evolution is still given by the solutions found above. To see this, assume that the late time behaviour of ϕ is given by

$$\phi(t) = \left(\frac{2}{3n}\right)^{\frac{1}{2}} t + Be^{-Ct} = \left(\frac{2}{3n}\right)^{\frac{1}{2}} t + O(e^{-Ct}) \quad , \tag{4.22}$$

where n, B, C are constants with (2/3) $\leq$ n. This implies that we take $\phi(t)$ to grow proportional to t asymptotically with exponentially small corrections. In this case, it is possible to repeat the above analysis and show that asymptotically (at late times), we have the following behaviour :

$$a(t) \approx t^n \exp\left[O\left(\frac{e^{-Ct}}{t}\right)\right] \quad , \tag{4.23}$$

and

$$V(\phi) \approx \frac{n}{4\pi G} \left(1 - \frac{2}{3n}\right)^{\frac{1}{2}} \frac{1}{\phi^2} \left[1 + O\left(\frac{e^{-Ct}}{t}\right)\right] \quad . \tag{4.24}$$

The asymptotic form for a(t) is essentially a power law found before with exponentially small corrections. The time scale for the validity of asymptotic solution is determined by (1/C) which is a parameter that can be fixed independent of n. This will allow one to ignore exponentially subdominant terms even when large values of n are invoked.

4.4 **Concluding Remarks:**

This analysis assumes that the asymptotic "velocity" of the scalar field is given by $\dot{\phi}(\infty) = (2/3n)^{\frac{1}{2}}$ if the expansion of the universe at late times is given by $a(t) = t^n$. This, in turn, requires the tachyonic potential $V(\phi)$ to have the asymptotic form A/ϕ^2 without any restriction on A. To have accelerated expansion, we must have reasonably large n requiring $\dot{\phi}(\infty)$ to be sufficiently less than unity- which is possible if A is non zero and arbitrary.

The situation is different if there are string theoretic reasons to expect $\dot{\phi}(\infty) = 1$ asymptotically. Since equation (4.16) can be equivalently written as

$$H^{-1}(t) = \frac{3}{2} \int \dot{\phi}^2 \, dt \quad , \tag{4.25}$$

$\dot{\phi}(\infty) = 1$ asymptotically will imply $a(t) \propto t^{2/3}$. Then we must have n = (2/3) which is just a dust dominated, zero pressure, expansion law. The cosmological implications are very different depending on whether string theory requires the condition $\dot{\phi}(\infty) = 1$ asymptotically or not. Let me discuss the different possibilities briefly.

If it is possible to have $\dot{\phi}(\infty) \neq 1$ asymptotically, then an attractive scenario emerges in which the tachyonic condensate could be driving the acceleration of the universe and mimicking the cosmological constant. In this case, one is interpreting the asymptotic evolution to be applicable during the current phase of the universe, in the range of redshifts, say, $0 < z < 3$. In a realistic model, one may need to worry about the timescales in string theory (presumably $\approx t_{\mathrm{Planck}}$) vis-a-vis the cosmological timescale. But this issue will arise in any attempt to use the tachyonic condensate in the late phases in the evolution of the universe, like-for example- as some kind of dark matter.

An easy way out is to use the solutions in the early universe. The solutions I have obtained, of course, could also be used to provide a power law inflation in the very early universe if $t \to \infty$ is interpreted merely as $t > t_{Planck}$. It may be easier to have non vanishing pressure for the tachyonic condensate for $t > t_{Planck}$ than during the current epochs of the universe. (I think it would be nicer if string theory could provide an effective cosmological constant at the current epoch- which has some observational support- rather than merely provide yet another inflation field. But the procedure outlined here can be used to construct a wide clas of inflationary solutions as well.)

If, on the other hand, string theory demands $\dot{\phi}(\infty) = 1$ asymptotically, then $V \to 0$ asymptotically in such a manner as to give finite ρ and zero pressure. The tachyonic condensate can then contribute, say, $\Omega = 4.7$ in the universe and- together with clustered normal matter contributing $\Omega = 0.3$ - can lead to a $\Omega = 1$ universe. Such a model is an extreme form of mixed dark matter model with a very smoothly distributed component at large scales. It is possible that the model is consistent with CMBR and galaxy clustering data but will contradict the supernova data if it is interpreted as indicating $\ddot{a} > 0$ in the recent past. (Because $\ddot{a} < 0$ when $n > 1$, this model will not be accelerating.) Given the observational uncertainties, it may still be worth studying the consequences of such a model.

Even in this case [with $\dot{\phi}(\infty) = 1$ asymptotically], it may be possible to provide an accelerated expansion for the universe in the recent past if one considers more complicated potentials. For example, one can arrange matters such that $n >> 1$ in the redshift range of $3 > z > 0$, say, with the asymptotic regime of $n = (2/3)$ coming into effect only in the future. The procedure developed in this chapter can be used to obtain suitable $V(\phi)$ which will ensure such a scenario.

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