Chapter 5

LRS Bianchi Type II String
Cosmological Models for Perfect
Fluid Distribution in General
Relativity

The present study deals with locally rotationally symmetric (LRS) Bianchi type II string cosmological models with perfect fluid distribution of matter. In present study we have considered two cases (i) \( \rho + \lambda = 0 \) and (ii) \( \rho - \lambda = 0 \), where \( \rho \) and \( \lambda \) are the rest energy density and the tension density of the string cloud respectively. The physical behaviour of these models are also discussed.
5.1 Introduction

In recent years, there has been considerable interest in string cosmology. Cosmic strings are topologically stable objects which might be found during a phase transition in the early universe (Kibble [1]). Cosmic strings play an important role in the study of the early universe. These arise during the phase transition after the big bang explosion as the temperature goes down below some critical temperature as predicted by grand unified theories (Zel'dovich et al. [2]; Kibble [1, 3]; Everett [4]; Vilenkin [5]). It is believed that cosmic strings give rise to density perturbations which lead to the formation of galaxies (Zel'dovich [6]). These cosmic strings have stress-energy and couple to the gravitational field. Therefore it is interesting to study the gravitational effects that arise from strings. The pioneering work in the formulation of the energy-momentum tensor for classical massive strings was done by Letelier [7] who considered the massive strings to be formed by geometric strings with particle attached along its extension. Letelier [8] first used this idea in obtaining cosmological solutions in Bianchi I and Kantowski-Sachs space-times. Stachel [9] has studied massive string.

Bali et al. [10 - 16] have obtained Bianchi types I, III and IX string cosmological models in general relativity. Yadav et al. [17] have studied some Bianchi type I viscous fluid string cosmological models with magnetic field. Recently Wang [18 - 21] has also discussed LRS Bianchi type I and Bianchi type III cosmological models for a cloud string with bulk viscosity. Recently Yadav, Pradhan and Rai [22] have obtained the integrability of cosmic string in Bianchi type III space-time in presence of bulk viscous fluid by applying a new technique.

The present day universe is satisfactorily described by homogeneous and isotropic models given by the FRW space-time. The universe in a smaller scale is neither homogeneous nor isotropic nor do we expect the Universe in its early stages to have these properties. Homogeneous and anisotropic cosmological models have been widely studied in the
framework of General Relativity in the search of a realistic picture of the universe in its early stages. Although these are more restricted than the inhomogeneous models which explain a number of observed phenomena quite satisfactorily. Bianchi type-II space-time has a fundamental role in constructing cosmological models suitable for describing the early stages of evolution of universe. Asseo and Sol [23] emphasized the importance of Bianchi type-II universe. A spatially homogeneous Bianchi model necessarily has a three-dimensional group, which acts simply transitively on space-like three-dimensional orbits. Here we confine ourselves to a locally rotationally symmetric (LRS) model of Bianchi type-II. This model is characterized by three metric functions $R_1(t), R_2(t)$ and $R_3(t)$ such that $R_1 = R_2 \neq R_3$. The metric functions are functions of time only. (For non-LRS Bianchi metrics we have $R_1 \neq R_2 \neq R_3$). For LRS Bianchi type-II metric, Einstein’s field equations reduce, in the case of perfect fluid distribution of matter, to three nonlinear differential equations.

Roy and Banerjee [24] have dealt with LRS cosmological models of Bianchi type-II representing clouds of geometrical as well as massive strings. Recently, Wang [25] studied the Letelier model in the context of LRS Bianchi type-II space-time. In this Letter, we have investigated LRS Bianchi type-II string cosmological models for perfect fluid distribution under two conditions (i) $p + \lambda = 0$ and (ii) $p = \lambda$.

### 5.2 The Metric and Field Equations

We consider the LRS Bianchi type II metric in the form

$$ds^2 = -dt^2 + (Bdx + Bzdy)^2 + A^2dy^2 + A^2dz^2, \quad (5.1)$$

where $A$ and $B$ are functions of $t$ only. The energy momentum tensor for a cloud of string with perfect fluid distribution is taken as

$$T_{ij} = (p + \rho)v_iv_j + pg_{ij} - \lambda x_ix_j, \quad (5.2)$$
where \( u_i \) and \( x_i \) satisfy condition
\[
v_i^j u_i = -x_i^j x_i = -1, \quad v_i^j x_i = 0, \tag{5.3}
\]
p is the isotropic pressure, \( \rho \) is the proper energy density for a cloud string with particles attached to them, \( \lambda \) is the string tension density, \( v^i \) the four-velocity of the particles, and \( x^i \) is a unit space-like vector representing the direction of string. In a co-moving co-ordinate system, we have
\[
v^i = (0, 0, 0, 1), \quad x^i = \left( \frac{1}{B}, 0, 0, 0 \right). \tag{5.4}
\]
If the particle density of the configuration is denoted by \( \rho_p \), then we have
\[
\rho = \rho_p + \lambda. \tag{5.5}
\]
The Einstein’s field equations
\[
R^i_j - \frac{1}{2} R g^i_j = -T^i_j, \tag{5.6}
\]
for the line-element (5.1) lead to
\[
2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} - \frac{3 B^2}{4 A^4} = p - \lambda, \tag{5.7}
\]
\[
\frac{\ddot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{A} \dot{B}}{AB} + \frac{1 B^2}{4 A^4} = p, \tag{5.8}
\]
\[
2\frac{\dot{A} \dot{B}}{AB} + \frac{\dot{A}^2}{A^2} - \frac{1 B^2}{4 A^4} = -\rho, \tag{5.9}
\]
where an overdot stands for the first and double overdot for the second derivative with respect to \( t \). The particle density \( \rho_p \) is given by
\[
\rho_p = \frac{\ddot{A}}{A} - \frac{\dot{B}}{B} - \frac{3 \dot{A} \dot{B}}{AB} - \frac{1 B^2}{4 A^4}. \tag{5.10}
\]

5.3 Solutions of the Field Equations

The field equations (5.7)-(5.9) are a system of three equations with five unknown parameters \( A, B, p, \rho \) and \( \lambda \). Two additional constraints relating these parameters are required
to obtain explicit solutions of the system. We assume that the expansion (θ) in the model is proportional to the shear (σ). This condition leads to

\[ A = B^m \]  

(5.11)

where \( m \) is proportionality constant. Eqs. (5.7)-(5.9) lead to

\[ \frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{A}{B} \left( \frac{\ddot{B}}{B} + \frac{2\dot{A}}{A} \right) - \frac{5B^2}{4A^2} = - (\lambda + \rho). \]  

(5.12)

From Eqs. (5.11) and (5.12), we obtain

\[ (m - 1) \frac{\dot{B}}{B} + 3m^2 \frac{\dot{B}^2}{B^2} = \frac{5}{4} B^{(2-4m)} - (\lambda + \rho). \]  

(5.13)

In order to overcome the under determinacy occurred here because of five unknowns involved in three independent field equations, we consider the following two cases:

(I) \( \rho + \lambda = 0 \), i.e. the sum of rest energy density and tension density for cloud of strings vanish (Reddy [26, 27], Mohanty et al. [28]) and

(II) \( \rho_p = 0 \). This corresponds to the state equation for a cloud of massless geometric (Nambu) strings given by \( \rho = \lambda \).

### 5.3.1 Case I:

In this case

\[ \lambda + \rho = 0. \]  

(5.14)

From (5.13) and (5.14), we obtain

\[ 2\dot{B} + \left( \frac{6m^2}{m - 1} \right) \frac{\dot{B}^2}{B} = \frac{5}{2(m - 1)} B^{(3-4m)}. \]  

(5.15)

Let \( \dot{B} = f(B) \) which implies that \( \ddot{B} = f' \), where \( f' = \frac{df}{dB} \).

\[ \frac{d(f^2)}{dB} + \frac{6m^2}{m - 1} \frac{f^2}{B} = \frac{5}{2(m - 1)} B^{(3-4m)}. \]  

(5.16)

Eq. (5.16), after integration, reduces to

\[ \frac{dB}{dt} = \sqrt{\frac{5B^{4(1-m)}}{4(m^2 + 4m - 2)} + c_1 B^{\frac{8m^2}{(1-m)}}}, \]  

(5.17)
where $c_1$ is an integrating constant. Hence the metric (5.1) reduces to

$$
\begin{align*}
\text{ds}^2 &= - \left[ \frac{dB^2}{\frac{5B^4(1-m)}{4(m^2+4m-2)} + c_1B^{\frac{6m}{1-m}}} \right] + B^2dx^2 \\
&\quad + [B^2z^2 + B^{2m}]dy^2 + 2B^2z\,dx\,dy + B^{2m}dz^2.
\end{align*}
$$

(5.18)

After using suitable transformation of coordinates i.e. ($B = \tau$), metric (5.18) reduces to

$$
\begin{align*}
\text{ds}^2 &= - \left[ \frac{d\tau^2}{\frac{5\tau^4(1-m)}{4(m^2+4m-2)} + c_1\tau^{\frac{6m}{1-m}}} \right] + \tau^2dx^2 \\
&\quad + [\tau^2z^2 + \tau^{2m}]dy^2 + 2\tau^2z\,dx\,dy + \tau^{2m}dz^2.
\end{align*}
$$

(5.19)

The pressure ($p$), the energy density ($\rho$), the string tension ($\lambda$), the particle density ($\rho_p$), the scalar of expansion ($\theta$), the shear ($\sigma$) and the proper volume ($V^3$) for the model (5.19) are given by

$$
p = \frac{(4m^2 + m - 3)}{(m^2 + 4m - 2)} \tau^{2(1-2m)} + c_1\tau^{\frac{6m}{1-m}} \left[ \frac{1}{\tau^2} + \frac{m^2(1 + m)}{(1 - m)} \right],
$$

(5.20)

$$
\rho = -\lambda = \frac{(3m^2 - 3m - 1)}{2(m^2 + 4m - 2)} \tau^{2(1-2m)} + c_1\tau^{\frac{2(2m^2 + m - 1)}{(1-m)}},
$$

(5.21)

$$
\rho_p = \frac{(3m^2 - 3m - 1)}{(m^2 + 4m - 2)} \tau^{2(1-2m)} + 2c_1\tau^{\frac{2(2m^2 + m - 1)}{(1-m)}},
$$

(5.22)

$$
\theta = \frac{(2m + 1)}{\tau} \left[ \frac{5}{4(m^2 + 4m - 2)} \tau^{4(1-m)} + c_1\tau^{\frac{m^2}{(1-m)}} \right]^\frac{1}{2},
$$

(5.23)

$$
\sigma^2 = \frac{(m - 1)^2}{3\tau^2} \left[ \frac{5}{4(m^2 + 4m - 2)} \tau^{4(1-m)} + c_1\tau^{\frac{m^2}{(1-m)}} \right],
$$

(5.24)

$$
V^3 = \tau^{(2m+1)}.
$$

(5.25)

From Eqs. (5.23) and (5.24), we obtain

$$
\frac{\sigma^2}{\theta} = \frac{(m - 1)^2}{3(2m + 1)^2} = \text{constant}.
$$

(5.26)

If we choose the suitable value of constant $m$ i.e. $m \geq \frac{4}{3}$, we find that the energy conditions $\rho \geq 0$, $\rho_p \geq 0$ are satisfied. The model (5.19) starts with a big bang at $\tau = 0$. The expansion in the model decreases as time increases.
stops at \( \tau = \infty \). When \( \tau \to 0 \) then \( \rho \to \infty, \lambda \to \infty \). When \( \tau \to \infty \) then \( \rho \to 0, \lambda \to 0 \).

Also \( p \to \infty \) when \( \tau \to 0 \) and \( p \to 0 \) when \( \tau \to \infty \). Since \( \frac{\gamma}{\theta} = \text{constant} \), hence the model does not approach isotropy in general.

According to Refs. [1, 29], when \( \rho_p / | \lambda | > 1 \), in the process of evolution, the universe is dominated by massive strings, and when \( \rho_p / | \lambda | < 1 \), the universe is dominated by the strings. In this case from Eqs. (5.21) and (5.22), we obtain

\[
\frac{\rho_p}{| \lambda |} = 2 > 1. \tag{5.27}
\]

Thus, in our model, the universe is dominated by massive strings throughout the whole process of evolution.

### 5.3.2 Case II:

In this case

\[ \lambda - \rho = 0. \tag{5.28} \]

Eqs. (5.7)-(5.9) lead to

\[
\frac{\dddot{B}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{A}B}{AB} - \frac{3B^2}{4A^4} = \rho - \lambda. \tag{5.29}
\]

From Eqs. (5.28) and (5.29), we obtain

\[
\frac{\dddot{B}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{A}B}{AB} - \frac{3B^2}{4A^4} = 0. \tag{5.30}
\]

Using (5.11) in (5.30) leads to

\[
2\ddot{B} + \left(\frac{2m(m-2)}{m-1}\right) \frac{\dot{B}^2}{B} = \frac{3}{2(m-1)} B^{3-4m}. \tag{5.31}
\]

Let \( \dot{B} = f(B) \) which implies that \( \ddot{B} = ff' \), where \( f' = \frac{df}{dB} \). Hence (5.31) reduces to

\[
\frac{d}{dB}(f^2) + \frac{2m(m-2)}{(m-1)} \frac{f^2}{B} = \frac{3}{2(m-1)} B^{3-4m}. \tag{5.32}
\]
5.3 Solutions of the Field Equations

Eq. (5.32), after integration, reduces to

\[
\frac{dB}{dt} = \sqrt{-\frac{3}{4(m^2 + 2)}} B^{4(1-m)} + c_2 B^{2m(4-m)}.
\]

(5.33)

where \( c_2 \) is an integrating constant.

After using suitable transformation of coordinates i.e \( (B = \tau) \), metric (5.1) reduces to

\[
ds^2 = -\left[ \frac{d\tau^2}{3 \frac{4}{(m^2 + 2)}} \tau^{4(1-m)} + c_2 \tau^{2m(4-m)} \right] + \tau^2 dx^2 + [\tau^2 z^2 + \tau^{2m}] dy^2 + 2\tau^2 z dx dy + \tau^{2m} dz^2.
\]

(5.34)

The pressure \( (p) \), the energy density \( (\rho) \), the string tension \( (\lambda) \), the particle density \( (\rho_p) \), the scalar of expansion \( (\theta) \), the shear \( (\sigma) \) and the proper volume \( (V^3) \) for the model (5.34) are given by

\[
p = \frac{(m^2 - 1)}{(m^2 + 2)} \tau^{2(1-2m)} - c_2 m \tau^{2(m^2 - 3m - 1)}.
\]

(5.35)

\[
\rho = \lambda = \frac{(2m^2 + 3m + 1)}{2(m^2 + 2)} \tau^{2(1-2m)} - m(m + 2) c_2 \tau^{2(m^2 - 3m - 1)}.
\]

(5.36)

\[
\rho_p = \rho - \lambda = 0,
\]

(5.37)

\[
\theta = \frac{(2m + 1)}{\tau} \left[ -\frac{3}{4(m^2 + 2)} \tau^{4(1-m)} + c_2 \tau^{2m(4-m)} \right]^{\frac{1}{2}},
\]

(5.38)

\[
\sigma^2 = \frac{(m - 1)^2}{3\tau^2} \left[ -\frac{3}{4(m^2 + 2)} \tau^{4(1-m)} + c_2 \tau^{2m(4-m)} \right]^{\frac{1}{2}}.
\]

(5.39)

From Eqs. (5.38) and (5.39), we obtain

\[
\frac{\sigma^2}{\theta^2} = \frac{(m - 1)^2}{3(2m + 1)^2} = \text{constant}.
\]

(5.40)

If we choose the suitable values of constants \( m \) and \( c_2 \) i.e. for \( m > 3 \), \( c_2 < 0 \) we find that the energy conditions \( \rho \geq 0 \), \( \rho_p \geq 0 \) are satisfied. Other physical behaviour of the model (5.34) are similar to the model (5.19).

From Eqs. (5.35) and (5.36), we obtain

\[
\frac{\rho_p}{\lambda} = 0.
\]

(5.41)

Hence, in this case the strings dominate over the particles.
Bibliography


Magnetized Anisotropic Bulk Viscous Cosmological Models with a Variable $A$-Term

Anirudh Pradhan, Saeed Otarod, and Sheel Kumar Singh

1 Department of Mathematics, Hindu Post-Graduate College, Zamania-282 331, Ghazipur, India
2 Department of Physics, Yasouj University, Yasouj, Iran
3 Department of Mathematics, Post-Graduate College, Ghazipur-233 001, India

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Some anisotropic homogeneous cosmological models with an electromagnetic field are obtained in the presence of a bulk viscous fluid. The source of the magnetic field is due to an electric current produced along the $x$-axis. The coefficient of the bulk viscosity is assumed to be a power function of the mass density ($\xi = \xi_0 \rho^n$). We obtain a cosmological term as a decreasing function of time, which is consistent with results from recent supernovae Ia observations. The behavior of the electromagnetic field tensor together with some physical aspects of the models are also discussed.

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I. INTRODUCTION

The problem of the cosmological constant is one of the most salient and unsettled problems in cosmology. The smallness of the recently observed effective cosmological constant ($\Lambda_0 \leq 10^{-56} \text{cm}^{-2}$) constitutes one of the most difficult problems involving cosmology and elementary particle physics theory. To explain the striking cancelation between the "bare" cosmological constant and the ordinary vacuum energy contributions of the quantum fields, many mechanisms have been proposed during the last few years. The "cosmological constant problem" can be expressed as the discrepancy between the negligible value $\Lambda$ for the present universe (as can be seen by the successes of Newton's theory of gravitation) and the values $10^{50}$ larger expected by the Glashow-Salam-Weinberg model, or by grand unified theory (GUT) where it should be $10^{107}$ larger. The cosmological term $\Lambda$ is then small at the present epoch simply because the universe is too old. The problem in this approach is to determine the right dependence of $\Lambda$ upon $R$ or $t$.

Models with a relic cosmological constant $\Lambda$ have received considerable attention recently among researchers for various reasons (see Refs. [5]–[7] and the references therein). Some of the recent discussions on the cosmological constant "problem" and on cosmology with a time-varying cosmological constant by Ratra and Peebles, Dolgov, and Sahni and Starobinsky point out that, in the absence of any interaction with matter or radiation, the cosmological constant remains a "constant"; however, in the presence of interactions with matter or radiation, a solution of the Einstein equations and the assumed equation of covariant conservation of stress-energy with a time-varying $\Lambda$ can be found. For these solutions, the conservation of energy requires a decrease in the energy density of the
vacuum component, to be compensated by a corresponding increase in the energy density of matter or radiation. Earlier investigations on this topic are contained in Zeldovich [13], Weinberg [1] and Carroll, Press, and Turner [14]. Recent observations by Perlmutter et al. [15] and Riess et al. [16] strongly favour a significant and positive value of $\Lambda$. Their finding arises from the study of more than 50 type Ia supernovae with red-shifts in the range $0.10 \leq z \leq 0.83$, and these suggest Friedmann models with negative pressure matter, such as a cosmological constant ($\Lambda$), domain walls, or cosmic strings (Vilenkin [17], Garnavich et al. [18]). Recently, Carmeli and Kuzmenko [19] have shown that the cosmological relativistic theory predicts the value for cosmological constant $\Lambda = 1.934 \times 10^{-35} \text{s}^{-2}$. This value of $\Lambda$ is in excellent agreement with the measurements recently obtained by the High-Z Supernova Team and the Supernova Cosmological Project (Garnavich et al. [18]; Perlmutter et al. [15]; Riess et al. [16]; Schmidt et al. [20]). The main conclusion of these observations is that the expansion of the universe is accelerating.

Several ansatz have been proposed in which the $\Lambda$ term decays with time (see Refs. Gasperini [21], Berman [22], Freese et al. [7], Özer and Taha [7], Peebles and Ratra [23], Chen and Hu [24], Abdussattar and Viswakarma [25], Gariel and Le Denmat [26], and Pradhan et al. [27]). Of special interest is the ansatz $\Lambda \propto S^{-2}$ (where $S$ is the scale factor of the Robertson-Walker metric) by Chen and Wu [24], which has been considered/modifed by several authors (Abdel-Rahaman [28], Carvalho et al. [7], Silveira and Waga [7], Vishwakarma [29]).

The occurrence of magnetic fields on the galactic scale is a well-established fact today, and their importance for a variety of astrophysical phenomena is generally acknowledged, as pointed out by Zeldovich et al. [30]. Also Harrison [31] has suggested that magnetic fields could have a cosmological origin. As a natural consequence, we should include magnetic fields in the energy-momentum tensor of the early universe. The choice of anisotropic cosmological models for the Einstein system of field equations leads to cosmological models more general than the Robertson-Walker model [32]. The presence of primordial magnetic fields in the early stages of the evolution of the universe has been discussed by several authors [33]–[42]. Strong magnetic fields can be created due to adiabatic compression in clusters of galaxies. Large-scale magnetic fields give rise to anisotropies in the universe. The anisotropic pressure created by the magnetic fields dominates the evolution of the shear anisotropy, and it decays slower than if the pressure was isotropic [43, 44]. Such fields can be generated at the end of an inflationary epoch [45]–[49]. Anisotropic magnetic field models have significant contributions in the evolution of galaxies and stellar objects. Bali and Ali [50] had obtained a magnetized cylindrically symmetric universe with an electrically neutral perfect fluid as the source of matter. Pradhan et al. [51] have investigated Bianchi type I cosmological models with a magnetic field in a different context.

A realistic treatment of the problem requires the consideration of material distributions other than the perfect fluid. It is well known that in an earlier stage of the universe, when the radiation in the form of photons as well as neutrinos decoupled from matter, it behaved like a viscous fluid. Misner [52] has studied the effect of viscosity on the evolution of cosmological models. A number of authors have discussed cosmological solutions with bulk viscosity in various contexts [53]–[61].
Roy and Prakash [62] have derived an anisotropic magnetohydrodynamic cosmological model in general relativity. Recently Pradhan et al. [63] have obtained magnetized anisotropic cosmological models with varying $\Lambda$. Motivated by the situations discussed above in regard to a variable cosmological term, we revisited their works. The paper is organized as follows. The metric and the field equations are presented in Section 2. In Section 3, we obtain the solutions for a universe filled with a bulk viscous fluid in the presence of a magnetic field. Section 4 includes the physical and geometrical features of the models. In Section 5, we obtain the solution of the field equations in the absence of magnetic fields, and some physical aspects of the models are also discussed. In Section 6, we discuss our main results and summarize our conclusions.

II. THE METRIC AND FIELD EQUATIONS

We consider the metric in the form of Marder [64],

$$ds^2 = A^2(dx^2 - dt^2) + B^2dy^2 + C^2dz^2,$$

(1)

where the metric potentials $A$, $B$, and $C$ are functions of $t$ alone. This is a transform of the metric of a Bianchi type I space time in co-moving coordinates, which has been studied by a number of authors e.g., (Heckmann and Schücking [65], Thorne [66], and Roy and Prakash [62]). In this paper we have considered the distribution of matter to consist of an electrically neutral bulk viscous fluid with an infinite electrical conductivity and a magnetic field. The energy momentum tensor in the presence of bulk stress has the form

$$T_{ij}^\eta = (\rho + \bar{p})v_i v_j + \bar{p} g_{ij} + E_i^j,$$

(2)

where $E_i^j$ is the electromagnetic field given by Lichnerowicz [67] as

$$E_i^j = \mu \left[ |h|^2 \left( \frac{1}{2} g_{ij} + \frac{1}{2} g_{ji} \right) - h_i h^j \right],$$

(3)

and

$$\bar{p} = \rho - \xi v_i v_i.$$  

(4)

Here $\rho$, $p$, $\bar{p}$, and $\xi$ are the energy density, isotropic pressure, effective pressure, and bulk viscous coefficient, respectively, and $v^i$ is the flow vector satisfying the relation

$$g_{ij} v^i v^j = -1.$$  

(5)

$\mu$ is the magnetic permeability and $h_i$ the magnetic flux vector defined by

$$h_i = \frac{1}{\mu} \star F_{ij} v^j,$$

(6)
where \( F_{ij} \) is the dual electromagnetic field tensor defined by Synge [68] to be
\[
F_{ij} = \frac{\sqrt{-g}}{2} \varepsilon_{ijkl} F^{kl}.
\]
(7)
\( F_{ij} \) is the electromagnetic field tensor and \( \varepsilon_{ijkl} \) is the Levi-Civita tensor density. Here, the co-moving coordinates are taken to be \( v^1 = 0 = v^2 = v^3 \) and \( v^4 = \frac{1}{4} \). We take the incident magnetic field to be in the direction of the x-axis, so that \( h_1 = 0, h_2 = 0 = h_3 = h_4 \). Due to the assumption of infinite conductivity of the fluid, we get \( F_{ij} = F_{23} \). The first set of Maxwell’s equations,
\[
F_{ij,k} + F_{kj,i} + F_{ki,j} = 0,
\]
(8)
where the semicolon represents a covariant differentiation, leads to \( F_{23} \) being a constant, say \( k \).

The Einstein field equations (in gravitational units \( c = 1, G = 1 \)) read
\[
R^l_l - \frac{1}{2} R g^l_l + \Lambda g^l_l = -8\pi T^l_l,
\]
(9)
for the line element (1) they are in detail
\[
8\pi A^2 \left( \bar{p} - \frac{k^2}{2\mu B^2 C^2} \right) = \frac{\dot{A}B}{AB} + \frac{\dot{A}C}{AC} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} - \frac{\dot{B}\dot{C}}{BC} - \Lambda A^2,
\]
(10)
\[
8\pi A^2 \left( \bar{p} + \frac{k^2}{2\mu B^2 C^2} \right) = \frac{\dot{A}^2}{A^2} - \frac{\dot{A}}{A} - \frac{\dot{C}}{C} - \Lambda A^2,
\]
(11)
\[
8\pi A^2 \left( \bar{p} + \frac{k^2}{2\mu B^2 C^2} \right) = \frac{\dot{A}^2}{A^2} - \frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \Lambda A^2,
\]
(12)
\[
8\pi A^2 \left( \rho + \frac{k^2}{2\mu B^2 C^2} \right) = \frac{\dot{A}B}{AB} + \frac{\dot{A}C}{AC} + \frac{\dot{B}C}{BC} + \Lambda A^2.
\]
(13)
Here and in the following expressions, a dot indicates ordinary differentiation with respect to \( t \).

III. SOLUTION OF THE FIELD EQUATIONS

Eqs. (10)–(13) represent a system of four equations in six unknowns \( A, B, C, \mu, \rho, \) and \( \Lambda \). For the complete determination of these unknowns two more conditions have to be imposed on them. Here we assume that the space time is of degenerate Petrov type I, the
degeneracy being in the \( y \) and \( z \) directions. This requires that \( C_{12}^{2} = C_{13}^{2} \). This condition is identically satisfied if \( B = C \). However, we shall assume the metric potentials to be unequal, owing to the assumed anisotropy. From Eqs. (10) and (11) we get

\[
\frac{d}{dt}\left( \frac{\dot{A}}{A} \right) + \frac{\dot{A}}{A} \left[ \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right] - \frac{\dot{B}}{B} \frac{\dot{B} \dot{C}}{BC} = -\frac{8\pi k^{2}A^{2}}{\mu B^{2}C^{2}}. \tag{14}
\]

From Eqs. (11) and (12) we get

\[
\frac{\dot{B}}{B} = \frac{\dot{C}}{C}. \tag{15}
\]

The condition \( C_{12}^{2} = C_{13}^{2} \) leads to

\[
\frac{\dot{B}}{B} - \frac{\dot{C}}{C} + 2\frac{\dot{A}}{A} \left[ \frac{\dot{C}}{C} - \frac{\dot{B}}{B} \right]. \tag{16}
\]

From Eqs. (15) and (16) we get

\[
\frac{\dot{A}}{A} \left[ \frac{\dot{C}}{C} - \frac{\dot{B}}{B} \right] = 0. \tag{17}
\]

Since \( B \neq C \), Eq. (17) on integration gives

\[
A = \text{constant} = A_{0}. \tag{18}
\]

From Eqs. (14) and (18) we get

\[
\frac{\dot{B}}{B} + \frac{\dot{B} \dot{C}}{BC} = \frac{8\pi k^{2}A_{0}^{2}}{\mu B^{2}C^{2}}. \tag{19}
\]

Eq. (15) on integration gives

\[
\dot{B} C - \dot{B} C = K, \tag{20}
\]

where \( K \) is an integration constant. Putting \( \ddot{B} = \alpha \) and \( BC = \beta \) in (20) reduces it to

\[
\frac{\dot{\alpha}}{\alpha} = K. \tag{21}
\]

Also Eq. (19) reduces to

\[
\frac{d}{dt} \left[ \left( \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} \right) \beta \right] = \frac{16\pi k^{2}A_{0}^{2}}{\mu B^{2}}. \tag{22}
\]

From Eqs. (21) and (22) we get

\[
\mu \beta \dot{\beta} = 16\pi k^{2}A_{0}^{2}, \tag{23}
\]
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which on integration gives
\[ \beta^2 = \frac{32 \pi k^2 A_0^2}{\mu \log \beta + L}, \]  
where \( L \) is a constant of integration. From Eqs. (21) and (24) we get
\[ \alpha = b \exp \left\{ \frac{2K}{a^2} \left\{ \left( a^2 \log \beta + L \right)^{\frac{1}{2}} - L^{\frac{1}{2}} \right\} \right\}, \]
where \( a^2 = \frac{32 \pi k^2 A_0^2}{\mu} \) and \( b \) is a constant of integration. Hence we have
\[ B^2 = b \beta \exp \left\{ \frac{2K}{a^2} \left\{ \left( a^2 \log \beta + L \right)^{\frac{1}{2}} - L^{\frac{1}{2}} \right\} \right\}, \]
and
\[ C^2 = \frac{\beta}{B} \exp \left\{ -\frac{2K}{a^2} \left\{ \left( a^2 \log \beta + L \right)^{\frac{1}{2}} - L^{\frac{1}{2}} \right\} \right\}. \]

Consequently the geometry of the line-element (1) takes the form
\[ ds^2 = A_0^2 \left[ dx^2 - \frac{d\beta^2}{(a^2 \log \beta + L) \, dx^2} + b\beta \exp \left\{ \frac{2K}{a^2} \left\{ \left( a^2 \log \beta + L \right)^{\frac{1}{2}} - L^{\frac{1}{2}} \right\} \right\} \, dy^2 \right. \]
\[ \left. + \frac{\beta}{B} \exp \left\{ -\frac{2K}{a^2} \left\{ \left( a^2 \log \beta + L \right)^{\frac{1}{2}} - L^{\frac{1}{2}} \right\} \right\} \, dx^2. \]

By a suitable transformation of coordinates the metric (28) is reduced to the form
\[ ds^2 = A_0^2 \left[ dX^2 - \frac{\exp (2T)}{(a^2 T + L) \, dT^2} + \exp \left\{ T + \frac{2K}{a^2} \left\{ \left( a^2 T + L \right)^{\frac{1}{2}} - L^{\frac{1}{2}} \right\} \right\} \, dY^2 \right. \]
\[ \left. + \exp \left\{ T - \frac{2K}{a^2} \left\{ \left( a^2 T + L \right)^{\frac{1}{2}} - L^{\frac{1}{2}} \right\} \right\} \, dZ^2. \]

The effective pressure and density for the model (29) are given by
\[ 8\pi \bar{p} = 8\pi (p - \xi \theta) = \frac{1}{4 A_0^2} \exp (-2T) \left[ L - K^2 + \frac{16 \pi k^2 A_0^2}{\mu} \left( 2T - 3 \right) \right] - \Lambda, \]
\[ 8\pi \rho = \frac{1}{4 A_0^2} \exp (-2T) \left[ L - K^2 + \frac{16 \pi k^2 A_0^2}{\mu} \left( 2T - 1 \right) \right] + \Lambda. \]

Here \( \theta \) is the scalar of expansion calculated for the flow vector \( \nu^i \) as
\[ \theta = \frac{1}{A_0} \exp (-T) \left[ L + \frac{32 \pi k^2 A_0^2 T}{\mu} \right]^{\frac{1}{2}}. \]
For the specification of $\xi$, we assume that the fluid obeys an equation of state of the form

$$p = \gamma \rho,$$

(33)

where $\gamma (0 \leq \gamma \leq 1)$ is a constant.

Thus, given $\xi(t)$ we can solve the system for the physical quantities. Therefore let us assume the following ad hoc law [56]–[58].

$$\xi(t) = \xi_0 \rho^n,$$

(34)

where $\xi_0$ and $m$ are real constants. For large value of $\rho$, $n$ is quite small, and Santos et al. [59] suggested one could get more realistic models if $n$ lies in the regime $0 \leq n \leq \frac{1}{2}$. For small density, $n$ may even be equal to unity, as was used in Murphy’s work [61] for simplicity. Also if $n = 1$, Eq. (34) may correspond to a radiative fluid [2].

On using Eq. (34) in Eq. (30), we obtain

$$8 \pi [p - \xi_0 \rho^n] = 8 \pi (p - \xi_0) = \frac{1}{4A_0^2} \exp (-2T) [L - K^2 + \frac{16 \pi k^2 A_0^2}{\mu} (2T - 3)] - \Lambda.$$

(35)

For simplicity and realistic models of physical importance, we consider the following two cases:

III-1. MODEL I: SOLUTIONS FOR $\xi = \xi_0$

When $n = 0$, Eq. (34) reduces to $\xi = \xi_0$. With the use of Eqs. (31), (32), and (33), Eq. (35) leads to

$$4\pi (1 + \gamma) \rho = \frac{4\pi \xi_0}{A_0} \frac{[L + 32 \pi k^2 A_0^2 T^\frac{3}{2}]}{\exp(T)} + \frac{(L - K^2)}{2A_0^2 \exp(2T)} + \frac{16 \pi k^2 (T - 1)}{\mu \exp(2T)}.$$

(36)

Eliminating $\rho(t)$ between Eqs. (31) and (36), we obtain

$$1 + \gamma) \Lambda = \frac{8\pi \xi_0}{A_0} \frac{[L + 32 \pi k^2 A_0^2 T^\frac{3}{2}]}{\exp(T)} +$$

$$\frac{[\mu (1 + \gamma)(L - K^2) + 16 \pi (\gamma - 3) k^2 A_0^2]}{4\mu A_0^2 \exp(2T)} + \frac{8\pi k^2 (1 - \gamma) T}{\mu \exp(2T)}.$$

(37)

From Eq. (36), it is observed that $\rho$ is always positive, and it decreases as time increases when $L - K^2 > 0$. From Eq. (37), we observe that for $L > K^2 + \frac{16 \pi (\gamma - 3) k^2 A_0^2}{\rho(1 + \gamma)}$, the cosmological term $\Lambda$ is found to be a decreasing function of time, and it approaches a small positive value at late time (i.e., the present epoch). This is in a good agreement with the results from recent supernovae la observations [15, 16, 18, 20].
III-2. MODEL II: SOLUTIONS FOR $\xi = \xi_0 \rho$

When $n = 1$, Eq. (34) reduces to $\xi = \xi_0 \rho$. With the use of Eqs. (31), (32), and (33), Eq. (35) leads to

$$\rho = \frac{(L - K^2)\mu + 32k^2A_0^3(T - 1)}{4\pi A_0 \exp(T) \left[(1 + \gamma)A_0 \exp(T) - \xi_0 (L + 32\pi k^2 A_0^3 T)^{\frac{1}{2}}\right]}.$$  \hfill (38)

Eliminating $\rho(t)$ between Eqs. (31) and (38), we obtain

$$[\left(1 + \gamma\right)A_0 - \xi_0 \exp(-T)(L + 32\pi k^2 A_0^3 T)^{\frac{1}{2}}] \Lambda = \frac{(L - K^2)\mu + 16\pi k^2 A_0^3 (2T - 3)}{4\pi A_0 \exp(2T)}$$

$$\frac{\left[\gamma A_0 - \xi_0 \exp(-T)(L + 32\pi k^2 A_0^3 T)^{\frac{1}{2}}\right]^2}{4A_0^3 \exp(2T)}.$$  \hfill (39)

From Eqs. (38), we observe that $\rho > 0$ if

$$K^2 + \frac{32k^2A_0^3}{\mu} < L < \frac{(1 + \gamma^2 A_0^3)}{\xi_0}.$$  

Here we also see that $\rho(t)$ is a decreasing function of time, which shows that the universe is expanding. It is also observed from Eq. (39) that $\Lambda(t)$ remains always negative but is a decreasing function of time. By decreasing we mean its absolute magnitude approaches zero, which is acceptable physically. A negative cosmological term adds to the attractive gravity of matter; therefore a universe with a negative cosmological term is invariably doomed to re-collapse. A positive cosmological term resists the attractive gravity of matter due to its negative pressure. For most universes the cosmological term eventually dominates over the attraction of matter and drives the universe to expand exponentially.

IV. SOME PHYSICAL ASPECTS OF THE MODEL

We shall now give the expressions for the kinematic quantities and the components of the conformal curvature tensor. With regard to the kinematics properties of the velocity $v^i$ in the metric (29), a straightforward calculation leads to the fact that the tensor of rotation $\omega_{ij}$ is zero and the components of the shear $\sigma_{ij}$ are given by

$$\sigma_{11} = -\frac{A_0}{3 \exp(T)} \left[L + \frac{32\pi k^2 A_0^3 T}{\mu}\right]^{\frac{1}{2}},$$  \hfill (40)

$$\sigma_{22} = \frac{1}{6A_0} \left[3K + \left(L + \frac{32\pi k^2 A_0^3 T}{\mu}\right)^{\frac{1}{2}}\right] \times$$
exp \left[ \frac{\mu K}{16\pi k^2 A_0} \left\{ \left( L + \frac{32\pi k^2 A_0^2 T}{\mu} \right)^{\frac{1}{2}} - L^{\frac{1}{2}} \right\} \right], \quad (41)

\sigma_{33} = \frac{1}{6A_0} \left[ \left( L + \frac{32\pi k^2 A_0^2 T}{\mu} \right)^{\frac{1}{2}} - 3K \right] x

\exp \left[ - \frac{\mu K}{16\pi k^2 A_0^2} \left\{ \left( L + \frac{32\pi k^2 A_0^2 T}{\mu} \right)^{\frac{1}{2}} - L^{\frac{1}{2}} \right\} \right], \quad (42)

the other components of the shear tensor \( \sigma_{ij} \) being zero. Hence

\sigma^2 = \frac{1}{18A_0^2} \frac{1}{\exp(2T)} \left[ 2L + 9K^2 + \frac{64\pi k^2 A_0^2 T}{\mu} \right].

The non-vanishing components of the conformal curvature tensor are obtained as

\begin{align}
C_{12}^{13} = C_{13}^{23} = \frac{1}{\exp(2T)} \left[ \frac{1}{12A_0^2} (L^2 - K^2) + \frac{4\pi k^2}{3\mu} (2T - 1) \right].
\end{align}

The model (29) apparently has a singularity at \( T = -\frac{L}{2} \). However it is not a real singularity but it occurs as such on account of the coordinates chosen. The expressions for the density, pressure, expansion, shear, and the conformal curvature tensor all remain finite for this value of \( T \). The Kreschmann scalar \( R_{ijkl}R^{ijkl} \) for this metric has the value

\frac{1}{4A_0^4} \exp(-4T) \left[ a^2 \left\{ a^2 + 2K^2 - 2(a^2T + L) \right\} \right],

which is finite when \( a^2T + L = 0 \). It is also observed that as \( T \) tends to infinity this scalar, as well as the scalar of expansion, the shear, and the conformal curvature tensor, tend to zero, so that the metric tends to become flat. From the form of the metric (29) it is clear that the model exists during the time \( T \geq T_1 \) where \( T_1 \geq -\frac{L}{2} \), and it has to be continued at time \( T = T_1 \) with another model valid during \( T \leq T_1 \) and representing the state prior to the evolution of the universe embodied by the metric (29).

V. SOLUTION IN THE ABSENCE OF A MAGNETIC FIELD

In the absence of the magnetic field the geometry of the space time is given by the metric

\begin{align}
\text{In the absence of the magnetic field the geometry of the space time is given by the metric}
\end{align}
The effective pressure and the density for the model (45) are obtained as

\[ 8\pi p = \frac{1}{4A_0^3 \exp(2T)} (L - K^2) - A, \] (46)

\[ 8\pi \rho = \frac{1}{4A_0^3 \exp(2T)} (L - K^2) + A. \] (47)

Here the value of the scalar of expansion \( \theta \) is obtained as

\[ \theta = \frac{\sqrt{L}}{A_0 \exp(-T)}. \] (48)

V-1. MODEL I: SOLUTIONS FOR \( \xi = \xi_0 \)

When \( n = 0 \), Eq. (34) reduces to \( \xi = \xi_0 \). With the use of Eqs. (31), (32), and (33), Eq. (46) leads to

\[ 8\pi (1 + \gamma) \rho = \frac{8\pi \xi_0 \sqrt{L}}{A_0 \exp(T)} + \frac{(L - K^2)}{2A_0^3 \exp(2T)}. \] (49)

Eliminating \( \rho(t) \) between Eqs. (47) and (49), we obtain

\[ (1 + \gamma)A = \frac{8\pi \xi_0 \sqrt{L}}{A_0 \exp(T)} + \frac{(1 - \gamma)(L - K^2)}{4A_0^3 \exp(2T)}. \] (50)

From Eq. (49), it is obvious that the condition \( L - K^2 > 0 \) guarantees that \( \rho \) is always a positive decreasing function of time. From Eq. (50), it is also observed that under the same condition \( L - K^2 > 0 \), \( A \) is always a decreasing function of \( T \) and it approaches a small positive value as \( T \) increases. This is in good agreement with the results from recent supernovae Ia observations [15, 16, 18, 20].

V-2. MODEL II: SOLUTIONS FOR \( \xi = \xi_0 \rho \)

When \( n = 1 \), Eq. (34) reduces to \( \xi = \xi_0 \rho \). With the use of Eqs. (31), (32), and (33), Eq. (46) leads to

\[ 16\pi \rho = \frac{1}{\left[1 + \gamma - \frac{\xi_0 \sqrt{L}}{A_0 \exp(T)}\right]} \frac{(L - K^2)}{4A_0^3 \exp(2T)}. \] (51)

Eliminating \( \rho(t) \) between Eqs. (47) and (51), we obtain

\[ \frac{1}{\left[1 + \gamma - \frac{\xi_0 \sqrt{L}}{A_0 \exp(T)}\right]} \frac{(L - K^2)}{4A_0^3 \exp(2T)} \left[1 + \gamma + \frac{\xi_0 \sqrt{L}}{A_0 \exp(T)}\right]. \] (52)

From Eq. (51), it is obvious that if \( 1 + \gamma > \frac{\xi_0 \sqrt{L}}{A_0} \) and \( L - K^2 > 0 \) then \( \rho \) will always be a positive decreasing function of \( T \), which confirms that our universe is expanding.
From Eq. (52), it is also observed that under the same conditions $\Lambda$ is always a decreasing function of $T$, and it approaches a small positive value as $T$ increases. Thus we see that the nature of the cosmological constant is consistent with the results from recent supernovae la observations.

Also for the model (45),

$$C_{12}^{12} = C_{13}^{13} = -C_{23}^{23} = \frac{(L^2 - K^2)}{12A_0^2 \exp (2T)}.$$  

(53)

Thus the magnetic field gives a positive contribution to the expansion, shear, and the free gravitational field, which die out for large values of $T$ at a slower rate than the corresponding quantities in the absence of the magnetic field.

VI. DISCUSSION AND CONCLUSION

We have obtained a new class of anisotropic cosmological models including an electromagnetic perfect fluid as the source. Generally the models represent an expanding, shearing, non-rotating, and Petrov type-I degenerate universe in which the flow vector is geodesic. In all these models, we observe that they do not approach isotropy for large values of time.

Recent observations of distant supernovae imply, in defiance of expectation, that the universe growth is accelerating, contrary to what has always been assumed, that the expansion is slowing down due to gravity. The cosmological constants in all these models given in Sections 3 and 5 are decreasing functions of time and they approach a small value as time increases (i.e., the present epoch). The values of the cosmological "constant" for these models are found to be small and positive, which are supported by the results from recent supernovae Ia observations recently obtained by the High-z Supernova Team and Supernova Cosmological Project (Garnavich et al. [18], Perlmutter et al. [15], Riess et al. [16], Schmidt et al. [20]). Our derived models confirm to these recent experimental results by showing that the universe now is definitely in a stage of accelerating expansion. Thus with our approach, we obtain a physically relevant decay law for the cosmological constant, unlike other investigators where ad hoc laws were used to arrive at mathematical expressions for the decaying vacuum energy. Thus our models are more general than those studied earlier.

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References

ome magnetized bulk viscous string cosmological models

1 General Relativity

ahesh Kumar Yadav · Anirudh Pradhan ·
lee Kumar Singh

Abstract Some Bianchi type-I viscous fluid string cosmological models with magnetic field are investigated. The viscosity coefficient of bulk viscous fluid is assumed to be a power function of mass density $\xi(t) = \xi_0 \rho^m$, where $\xi_0$ and $m$ are constants. To get a determinate model, we assume $p = (1 + \omega)\lambda$, where $\rho$ is rest energy density, $\omega$ a positive constant and $\lambda$ the string tension density and expansion $\theta$ is proportional to eigen value $\sigma_1$ of the shear tensor $\sigma^I$. The behaviour of the models from physical and geometrical aspects in presence and absence of magnetic field is discussed.

Keywords Cosmic string · Viscous models · Magnetic field

Abstract Some Bianchi type-I viscous fluid string cosmological models with magnetic field are investigated. The viscosity coefficient of bulk viscous fluid is assumed to be a power function of mass density $\xi(t) = \xi_0 \rho^m$, where $\xi_0$ and $m$ are constants. To get a determinate model, we assume $p = (1 + \omega)\lambda$, where $\rho$ is rest energy density, $\omega$ a positive constant and $\lambda$ the string tension density and expansion $\theta$ is proportional to eigen value $\sigma_1$ of the shear tensor $\sigma^I$. The behaviour of the models from physical and geometrical aspects in presence and absence of magnetic field is discussed.

Introduction

Cosmic strings play an important role in the study of the early universe. These arise during the phase transition after the big bang explosion as the temperature goes down below some critical temperature as predicted by grand unified theories (Zel’dovich et al. 1975a, 1975b; Kibble, 1976, 1980; Everett 1981; Vilenkin 1981). It is believed that cosmic strings give rise to density perturbations which lead to formation of galaxies (Zel’dovich 1980). These cosmic strings have stress energy and couple to the gravitational field. Therefore, it is interesting to study the gravitational effect which arises from strings. The general treatment of strings was initiated by Letelier (1979, 1983) and Stachel (1980). The occurrence of magnetic fields on galactic scale is well-established fact today, and their importance for a variety of astrophysical phenomena is generally acknowledged as pointed out by Zel’dovich et al. (1993). Also Harrison (1973) has suggested that magnetic field could have a cosmological origin. As a natural consequences, we should include magnetic fields in the energy-momentum tensor of the early universe. The choice of anisotropic cosmological models in Einstein system of field equations leads to the cosmological models more general than Robertson-Walker model (Robertson and Walker 1936). The presence of primordial magnetic fields in the early stages of the evolution of the universe has been discussed by several authors (Misner et al. 1973; Asseo and Sol 1987; Pudritz and Silk 1989; Kim et al. 1991; Perley and Taylor 1991; Kronberg et al. 1991; Wolfe et al. 1992; Kulsrud et al. 1997; Barrow 1997). Melvin (1975), in his cosmological solution for dust and electromagnetic field suggested that during the evolution of the universe, the matter was in a highly ionized state and was smoothly coupled with the field, subsequently forming neutral matter as a result of universe expansion. Hence the presence of magnetic field in string dust universe is not unrealistic. Banerjee et al. (1990) have investigated an axially symmetric Bianchi type I string dust cosmological model in presence and absence of magnetic
field using a supplementary condition $\alpha = a/3$ between metric potential where $\alpha = \alpha(t)$ and $\beta = \beta(t)$ and $a$ is constant. The string cosmological models with a magnetic field are also discussed by Chakraborty (1991), Tikekar and Patel (1992, 1994). Patel and Maharaj (1996) investigated stationary rotating world model with magnetic field. Ram and Singh (1995) obtained some new exact solution of string cosmology with and without a source free magnetic field for Bianchi type I space-time in the different basic form considered by Carminati and McIntosh (1980). Singh and Singh (1999) investigated string cosmological models with magnetic field in the context of space-time with $G_3$ symmetry. Lidsey et al. (2000) have reviewed aspects of super string cosmology with the emphasis on the cosmological implications of duality symmetries in the theory.

The distribution of matter can be satisfactorily described by a perfect fluid due to the large scale distribution of galaxies in our universe. However, observed physical phenomena such as the large entropy per baryon and the remarkable degree of isotropy of the cosmic microwave background radiation, suggest analysis of dissipative effects in cosmology. Furthermore, there are several processes which are expected to give rise to viscous effects. These are the decoupling of neutrinos during the radiation era and the decoupling of radiation and matter during the recombination era. Bulk viscosity is associated with the GUT phase transition and string creation. Misner (1967, 1968) has studied the effect of viscosity on the evolution of cosmological models. The role of viscosity in cosmology has been investigated by Weinberg (1971), Nightingale (1973), Heller and Klimek (1975) have obtained a viscous universe without initial singularity. The model studied by Murphy (1973) possessed an interesting feature in which big bang type of singularity of infinite space-time curvature does not occur to be a finite past. However, the relationship assumed by Murphy between the viscosity coefficient and the matter density is not acceptable at large density. These motivate to study cosmological bulk viscous fluid string model. Recently Bali and Pradhan (2007) have investigated Bianchi type-III cosmological models with bulk viscous fluid for massive string. Bulk viscous string cosmological models in different space-times have been studied by several authors (Maharaj et al. 1995; Yadav et al. 2007; Wang 2004a, 2004b, 2005, 2006).

Recently Bali and Upadhaya (2003), Bali and Anjali (2006) have investigated Bianchi type I magnetized string cosmological models. Motivated by the situation discussed above, in this paper, we have generalized these solutions by including bulk viscosity in electromagnetic field tensor. This paper is organized as follows: The metric and field equations are presented in Sect. 2. In Sect. 3, we deal with the solution of the field equations in presence of viscous fluid. In Sects. 3.1, 3.2 and 3.3 we deal the solutions in three different cases by choosing $m = 0, 1, 1/2$. Section 4 includes the solution in absence of magnetic field. In Sect. 5, we have given the concluding remarks.

### 2 The metric and field equations

We consider the Bianchi Type I metric in the form

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2.$$  

The energy momentum tensor for the string in presence of bulk stress and magnetic field has the form

$$T_{ij} = (\rho + p)v_iv_j + \rho g_{ij} - \lambda x_i x_j + E_{ij},$$

where $v_i$ and $x_i$ satisfy condition

$$v_i v_i = -x_i x_i = -1, \quad v_i x_i = 0.$$  

In (2), $\rho$ is rest energy density for a cloud string, $\lambda$ is the string tension density, $x_i$ is a unit space-like vector representing the direction of string, $p$ is isotropic pressure, $\rho$ is effective pressure and $v_i$ is the four velocity vector satisfying the relation

$$g_{ij} v_i v_j = -1.$$  

Here $E_{ij}$ is the electromagnetic field given by Lichnerowicz (1967)

$$E_{ij} = \mu \left[ h_i h_j \left( v_i v_j + \frac{1}{2} g_{ij} \right) - h_i h_j \right]$$

and $\rho$ is given by

$$\rho = p - \xi v_i v_i.$$  

where $\xi$ is the coefficient of bulk viscosity, $\mu$ is the magnetic permeability and $h_i$ the magnetic flux vector defined by

$$h_i = \frac{1}{\mu} F_{ij} v^j,$$

where the dual electromagnetic field tensor $*F_{ij}$ is defined by Synge (1960)

$$*F_{ij} = \frac{\sqrt{-g}}{2} \epsilon_{ijkl} F^{kl}.$$  

$F_{ij}$ is the electromagnetic field tensor and $\epsilon_{ijkl}$ is the Levi Civita tensor density. The semicolon represents a covariant differentiation. Here, the comoving coordinates are taken to be $v^0 = 0 = v^2 = v^3$ and $v^1 = 1$. We consider the current to be flowing along the $x$-axis so that $h_1 \neq 0, h_2 = h_3 = h$.
id consequently $F_{23}$ is the only non-vanishing component $F_{ij}$. Maxwell's equations

\begin{align}
    \{i; j\} = 0, \\
    \frac{1}{\mu} F_{ij} = j^j
\end{align}

quire

\begin{equation}
    F_{23} = \text{constant}.
\end{equation}

The field equations (with $\frac{\text{d}g}{\text{d}t^2} = 1$)

\begin{align}
    \nabla^2 F_{ij} + \lambda g_{ij} = -T_{ij}, \\
    \nabla^2 \phi + \frac{\partial \phi}{\partial t} = 0,
\end{align}

where $\phi$ is a constant of integration.

\subsection*{3 Solutions of the field equations}

The field equations (13–16) are a system of four equations with seven unknown parameters $A, B, C, \rho, \lambda, \xi$ and $\phi$. We need three additional conditions to obtain explicit solutions of the system.

We first assume that the expansion $\theta$ is proportional to the eigen value $\sigma_1^2$ of the shear tensor $\sigma_1^2$. This condition leads to

\begin{equation}
    A = \ell (BC)^n,
\end{equation}

where $n = \frac{L+3}{2L-3}$ and $\ell$ & $L$ are constants of integration. Without any loss of generality we can choose $\ell = 1$. Hence (23) reduces to

\begin{equation}
    A = (BC)^n.
\end{equation}

Using (24) in (14) and (15), we obtain

\begin{equation}
    \frac{C_{44}}{C} - \frac{B_{44}}{B} = n \left( \frac{B_4}{B} + \frac{C_4}{C} \right) \left( \frac{B_4}{B} - \frac{C_4}{C} \right),
\end{equation}

which on integration leads to

\begin{equation}
    \frac{C_2}{(BC)^4} = L \frac{B}{(BC)^n}.
\end{equation}

Let us consider

\begin{equation}
    BC = \mu, \quad \frac{B}{C} = \nu.
\end{equation}

Equations (26) and (27) lead to

\begin{equation}
    \frac{v_4}{v} = \frac{L}{\mu^{n+1}}.
\end{equation}

From (13) and (14), we have

\begin{equation}
    \frac{B_{44}}{B} + \frac{B_4C_4}{BC} - \frac{A_{44}}{A} - \frac{A_4C_4}{AC} = \lambda + \frac{2k}{B^2C^2},
\end{equation}

where

\begin{equation}
    k = F_{23}^2.
\end{equation}

Here we consider the equation of the state

\begin{equation}
    \rho = (1 + \omega)\lambda,
\end{equation}

where $\omega$ is a positive constant. Using (29) in (16) with (31), we obtain

\begin{equation}
    \frac{A_4B_4}{AB} = \Delta + \frac{k(1 + 2\omega)}{B^2C^2}.
\end{equation}
From (24), (27) and (32), we obtain
\[
2(1 + \omega)(2n - 1)\frac{\mu_{44}}{\mu} - 2(1 + \omega)\frac{\nu_{44}}{\nu} + [n^2(4 + 3\omega) + 2n + 1] \frac{\mu^2}{\mu^2} + [2n^2 - 1] \frac{\nu^2}{\nu^2} + [2n^2\omega - n(2 + 3\omega) - 2(1 + \omega)] \frac{\mu^4 + \nu^4}{\mu \nu} + 4\lambda + \frac{4k(1 + 2\omega)}{\mu^2} = 0.
\]

(33)

Using (28) in (33), we get
\[
2(1 + \omega)(2n - 1)\frac{\mu_{44}}{\mu} + [n^2(4 + 3\omega) + 2n + 1] \frac{\mu^2}{\mu^2} + Ln\omega(2n - 1) \frac{\mu^4}{\mu^{n+2}} = \frac{n^2(\omega + 1)L^2}{\mu^{(n+1)}} - 4\Lambda - \frac{4k(1 + 2\omega)}{\mu^2}.
\]

(34)

Equations (31) and (34) with \(\omega = 0\) (Nambu strings) leads to
\[
2ff' + \frac{Mf^2}{\mu} = \frac{1}{2n - 1} \left[ \frac{n^2L^2}{\mu^{2n+1}} - \frac{4k}{\mu} - 4\Lambda \mu \right],
\]

(35)

where
\[
\mu_4 = f(\mu), \quad f' = \frac{df}{d\mu} \quad \text{and} \quad M = \frac{4n^2 + 2n + 1}{2n - 1}.
\]

(36)

Equation (35) can be rewritten in the form
\[
\frac{d}{d\mu}(f^2) + \frac{Mf^2}{\mu} = \frac{1}{2n - 1} \left[ \frac{n^2L^2}{\mu^{2n+1}} - \frac{4k}{\mu} - 4\Lambda \mu \right],
\]

(37)

which after integration gives
\[
f^2 = \frac{1}{(2n - 1)} \left[ \frac{nL^2}{(M - 2n)\mu^{2n}} - \frac{4k}{M} - \frac{4\Lambda \mu^2}{M + 2} \right] + \frac{N}{\mu^M},
\]

(38)

where \(N\) is constant. From (28) and (38), we have
\[
\log v = \int \frac{Ld\mu}{\mu^{n+1} \sqrt{\frac{1}{2n-1} \left[ \frac{L}{(M - 2n)\mu^{2n}} - \frac{4k}{M} - \frac{4\Lambda \mu^2}{M + 2} \right] + \frac{N}{\mu^M}}} + \log b,
\]

(39)

After suitable transformation of coordinates, the metric (1) reduces to the form
\[
ds^2 = -\frac{L^2}{(4n+1)^2} \frac{dT^2}{P k - Q \Lambda T^2 + \frac{N^2}{T^2}} + T^2 dX^2 + T v dY^2 + \frac{T}{v} dZ^2,
\]

(4)

where \(v\) is determined by (39) and \(\mu = T, x = X, y = \zeta = Z\) and
\[
P = \frac{4}{4n^2 + 2n + 1}, \quad Q = \frac{4}{4n^2 + 6n - 1}.
\]

The rest energy density \((p)\), the string tension \((\lambda)\) and the pressure \((p)\) for the model (40) are given by
\[
p = \frac{\lambda}{\frac{4n+1}{4}} \left[ \frac{N}{T^2} + k_2 \Lambda - \frac{kk_3}{T^2} \right],
\]

(4)

\[
p = \xi \theta + \frac{n(4n + 1)}{2n - 1} \left( \frac{N}{T^2} - k_2 \Lambda - \frac{kk_3}{T^2} \right),
\]

(4)

where
\[
k_1 = \frac{4n^2 + 6n - 1}{2n - 1},
\]

(4)

\[
k_2 = \frac{4n^2 + 6n - 1}{4n^2 + 2n + 1},
\]

(4)

\[
k_3 = \frac{4n^2 + 2n + 1}{4n^2 + 2n + 1}.
\]

(4)

Thus, given \(\xi(t)\), we can solve for the physical parameters. In most investigations involving bulk viscosity, it is assumed to be a simple power function of the energy density (Pavon et al. 1991, 1996; Maartens 1995; Zimdahl 1996; Santos et al. 1985)
\[
\xi(t) = \xi_0 \rho^m,
\]

(45)

where \(\xi_0\) and \(m\) are constants. For small density, \(m\) may even be equal to unity as used in Murphy’s work (Murphy 1973). If \(m = 1\), (45) may correspond to a radiative fluid (Wember 1972). Near the big bang, \(0 \leq m \leq \frac{1}{2}\) is a more appropriate assumption (Belinskii and Khalatnikov 1976) to obtain realistic models.

For simplicity and realistic models of physical importance, we consider the following three cases \((m = 0, \frac{1}{2}, 1)\).
Model I: solution for $\xi = \xi_0$

When $m = 0$, (45) reduces to $\xi = \xi_0 = \text{constant}$. Hence in this case (43), with the use of (17) leads to

$$n(4n + 1) \frac{N}{T^{k_1}} - k_2 \Lambda - \frac{k_3}{T^2}$$

$$+ (n + 1)\xi_0 \left[ \frac{L^2}{(4n + 1)T^{2(n+1)}} - \frac{Q\Lambda + N}{T^{k_1}} - \frac{Pk}{T^2} \right]^{\frac{1}{2}}.$$

(46)

Model II: solution for $\xi = \xi_0\rho$

When $m = 1$, (45) reduces to $\xi = \xi_0\rho$. Hence in this case (43), with the use of (17) and (42) leads to

$$n(4n + 1) \frac{N}{T^{k_1}} - k_2 \Lambda - \frac{k_3}{T^2}$$

$$+ (n + 1)\xi_0 \left[ \frac{L^2}{(4n + 1)T^{2(n+1)}} - \frac{Q\Lambda + N}{T^{k_1}} - \frac{Pk}{T^2} \right]$$

$$\times \left[ \frac{(4n + 1)N}{4T^{k_1}} + k_2 \Lambda - \frac{k_3}{T^2} \right]^{\frac{1}{2}}.$$  

(47)

Model III: solution for $\xi = \xi_0\rho^{\frac{1}{2}}$

When $m = \frac{1}{2}$, (45) reduces to $\xi = \xi_0\rho^{\frac{1}{2}}$. Hence in this case (43), with the use of (17) and (42) leads to

$$n(4n + 1) \frac{N}{T^{k_1}} - k_2 \Lambda - \frac{k_3}{T^2}$$

$$+ (n + 1)\xi_0 \left[ \frac{L^2}{(4n + 1)T^{2(n+1)}} - \frac{Q\Lambda + N}{T^{k_1}} - \frac{Pk}{T^2} \right]$$

$$\times \left[ \frac{(4n + 1)N}{4T^{k_1}} + k_2 \Lambda - \frac{k_3}{T^2} \right]^{\frac{1}{2}}.$$  

(48)

The physical aspects of the models: With regard to the kinematic properties of the velocity vector $v'$ in the metric $g$, a straightforward calculation leads to the following expressions for the scalar of expansion ($\theta$) and shear stress ($\sigma_i^j$)

$$\sigma_i^j = -\frac{1}{6} \left[ (2n - 1) \left\{ \frac{L^2}{(4n + 1)T^{2(n+1)}} \right\} - \frac{Q\Lambda + N}{T^{k_1}} - \frac{Pk}{T^2} \right]^{\frac{1}{2}} - \frac{3L}{T^{2(n+1)}}.$$  

(51)

$$\sigma_i^3 = -\frac{1}{6} \left[ (2n - 1) \left\{ \frac{L^2}{(4n + 1)T^{2(n+1)}} \right\} - \frac{Q\Lambda + N}{T^{k_1}} - \frac{Pk}{T^2} \right]^{\frac{1}{2}} + \frac{3L}{T^{2(n+1)}}.$$  

(52)

$$\sigma_4^4 = 0.$$  

(53)

The model (40) starts with a big bang at $T = 0$. When $T \rightarrow 0$ then $\rho = \lambda \rightarrow \infty$ and $p \rightarrow \infty$ if $\frac{L}{k_1} > 0$. When $T \rightarrow \infty$ then $\rho = \lambda \rightarrow \frac{(4n^2 - 2\lambda - 2\Lambda)}{(4n + 2\Lambda - 1)} A$ and $p \rightarrow \frac{(4n^2 - 2\lambda - 2\Lambda)}{(4n + 2\Lambda - 1)} A$. Since $\lim_{T \rightarrow \infty} \frac{L}{k_1} \neq 0$, hence the models do not approach isotropy for large values of $T$. However, if $n = \frac{1}{2}$ then the model (40) isotropizes. There is a Point Type singularity (MacCallum 1971) at $T = 0$ in the model.

For $\xi = 0$, we get the same model obtained by Bali and Anjali (2006). If we set $n = 1, B = C$ and $\xi = 0$, we get the LRS Bianch type I model obtained by Bali and Upadhaya (2003). Thus the effect of bulk viscosity is to introduce a change in the perfect fluid model.

4 Solutions in absence of magnetic field

In absence of magnetic field, i.e., when $k = 0$, the metric (40) reduces to

$$ds^2 = -\frac{dT^2}{\left( \frac{L^2}{(4n + 1)T^{2(n+1)}} - \frac{Q\Lambda T^2 + N}{T^{k_1}} \right)}$$

$$+ r^{2n}dX^2 + Tv dY^2 + \frac{r^{2n}}{\nu} dZ^2,$$  

(54)

where $v$ is determined by

$$\log v = \int \frac{Ld\mu}{\mu^{n+1}} \left( \frac{1}{2n-1} \left[ (M-2n)\mu^{2n} \right. \right.$$  

$$- \frac{4\Lambda \mu^2}{M+2} \left. \right] + \frac{N}{\mu^{n+1}} \right) + \log b.$$  

(55)

The rest energy density ($\rho$), the string tension ($\lambda$) and the pressure ($p$) for the model (54) are given by

$$\rho = \lambda = \left( \frac{4n + 1}{4} \right) \frac{N}{T^{k_1}} + k_2 \Lambda,$$  

(56)

$$p = \xi \theta + \frac{n(4n + 1)}{2n - 1} \frac{N}{T^{k_1}} - k_2 \Lambda.$$  

(57)
4.1 Model I: solution for $\xi = \xi_0$

When $m = 0$, (45) reduces to $\xi = \xi_0 = \text{constant}$. Hence in this case (57), with the use of (17) leads to

$$p = \frac{n(4n + 1)}{2n - 1} \frac{N}{T^{k_1}} - k_2 \Lambda + (n + 1)\xi_0 \left[ \frac{L^2}{(4n + 1)T^{2(n+1)}} - \frac{T^{k_1}}{Q \Lambda + N} \right]^{\frac{1}{2}}. \tag{58}$$

4.2 Model II: solution for $\xi = \xi_0 \rho$

When $m = 1$, (45) reduces to $\xi = \xi_0 \rho$. Hence in this case (57), with the use of (17) and (56) leads to

$$p = \frac{n(4n + 1)}{2n - 1} \frac{N}{T^{k_1}} - k_2 \Lambda + (n + 1)\xi_0 \left[ \frac{L^2}{(4n + 1)T^{2(n+1)}} - \frac{T^{k_1}}{Q \Lambda + N} \right]^{\frac{1}{2}} \times \left[ \frac{(4n + 1)N}{4T^{k_1}} + k_2 \Lambda \right]. \tag{59}$$

4.3 Model III: solution for $\xi = \xi_0 \rho^\frac{1}{2}$

When $m = \frac{1}{2}$, (45) reduces to $\xi = \xi_0 \rho^{\frac{1}{2}}$. Hence in this case (57), with the use of (17) and (56) leads to

$$p = \frac{n(4n + 1)}{2n - 1} \frac{N}{T^{k_1}} - k_2 \Lambda + (n + 1)\xi_0 \left[ \frac{L^2}{(4n + 1)T^{2(n+1)}} - \frac{T^{k_1}}{Q \Lambda + N} \right]^{\frac{1}{2}} \times \left[ \frac{(4n + 1)N}{4T^{k_1}} + k_2 \Lambda \right]. \tag{60}$$

Some physical aspects of the models: The expressions for the scalar of expansion ($\theta$) and for the shear tensor ($\sigma_i^j$) for model (54) are given by

$$\theta = (n + 1) \left[ \frac{L^2}{(4n + 1)T^{2(n+1)}} - \frac{T^{k_1}}{Q \Lambda + N} \right]^{\frac{1}{2}}, \tag{61}$$

$$\sigma_1^1 = \frac{(2n - 1)}{3} \left[ \frac{L^2}{(4n + 1)T^{2(n+1)}} - \frac{T^{k_1}}{Q \Lambda + N} \right]^{\frac{1}{2}}, \tag{62}$$

$$\sigma_2^2 = -\frac{1}{6} \left[ (2n - 1) \right] \left[ \frac{L^2}{(4n + 1)T^{2(n+1)}} - \frac{T^{k_1}}{Q \Lambda + N} \right]^{\frac{1}{2}}, \tag{63}$$

5 Concluding remarks

Some Bianchi type I string cosmological models with a perfect fluid as the source of matter are obtained in presence and absence of magnetic field. Generally, the models are expanding, shearing and non-rotating. In all these models, we observe that they do not approach isotropy for large value of time $T$. Our solutions generalize the solutions recently obtained by Bali and Anjali (2006) and Bali and Upadhay (2003). Thus the effect of bulk viscosity is to introduce change in the perfect fluid model.

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References

Cylindrically symmetric inhomogeneous universe with electromagnetic field in string cosmology

nirudh Pradhan · Anju Rai · Sheel Kumar Singh

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Abstract Cylindrically symmetric inhomogeneous string cosmological model in presence of electromagnetic field is investigated. We have assumed that $F_{23}$ is the only non-vanishing component of $F_{ij}$. To get the deterministic solution, it has been assumed that the expansion ($\theta$) in the model proportional to the eigen value $\sigma^1_j$ of the shear tensor $\sigma^j_j$. The physical and geometric aspects of the model are also discussed.

Keywords Cosmic string · Electromagnetic field · Inhomogeneous universe

Introduction

Recent years, cosmic strings have drawn considerable attention among researchers for various aspects such the study of the early universe. These strings arise during the phase transition after the big bang explosion as the temperature goes down below some critical temperature as predicted by grand unified theories (Zel'dovich et al. 1975; Kibble 1976, 1980; Everett 1981; Vilenkin 1981). It is believed that cosmic strings give rise to density perturbations which lead to formation of galaxies (Zel'dovich 1980). These cosmic strings have stress energy and couple to the gravitational field. Therefore, it is interesting to study the gravitational effect which arises from strings. The general treatment of strings was initiated by Letelier (1979, 1983) and Stachel (1980). The occurrence of magnetic fields on galactic scale is well-established fact today, and their importance for a variety of astrophysical phenomena is generally acknowledged as pointed out Zel'dovich et al. (1983). Also Harrison (1973) has suggested that magnetic field could have a cosmological origin. As a natural consequences, we should include magnetic fields in the energy-momentum tensor of the early universe. The choice of anisotropic cosmological models in Einstein system of field equations leads to the cosmological models more general than Robertson-Walker model (Robertson and Walker 1936). The presence of primordial magnetic fields in the early stages of the evolution of the universe has been discussed by several authors (Misner et al. 1973; Asseo and Sol 1987; Pudritz and Silk 1989; Kim et al. 1991; Perley and Taylor 1991; Kronberg et al. 1991; Wolfe et al. 1992; Kulsrud et al. 1997; Barrow 1997). Melvin (1975), in his cosmological solution for dust and electromagnetic field suggested that during the evolution of the universe, the matter was in a highly ionised state and was smoothly coupled with the field, subsequently forming neutral matter as a result of universe expansion. Hence the presence of magnetic field in string dust universe is not unrealistic.

Banerjee et al. (1990) have investigated an axially symmetric Bianchi type I string dust cosmological model in presence and absence of magnetic field using a supplementary condition $\alpha = \alpha(t)$ and $\beta = \beta(t)$ are...

Cylindrically symmetric space-time play an important role in the study of the universe on a scale in which anisotropy and inhomogeneity are not ignored. Inhomogeneous cylindrically symmetric cosmological models have significant contribution in understanding some essential features of the universe such as the formation of galaxies during the early stages of their evolution. Bali and Tyagi (1989) and Pradhan et al. (2001, 2006) have investigated cylindrically symmetric inhomogeneous cosmological models in presence of electromagnetic field. Barrow and Kunze (1997, 1998) found a wide class of exact cylindrically symmetric flat and open inhomogeneous string universes. In their solutions all physical quantities depend on at most one space coordinate and the time. The case of cylindrical symmetry is natural because of the mathematical simplicity of the field equations whenever there exists a direction in which the pressure equal to energy density. Baysal et al. (2001), Kilinc and Yavuz (1996) have investigated some string cosmological models in cylindrically symmetric inhomogeneous universe. Kilinc and Yavuz (2000) have also studied string cosmology with magnetic field in cylindrically symmetric space-time.

Recently Pradhan et al. (2007) have investigated a new class of string cosmological models in cylindrically symmetric inhomogeneous universe. Motivated by the situation discussed above, in this paper, we have generalized one case of these solutions by including electromagnetic field tensor. We have taken string and electromagnetic field together as the source gravitational field as magnetic field are anisotropic stress source and low strings are one of anisotropic stress source as well. This paper is organized as follows: The metric and field equations are presented in Sect. 2. In Sect. 3, we deal with the solution of the field equations in presence of electromagnetic field for perfect fluid distribution. In Sect. 4, we have given the concluding remarks.

2 The metric and field equations

We consider the metric in the form

$$ds^2 = A^2(dx^2 - dt^2) + B^2dy^2 + C^2dz^2,$$

where $A$, $B$ and $C$ are functions of $x$ and $t$. The energy momentum tensor for the cloud of strings with electromagnetic field has the form

$$T^i_j = p u^i u^j - \lambda_{x^i x^j} + E^i_j,$$

where $u^i$ and $x^i$ satisfy conditions

$$u^i u_i = -x^i x_i = -1,$$

and

$$u^i x_i = 0.$$

Here $\rho$ being the rest energy density of the system of strings, $\lambda$ the tension density of the cloud of strings, $x^i$ is a unit space-like vector representing the direction of strings so that $x^1 \neq 0$ and $x^2 = x^3 = x^4$ and $u^i$ is the four velocity vector satisfying the following conditions

$$g_{ij} u^i u^j = -1.$$

In (2), $E^i_j$ is the electromagnetic field given by Lichnerowicz (1967)

$$E^i_j = \tilde{\mu} \left[ h_i h^i \left( u_i u^j + \frac{1}{2} g^j_i \right) - h_i h^j \right].$$

where $\tilde{\mu}$ is the magnetic permeability and $h_i$ the magnetic flux vector defined by

$$h_i = \frac{1}{\tilde{\mu}} \ast F_{i\mu} u^\mu,$$

where the dual electromagnetic field tensor $\ast F_{ij}$ is defined by Synge (1960)

$$\ast F_{ij} = \frac{\sqrt{-g}}{2} \epsilon_{ijkl} F^{kl}.$$  

Here $F_{ij}$ is the electromagnetic field tensor and $\epsilon_{ijkl}$ is the Levi-Civita tensor density.

In the present scenario, the comoving coordinates are taken as

$$u^i = \left( 0, 0, 0, \frac{1}{A} \right).$$
c incident magnetic field is taken along x-axis so that
\[ \neq 0, \quad h_2 = 0 = h_3 = h_4. \]  
(10)

e first set of Maxwell's equation
\[ i_{;k} + F_{jk;i} + F_{ki;j} = 0, \]  
(11)

reduces to
\[ 3 = \text{constant} = H \text{ (say)}. \]  
(12)

The Einstein's field equations (with \( \kappa = 1 \))
\[ -\nabla_i R_{ij} = -T_{ij}, \]  
(13)

with
\[ \rho = \frac{A_1}{A}, \frac{A_2}{A}, \frac{B_1}{B}, \frac{B_2}{B}, \frac{C_1}{C}, \frac{C_2}{C} \]  
(25)

and
\[ \lambda = \frac{A_3}{A}, \frac{A_4}{A}, \frac{B_3}{B}, \frac{B_4}{B}, \frac{C_3}{C}, \frac{C_4}{C} \]  
(26)

Thus, due to all the three (strong, weak and dominant) energy conditions, one finds \( \rho \geq 0 \) and \( \rho_\rho \geq 0 \), together with the fact that the sign of \( \lambda \) is unrestricted, it may take values positive, negative or zero as well.

3 Solutions of the field equations

As in the case of general-relativistic cosmologies, the introduction of inhomogeneities into the string cosmological equations produces a considerable increase in mathematical difficulty: non-linear partial differential equations must now be solved. In practice, this means that we must proceed either by means of approximations which render the non-linearities tractable, or we must introduce particular symmetries into the metric of the space-time in order to reduce the number of degrees of freedom which the inhomogeneities can exploit.

The research on exact solutions is also based on some physically reasonable restrictions used to simplify the Einstein equation. Equation (18) contains three unknowns \( A, B \) and \( C \). Hence to get a deterministic solution, let us assume that expansion \( (\theta) \) in the model is proportional to the eigen value \( \sigma^2 \) of the shear tensor \( \sigma^i j \). This condition leads to
\[ A = (BC)^{\sigma^2}. \]  
(27)

\[ \dot{\rho} = \frac{A_1}{A} + \left( \frac{A_2}{A} + \frac{B_2}{B} + \frac{C_2}{C} \right) \rho = 0 \]  
(25)

With the help of (1–4, 9, 10), the Bianchi identity \( (T_{ij}) \) reduced to two equations:

\[ \frac{A_2}{A} + \left( \frac{B_2}{B} + \frac{C_2}{C} \right) = 0, \]  
(18)

where \( \dot{\rho} \) is the derivative of the metric (1). Using the field equations and the relations (19) and (20) one obtains the Raychaudhuri's equation as
\[ \frac{\dot{\rho}}{\rho} = \frac{2}{3} g^2 - \frac{1}{2} \rho_\rho, \]  
(23)

where \( g \) is the determinant of the metric (1). Using the field equations and the relations (19) and (20) one obtains the Raychaudhuri's equation as

\[ \frac{\dot{\rho}}{\rho} = \frac{2}{3} g^2 - \frac{1}{2} \rho_\rho. \]  
(23)

where dot denotes differentiation with respect to \( t \) and
\[ R_{ij}\dot{u}^i = \frac{1}{2} \rho_\rho. \]  
(24)
where $n$ is a constant. The above condition has also been used by Bali and Anjali (2006) as an ad hoc basis to get the solution. Equations (15) and (16) lead to
\[
\frac{B_{tt}}{B} - \frac{B_{xx}}{B} = \frac{C_{tt}}{C} - \frac{C_{xx}}{C} = \text{(constant)}. \tag{28}
\]
Using (27) in (18) reduces to
\[
\frac{B_{tt}}{B} + \frac{C_{xx}}{C} - 2n\left(\frac{B_t}{B} + \frac{C_t}{C}\right)\left(\frac{B_x}{B} + \frac{C_x}{C}\right) = 0. \tag{29}
\]
To get the deterministic solution, we assume
\[
B = f(x)g(t) \quad \text{and} \quad C = h(x)k(t)
\]
and discuss its consequences below in this paper.

In this case (29) reduces to
\[
\frac{f_{xx}}{f} \frac{h_{xx}}{h} = \frac{(2n-1)(k_t/k) + 2n(g_t/g)}{(2n-1)(g_t/g) + 2n(k_t/k)} = K \text{ (constant)}, \tag{30}
\]
which leads to
\[
\frac{f_x}{f} = K \frac{h_x}{h}, \tag{32}
\]
and
\[
k_t/k - K - 2nK - 2n = a \text{ (constant)}. \tag{33}
\]
From (32) and (33), we obtain
\[
f = ah^K, \tag{34}
\]
and
\[
k = bg^a. \tag{35}
\]
where $a$ and $b$ are integrating constants. Equation (28) reduces to
\[
\frac{g_{tt}}{g} - \frac{k_{tt}}{k} = \frac{f_{xx}}{f} - \frac{h_{xx}}{h} = N, \tag{36}
\]
where $N$ is a constant. From (33) and (36) we obtain
\[
8gn + ag^2 = -\frac{N}{a-1} g^2 \tag{37}
\]
which leads to
\[
g = \beta^{\frac{1}{a+1}} \cosh^\frac{a}{a+1} (bt + t_0), \tag{38}
\]
where $b = \sqrt{\frac{N(a+1)}{(1-a)}}$ and $\beta$, $t_0$ are constants of integration. Thus, from (35) we get
\[
k = \delta^\frac{a}{a+1} \cosh^\frac{a}{a+1} (bt + t_0). \tag{39}
\]
From (32) and (36), we obtain
\[
h_{xx} + Kh^2 = -\frac{N}{K - 1} h^2, \tag{40}
\]
which leads to
\[
h = \ell \xi_1 \cosh \xi_1 (rx + x_0), \tag{41}
\]
where $r = \sqrt{\frac{N(a+1)}{(K-1)}}$ and $\ell$, $x_0$ are constants of integration. Hence, from (34) we have
\[
f = \alpha \ell \xi_1 \cosh \xi_1 (rx + x_0). \tag{42}
\]
Hence, we obtain
\[
B = fg = Q \cosh \xi_1 (rx + x_0) \cosh \xi_1 (bt + t_0), \tag{43}
\]
\[
C = hk = R \cosh x + r \cosh \xi_1 (bt + t_0), \tag{44}
\]
\[
A = (BC)^a = M \cosh^a (rx + x_0) \cosh^a (bt + t_0), \tag{45}
\]
where
\[
Q = \alpha \beta^{\frac{1}{a+1}} \ell \xi_1, \tag{46}
\]
\[
R = \delta \beta^{\frac{a}{a+1}} \ell \xi_1, \tag{47}
\]
\[
M = (QR)^a. \tag{48}
\]
Hence, the metric (1) takes the form
\[
ds^2 = M^2 \cosh^{2n}(rx + x_0) \cosh^{2n}(bt + t_0)(dx^2 - dt^2) + Q^2 \cosh^{2n}(rx + x_0) \cosh^{2n}(bt + t_0)dy^2 + R^2 \cosh^{2n}(rx + x_0) \cosh^{2n}(bt + t_0)dz^2. \tag{49}
\]
By using the following transformation
\[
X = x + x_0, \tag{50}
\]
\[
Y = Qy, \tag{51}
\]
\[
Z = Rz, \tag{52}
\]
\[
T = t + t_0, \tag{53}
\]
the metric (46) reduces to
\[
ds^2 = M^2 \cosh^{2n}(rX) \cosh^{2n}(bt + t_0)(dX^2 - dT^2) + Q^2 \cosh^{2n}(rX) \cosh^{2n}(bt + t_0)dy^2 + R^2 \cosh^{2n}(rX) \cosh^{2n}(bt + t_0)dz^2. \tag{54}
\]
The energy density ($\rho$), the string tension density ($\lambda$), the particle density ($\rho_p$), the scalar of expansion ($\theta$), shear ten-
\[ \left( \frac{b^2(n + \frac{a}{(a+1)^2}) \tanh^2(bT)}{M^2 \cosh^{2n}(rX) \cosh^{2M}(bT)} \right) \left( \frac{1}{\cosh^2(bT) \cosh^2(rX)} \right) \]

(49)

\[ \left( \frac{b^2(n + \frac{a}{(a+1)^2}) \tanh^2(bT)}{M^2 \cosh^{2n}(rX) \cosh^{2M}(bT)} \right) \left( \frac{1}{\cosh^2(bT) \cosh^2(rX)} \right) \]

(50)

here

\[ \frac{H^2}{2 \mu} = \frac{b(n + 1) \tanh(bT)}{M \cosh^n(rX) \cosh^M(bT)} \]

(52)

\[ \frac{b^2 \tanh^2(bT) [(a + 1)^2(n^2 - n + 1) - 3a]}{3(a + 1)^2 M^2 \cosh^{2n}(rX) \cosh^{2M}(bT)} \]

(53)

\[ \frac{\sqrt{-\rho - \lambda}}{\rho - \lambda} = \frac{b^2 \tanh^2(bT) [(a + 1)^2(n^2 - n + 1) - 3a]}{3(a + 1)^2 M^2 \cosh^{2n}(rX) \cosh^{2M}(bT)} \]

(54)

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References


