CHAPTER-1

INTRODUCTION AND THEORY OF FILTERS
# CHAPTER-1

## Introduction and Theory of Filters

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INTRODUCTION AND THEORY OF FILTERS

1.0 Introduction:

A filter is basically a device designed to isolate, suppress or pass a group of signals from a combination of signals. Filters are widely used in signal processing, communications systems and in the electronic instrumentation.

Electrical Filters can be categorized at many stages based on the type of signals processed into analog and digital filters, based on the type of elements used in their construction into passive and active filters, and based on the type of function achieved into low pass, band pass, high pass and band stop (reject) filters.

Passive and active filters are distinguished by the passivity of the elements used in their construction. If an element needs power or unable of power gain, then it is known as a passive element. An element that is not passive is known as active elements.

Passive filters are constructed completely with passive elements i.e. (resistors, capacitors, and inductors) and do not use any active elements or power supply. An active filter uses an active element such as operational amplifiers, transistors, and operational transconductance amplifiers along with or without use any passive elements. The most used of filters are low pass, band pass, high pass, band stop (band reject) and all pass filters.

Electrical filters permeate modern technology so much that it is difficult to find any electronic system that does not employ a filter in one form or another [1]. The frequencies that separate the various pass and attenuation bands are called the cut-off frequencies [2]. Active filter without the capacitor and inductor is called an active-R filter and has received much attention due to its potential advantages in terms of miniaturization, ease of design and high-frequency performance [3-8]. The patterns of pass-bands and stop-bands give rise to most common filters as low pass, high pass, band pass, and band stop [9]. A switched-capacitor can replace a resistor [10]. MOSFET technology can be used for designing the switched-capacitor circuits [11-12]. The filter circuits using switched-capacitor allow very sophisticated, accurate and tunable analog circuits to be manufactured. Many of the circuits proposed the working of only one type
of operation [13]. Analog filters are important building blocks and widely used for continuous-time signal processing [14].

In recent years, the current-mode analog signal processing circuit techniques have received wide attention due to the high accuracy, the wide signal bandwidth and the simplicity of implementing signal operations [15,16]. Current-mode filter theoretically should exhibit high output impedance (Ideally infinite) and low input impedance (Ideally Zero) [17].

The operational transconductance amplifier (OTA) is a differential voltage-controlled current source (DVCCS). Its transconductance is used as a design parameter in the same way as the R’s are used in conventional active-RC filters, which represents the ratio of the output current to the input voltage, i.e., \( \frac{I_{out}}{V_d} \). It is adjustable over several decades by a supplied \( I_{abc} \) (amplifier bias current) which lends electronic tenability to circuit parameters [18, 19].

Analogue continuous time active filters utilizing an operational amplifier (OA) pole and the transconductance control property of the operational transconductance amplifier (OTA) have received considerable attention recently. These filters do not need to employ additional passive elements and are therefore sometimes called active-only filters [20].

### 1.1 Formulation of the Research Problem:

#### 1.1.1 Statement of the Research Problem:

Any electrical active filter design by an engineer should aim to be extremely stable in practical use.

#### 1.1.2 Motivations of the Research Work:

Any attempt of creating a research work intends to carry some motivations which are the reasons behind that work. In recent years, there has been considerable much attention and interest in the development of active filter and its applications because of the increasing concern over power quality, at both distribution and consumer levels. There is a need to control reactive power and voltage stability at transmission levels, and its potential advantages in terms of miniaturization, easy design, high frequency performance, and signal processing applications. The voltage-mode and current-mode active filters are attractive because of the higher slew rate, the low power
consumption, and the wide bandwidth. Active filters are extensively used in communication system, instrumentation system, control systems entertainment electronics, sonar system, etc.

1.1.3 Aims of the Research Work:

The general aim of this research work is to design, implement, and study the stability of 3\textsuperscript{rd}-order voltage-mode and current-mode active filters.

1.1.3.1 Objectives of the Research Work:

The research work presented in this thesis is an attempt to achieve the following objectives:

- To design, implement and study the stability of 3\textsuperscript{rd}-order voltage-mode active-R filter.
- To design, implement and study the stability of 3\textsuperscript{rd}-order voltage-mode switched-capacitor filter.
- To design, implement and study the stability of 3\textsuperscript{rd}-pole current-mode active-R filter.
- To design, implement and study the stability of 3\textsuperscript{rd}-pole current-mode OTA filter.

1.1.4 Significance of the Research Work:

In the previous decades, the stability of a higher-order active filter problem has no tackled by the interested researchers and engineers. The present thesis suggests solutions for the stability of third-order active filters. A novel four active filters’ circuits have designed and studied for different parameters, which are easier to design and easier to adjust the gain and frequency. The proposed active filters circuits implement three filter functions low pass, band pass, and high pass simultaneously in each single circuit, except for OTA filter, they are getting by adjusting the parameters of OTA. The active filters circuits yield very good accuracy with a steep roll-off and good performance characteristics.

1.2 Review of Literature:

An active-R filter with only one capacitor has been proposed (1972) [21]. A general second order active-R filter using only resistance and operational amplifier has been proposed (1975) [22]. An active-R filter using only resistance and an operational
amplifier which realizes all the low pass, high pass, and band pass responses simultaneously at three different terminals has been proposed [23]. An active-R filter with four output nodes has been proposed [24]. Network theory and filter design has been studied in 1986 [25]. Second order active-R filter by using only inductor and capacitors has been studied (1989) [26]. An active-R biquadratic filter with positive feedback and feedforward signal has been suggested (2002) [27]. Second order active-R filter with multiple feedback has been proposed (2003) [28]. A third order active-R filter with feedforward input signal has been suggested (2003) [29]. An analysis of Switched-Capacitor Common-Mode Feedback Circuit has been studied (2003) [30]. Analysis and Design of a New Current Controlled Switched Capacitor Filter has been suggested (2005) [31]. New switched capacitor fractional order integrator has been suggested in 2007 [32]. Second order current-mode active-R filter by using operational amplifiers and resistors has been proposed (2008) [33]. Active low-pass Butterworth filters and design methods have been proposed (2009) [34]. A new active element for the realization of current-mode analog blocks current follower transconductance amplifier (CFTA) has been suggested (2009) [35]. Active-only universal filter using four dual current output operational transconductance amplifiers (OTAs) and three operational amplifiers (OAs) has been studied (2011) [36]. Differential operation transconductance amplifier (FD-OTA) using the commercially available ICs has been proposed (LT1228) (2012) [37]. A continuously high tunable lumped bandpass filter has been proposed (2012) [38]. Telescopic operational transconductance amplifier has been designed (2013) [39]. Operational transconductance amplifier using pspice has been implemented (2014) [40]. A new OTA-C current-mode second-order filters with single input current using dual-output OTA has been proposed (2014) [41]. An active second order RC band-pass filter for different values of quality factor Q has been implemented (2014) [42]. U.S. Patent No. 3,569,851 to Jaumann; Andreas (Ebenhausen, DT) discloses an electrical filter circuit, which comprises an amplifier with a pair of push-pull terminals on its input and/or output side [43]. U.S. Patent No. 3,707,685 to Geffen; Philip R. (Laurel, MD) discloses Q-Invariant Active Filters. It produced three forms; low pass, band pass, and high pass. In each form, the element values are preselected to provide zero sensitivity of the pole Q to the frequency-determining passive elements [44]. U.S. Patent No. 3,919,658 to Friend; Joseph John (Freehold Twp., Monmouth County, NJ) discloses an active-RC filter circuit, which uses an operational amplifier and a plurality of resistors and
capacitors to obtain a variety of filter transfer functions [45]. U.S. Patent No. 3,955,150 to Soderstrand; Michael A. (San Francisco, CA) discloses an active-R filter [46]. U.S. Patent No. 3,972,006 to Ruegg; Frank A. (Brea, CA) discloses a band-pass filter. The present invention relates to active filter circuits, and more particularly to an improved pre-tuned single frequency band-pass filter utilizing two stages, each comprised of a differential input high-performance operational amplifier [47]. U.S. Patent No. 4,189,681 to Lawson; Kenneth D. (Cataumdet, MA), Brown; Neil L. (Falmouth, MA) discloses a band-pass filter having low passband phase shift [48]. U.S. Patent No. 4,356,451 to Wilson; Harold E. (Miami, FL) discloses an active band-pass filter, which comprises a differential amplifier having a pair of input resistors [49]. U.S. Patent No. 4,382,233 to Hofer; Bruce E. (Beaverton, OR) discloses a multiple-feedback path band-pass/band-reject filter, which exhibits a nearly constant Q over a predetermined frequency range, and which is tunable by means of a single variable impedance element is provided [50]. U.S. Patent No. 4,383,230 to Manzolini; David B. (Rome, NY) discloses voltage tuned active filter and circuitry simulating a capacitance and an inductance [51]. U.S. Patent No. 4,659,995 to Feistel; Karl H. (Nuremberg, DE) discloses an active fourth-degree filter element. It connected in series with other such elements in forming a band-pass or band-stop filter with adjustable bandwidth [52]. U.S. Patent No. 6,433,706 to Guimaraes; Homero L. (Gurnee, IL) discloses a current-mode filter with a transfer function having complex zeros [53]. U.S. Patent No. 7,682,703 to Cavazzoni; Roberto (42044 Gualtieri (Reggio Emilia), IT) discloses an active filter. An active filter comprising a first stage, a second stage and a third stage, each of them being provided with a respective operational amplifier [54]. U.S. Patent No. 8,502,597 to Khatibi; Arezou (San Jose, CA), Bicakci; Ara (San Jose, CA), Gaethke; Rainer (San Francisco, CA) discloses a low-pass filter design, with high-quality factor (Q) [55]. U.S. Patent No. 8,682,620 to Granger-Jones; Marcus (Scotts Valley, CA), Khlat; Nadim (Midi-Pyrenees, FR), Bauder; Ruediger (Feldkirchen-Westerham, DE) discloses a power amplifier with tunable band-pass and notch filter, which relates to a multi-band RF power amplifier (PA) module [56].

1.3 Classifications of Filters:

Filters can be classified based on the type of:

- Signals processed
- Elements used in their construction, and
- Functions performed

The classifications of filters are shown in Fig. 1.1.

Fig. 1. 1: Classifications of filters.

1.3.1 Classifications of Filters Based on the Type of Signals Processed:
Filters can be classified based on the type of signals processed into

1.3.1.1 Analog Filters:
Analog filters are designed to process analog signals. The analog filters can be classified based on the type of elements used in their construction into
A. Passive Filters:

A passive filter is a type of analog electronic filter which constructed completely with passive elements i.e., (resistors, capacitors, and inductors). The passive filters can be classified based on the type of functions performed by

a. Passive Low Pass Filter (PLPF)
b. Passive High Pass Filter (PHPF)
c. Passive Band Pass Filter (PBPF), and
d. Passive Band Stop (Reject) Filter (PBSF) or (PBRF)

B. Active Filters:

An active filter is also a type of analog electronic filter which constructed completely with active elements i.e. (transistors, operational amplifiers, and operational transconductance amplifiers) along with or without passive components. There are two types of active filter, one is called active-only filter, which is constructed completely with active components, such as OTA filter, and the other is called passive and active combinations filter, which is constructed with passive and active components, such as active-R and switched-capacitor filters.

- **Active-only Filters:**

  The active-only filters can be classified based on the type of functions performed by

  a. Active-Only Low Pass Filter (AOLPF)
b. Active-Only High Pass Filter (AOHPF)
c. Active-Only Band Pass Filter (AOBPF), and
d. Active-only Band Stop (Reject) Filter (AOB SF) or (AOBRF)

- **Passive & Active Combinations Filters:**

  The passive & active combinations filter can be classified based on the type of functions performed by

  a. Passive & Active Combinations Low Pass Filter (PACLPF)
b. Passive & Active Combinations High Pass Filter (PACHPF)
c. Passive & Active Combinations Band Pass Filter (PACBPF), and
d. Passive & Active Combinations Band Stop (Reject) Filter (PACBSF) or (PACBRF)
Active filters are most extensively used in the field of instrumentation system, communication system, audio system, biomedical system, and signal processing.

1.3.1.2 Digital Filters:

Digital filters are designed to process digital signals. A digital filter is any electronic filter that works by performing digital mathematical operations on an intermediate form of a signal. Digital filters can achieve virtually any filtering effect that can be expressed as a mathematical function or algorithm.

A digital filter is simply a discrete-time, discrete-amplitude convolver.

A digital filter uses a digital processor to perform numerical calculations on sampled values of the signal.

1.4 Characteristics of Basic Filters:

There are four basic filters as following:

- Low pass filter (LPF)
- High pass filter (HPF)
- Band pass filter (BPF), and
- Band reject/stop filter (BRF or BSF)

1.4.1 Characteristics of Low Pass Filter:

The low pass filter passes all frequencies below cut-off frequency i.e. -3dB frequency. Low pass filter has one pass band region and one stop band region.

Fig. 1. 2: Characteristics of a low pass filter.

The characteristics of low pass filter (LPF) are shown in the Fig. 1.2.
1.4.2 Characteristics of High Pass Filter:

The high pass filter passes all frequencies above cut-off frequency i.e. -3dB frequency. High pass filter has one pass band region and one stop band region.

Fig. 1. 3: Characteristics of a high pass filter.

The characteristics of high pass filter (HPF) are shown in the Fig. 1.3.

1.4.3 Characteristics of Band Pass Filter:

The band pass filter passes a certain band of frequencies (all frequencies between lower \( f_{c1} \) and higher \( f_{c2} \) cut-off frequencies). Band pass filter has one pass band region and two stop band regions.

Fig. 1. 4: Characteristics of a band pass filter.

The characteristics of band pass filter (BPF) are shown in the Fig. 1.4.
1.4.4 Characteristics of Band Reject Filter:

The band reject filter attenuates a certain band of frequencies (all frequencies between lower \( f_{C1} \) and higher \( f_{C2} \) cut-off frequencies). Band reject filter has two pass band regions and one stop band region. This filter is also called as band stop (BSF) or band elimination filters (BEF).

Fig. 1. 5: Characteristics of a band reject filter.

The characteristics of band reject filter (BRF) are shown in the Fig. 1.5.

1.5 Principle of Positive and Negative Feedback Amplifiers:

An operational amplifier, which uses feedback is called a feedback amplifier. A feedback amplifier is sometimes referred to as a closed-loop amplifier because the feedback forms a closed-loop between the input and output terminals [57]. A feedback amplifier basically involves of two parts one is an operational amplifier and the others is a feedback circuit. The feedback circuit can be constructed using passive elements, active elements, or passive and active combinations elements. A closed-loop amplifier can be represented using two blocks, one for the operational amplifier, and the other for a feedback circuit.

For an ordinary amplifier, i.e. without feedback, \( v_i \) and \( v_o \) are the input voltage and output voltage respectively. If \( A \) is the voltage gain of the operational amplifier, then

\[
A = \frac{v_o}{v_i} \quad (1.1)
\]

The gain \( A \) is also called as open-loop gain.

The principle of positive and negative feedback of an operational amplifier is shown in Fig. 1.6.
The Fig. 1.6 has two parts: an operational amplifier with gain (A) and a feedback circuit with a feedback ratio (B), it is connected to the input voltage, so that the input voltage becomes \( v_i + Bv_o \) depending whether the feedback is positive or negative. This voltage is amplified by amplifier (A) times. Suppose negative feedback, we have
\[
A (v_i - Bv_o) = v_o
\]
Therefore
\[
\frac{v_o}{v_i} = \frac{A}{1 + BA} \tag{1.3}
\]

The left-hand side of eq (1.3) represents the amplifier gain with feedback, i.e.
\[
A_f = \frac{A}{1 + AB} \tag{1.4} \quad for \ negative \ feedback,
\]

The negative feedback amplifier is an electronic amplifier, which subtracts a fraction of its output from its input so that the negative feedback resists the original signal.
\[
A_f = \frac{A}{1 - AB} \tag{1.5} \quad for \ positive \ feedback
\]

The positive feedback amplifier is an electronic amplifier, which adds a fraction of its output to its input so that the positive feedback does not resist the original signal. Where the term \( AB \) is called as feedback factor and \( B \) is as feedback ratio. The term \( 1 \pm AB \) is known as loop gain \( A_f \) with feedback is closed loop gain.

**1.5.1 Benefits of Negative Feedback:**

The benefits of negative feedback are as the following:

a. Highly stabilized gain.
b. Reduction in non-linear distortion, noise, and gain.
c. Increased bandwidth, and circuit stability.
d. Increases input impedance and decreases output impedance.
e. Less amplitude, frequency, harmonic, and phase distortion.

1.6 Transfer Function of Filter:

An analog filter is a linear system that has an input \( x_i(t) \) and output \( x_o(t) \) signals.

![Fig. 1.7: Block diagram of a system.](image)

The relation between a particular input \( x_i(t) \) and output \( x_o(t) \) signals of a system can be written as

\[
\begin{align*}
    a_n \frac{d^n x_o(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x_o(t)}{dt^{n-1}} + \cdots + a_1 \frac{dx_o(t)}{dt} + a_0 x_o(t) \\
    = b_m \frac{d^m x_i(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x_i(t)}{dt^{m-1}} + \cdots + b_1 \frac{dx_i(t)}{dt} + b_0 x_i(t)
\end{align*}
\]

Where \( a \)'s and \( b \)'s are combinations of system parameters. The transfer function of a system can be determined by taking the Laplace transform of the transfer function of a system. Taking Laplace transform and assuming all initial condition equal to zero, we get

\[
G(s) = \frac{x_o(s)}{x_i(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}
\]

Where \( G(s) \) is the transfer function of a filter, and it is defined as the ratio of Laplace transformation of output \( x_o(s) \) signal of a system to Laplace transformation of input \( x_i(s) \) signal of a system, i.e.

\[
G(s) = \frac{x_o(s)}{x_i(s)}
\]
Where the order of \( x_o(s) \) never exceeds the order of \( x_i(s) \). The order of \( x_i(s) \) is called the order of a filter.

The transfer function may take the following form:

I. The ratio of transfer of a voltage to transfer of another voltage, it is called the voltage transfer function.

II. The ratio of transfer of a voltage to transfer of a current, it is called impedance transfer function.

III. The ratio of transfer of a current to transfer of another current, it is called the current transfer function.

IV. The ratio of transfer of a current to transfer of a voltage, it is called admittance transfer function.

1.6.1 Zero Order System:

If all \( a \)'s and \( b \)'s, except \( a_0 \) and \( b_0 \) are equal to zero the eq (1.7) becomes

\[
G(s) = \frac{x_o(s)}{x_i(s)} = \frac{b_0}{a_0} = k
\]

(1.9)

Where \( k \) is a constant, so this system is called zero-order system.

1.6.2 First Order System:

If all \( a \)'s and \( b \)'s, except \( a_0 \), \( a_1 \) and \( b_0 \) are equal to zero the eq (1.7) becomes

\[
G(s) = \frac{x_o(s)}{x_i(s)} = \frac{b_0}{a_1s + a_0} = \frac{k}{Ts + 1}
\]

(1.11)

\[
G(s) = \frac{x_o(s)}{x_i(s)} = \frac{k/\tau}{s + 1/\tau}
\]

(1.12)

Any system obeys eq. (1.10), this system is called first order system.

1.6.3 Second Order System:

If all \( a \)'s and \( b \)'s, except \( a_0 \), \( a_1 \), \( a_2 \) and \( b_0 \) are equal to zero the eq (1.7) becomes

\[
G(s) = \frac{x_o(s)}{x_i(s)} = \frac{b_0}{a_2s^2 + a_1s + a_0}
\]

(1.13)

\[
G(s) = \frac{x_o(s)}{x_i(s)} = \frac{b_0/a_2}{s^2 + \frac{a_1}{a_2}s + \frac{a_0}{a_2}}
\]

(1.14)
\[ G(s) = \frac{x_o(s)}{x_i(s)} = \frac{kw_n^2}{s^2 + 2\zeta w_n s + w_n^2} \]  

(1.15)

Where \( k = \frac{b_0}{a_0} \) is constant, \( w_n \) is natural frequency, and \( \zeta = \frac{1}{2Q} \) is damping ratio.

Any system obeys eq. (1.13), this system is called second order system.

### 1.7 Coefficient-Matching Technique:

The coefficient-matching technique is a process of realizing a network transfer functions. This technique is only useful for simple structures, as the derivations of transfer functions become more complex as the order of filter increases [58].

In this method, a network or a filter structure is expected and its transfer function is assessed. A set of equations is obtained by comparing the coefficients of the denominator of the complex frequency variable (S) of a transfer function of a system with the coefficients of the denominator of the complex frequency variable (S) of the general transfer function of a filter.

The solution of these equations yields the required elements values. If the required elements values are not practical to implement (very low or very high values), these can be easily scaled up to get practical values.

This method is called as coefficient-matching technique (CMT). This coefficient-matching technique is applicable to passive and active filters.

### 1.8 Open-loop and Closed-loop Gains of an Op-amp:

This type of an op-amp, circuit is called the open-loop op-amp, circuit because the closed-circuit path does not exist between the output \( (v(s)_{out}) \) and the input \( (v(s)_{int}) \) signals. If an op-amp, circuit without the feedback loop is called an open-loop amplifier.

Consider the open-loop op-amp, circuit is shown in Fig. 1.8.
The open-loop voltage gain \( G(s)_{OL} \) of this op-amp, circuit is given by:

\[
G(s)_{OL} = \frac{v(s)_{out}}{v(s)_{int}} = -A_{OP} \quad (1.16)
\]

Where \( A_{OP} \) is gain of an op-amp.

Therefore, the gain of an op-amp, is often alternatively referred to as the open-loop gain.

If we apply the negative feedback to the open-loop gain, the closed-loop gain can be calculated by:

\[
G(s)_{CL} = \frac{G(s)_{OL}}{1 + G(s)_{OL}H_{FB}} \quad (1.17)
\]

Where \( G(s)_{CL} \) is called closed-loop gain, \( G(s)_{OL} \) is called open-loop gain, and \( H_{FB} \) is called feedback factor.

If \( G(s)_{OL} \gg H_{FB} \) then

\[
G(s)_{CL} = \frac{1}{H_{FB}} \quad (1.18)
\]

Consider the closed-loop op-amp, circuit is shown in Fig. 1.9.

This type of op-amp, circuit is called the closed-loop op-amp, circuit because the closed-circuit path exists between the output \( (v(s)_{out}) \) and the input \( (v(s)_{int}) \) signals.
Fig. 1. 9: Closed-loop voltage gain of an op-amp.

The closed-loop voltage gain \( G(s)_{\text{cl}} \) of this op-amp, circuit is given by:

\[
G(s)_{\text{cl}} = \frac{v(s)_{\text{out}}}{v(s)_{\text{int}}} = -\frac{R_2}{R_1} \quad (1.19)
\]

1.9 Stability Criteria of a System:

It is necessary that any filter designed by an engineer has to be extremely stable in practical use. Therefore, we always must analyze the stability of a system under various operating conditions and environments [59].

The gain and phase margins are one way to examine the stability of a feedback system. The gain margin is defined as the variation in open loop gain required to make a closed loop system unstable. It refers to the amount of gain, which can be increased or decreased without making the system unstable. The phase margin is defined as the variation in open loop phase shift required to make a closed loop system unstable. It refers to the amount of phase, which can be increased or decreased without making the system unstable.

The gain margin is the difference between the gain or magnitude curve at the phase crossover frequency \( f_{\text{PCO}} \), which gives us a phase of 180°, or \(-180°\) and 0 dB. Similarly, the phase margin is the difference between the phase curve at the gain crossover frequency \( f_{\text{GCO}} \), which gives us a gain of 0 dB and 180°. That are shown in Fig. 1.10.
Fig. 1.10: The difference of gain \((G_m)\) and phase \((P_m)\) margins.

The gain margin \((G_m)\) and phase margin \((P_m)\) can be obtained using, whether the Bode diagram or Nyquist diagram.

1.9.1 Estimating \(G_m\) and \(P_m\) Using Bode Diagram:

Now consider the Bode diagram of a system is as shown in Fig. 1.11. To obtain the gain margin \((G_m)\) and phase margin \((P_m)\) as the following.

A. To estimate the gain margin \((G_m)\):

1- Find the frequency, which gives us a phase of \(180^\circ\), or \(-180^\circ\). This frequency is called the phase cross over frequency \((f_{PCO})\).

2- Find the gain (in dB), at that frequency \((f_{PCO})\).

3- Then, we defined the gain margin \((G_m)\) as:

\[
\text{Gain Margin} \ (G_m) = 0 - G \text{ dB} \quad (1.20)
\]

B. To estimate the phase margin \((P_m)\):

1- Find the frequency, which gives us a gain of 0 dB. This frequency is called the gain crossover frequency \((f_{GCO})\).

2- Find the phase (in degree), at that frequency \((f_{GCO})\).

3- Then, we defined the phase margin \((P_m)\) as:

\[
\text{Phase Margin} \ (P_m) = 180^\circ + P \text{ (in deg)} \quad (1.21).
\]
1.9.2 Estimating $G_m$ and $P_m$ Using Nyquist Diagram:

The Nyquist plot of a system is a plot of the frequency response of the contour (return ration) with the real part plotted against the imaginary part of the transfer function of a feedback system, and it is another way to examine the stability of a feedback system.

Now consider the Nyquist diagram of a feedback system is as shown in Fig. 1.12. To obtain the gain margin ($G_m$) and phase margin ($P_m$) as the following.

A. To estimate the gain margin ($G_m$):
   1- Find the intersection between the contour of Nyquist plot and negative real axis. On our figure, this is at the gain margin point ($-a$).
   2- Take the reciprocal of the gain margin point ($a$).
   3- Then, we defined the gain margin ($G_m$) as:
      \[
      \text{Gain Margin (} G_m \text{) (in dB) } = 20 \log_{10} \left( \frac{1}{a} \right) \quad (1.22)
      \]

B. To estimate the phase margin ($P_m$):
   1- Here, we have to draw a circle through the points ((1,0), (-1,0), (0, j), (0, -j)) this circle is called unit circle.
2- Draw a line from original point (0,0) to the intersection point between the contour of Nyquist plot and unit circle.

3- Get the angle $\alpha$.

4- Then, we defined the phase margin ($P_m$) as:

$$\text{Phase Margin}(P_m) = \alpha = \tan^{-1} \frac{y}{x} \quad (1.23)$$

Thus, the gain margin ($G_m$) is the reciprocal of the gain margin point of the open-loop voltage gain at the phase cross over frequency ($f_{PCO}$), where the open-loop phase shift reaches $-180^\circ$. Phase margin ($P_m$) is the absolute value of the open-loop phase shift at gain cross over frequency ($f_{GCO}$), where the open-loop gain first equals zero.

![Diagram of Nyquist plot and stability analysis](image)

**Fig. 1.12**: Stability response of a system.

### 1.10 Stability Analysis Using the Nyquist Theorem:

The stability of a system can be concluded using the Nyquist theorem [60].

The Nyquist theorem can be interpreted for the following two stages:

1- The closed-loop system with unity negative feedback is stable if and only if the Nyquist plot of a stable open-loop model $G(s)$ does not enclose the critical point (-1, j0). If the full Nyquist plot encloses the critical point
(-1, j0) t times in a clockwise direction, then there will be t unstable closed-loop poles.

2- The closed-loop system is stable if and only if the Nyquist plot of an unstable open-loop mode $G(s)$ encloses the critical point (-1, j0) t times in an anticlockwise (counterclockwise) direction. If there are q anti-clockwise encirclements of the critical point, then there will be (q-t) unstable closed-loop poles.

If the phase margin ($P_m$) < 0, the closed-loop system is unstable. The following special cases should also be considered [46].

- If the contour of Nyquist plot does not intersect the negative real axis, the gain margin is infinite.
- If the contour of Nyquist plot intersects the negative real axis many times between critical point (-1, j0) and original point (0, j0), the one which is nearest to the critical point (-1, j0) can be considered as the gain margin point (a).
- If the contour of Nyquist plot does not intersect the unit circle, the phase margin is infinite.
- If the contour of Nyquist plot intersects the unit circle many times, the one which is nearest to the negative real axis can be considered as the phase margin point.

1.11 Characteristic Equation:

The standard block diagram of a feedback system is shown in Fig. 1.13. There are two basic blocks $G(s)$ and $H(s)$, the open-loop transfer function (the plant and controller) and the closed-loop transfer function, respectively. $X(s)$, and $Y(s)$ are the Laplace transforms of the input $x(t)$ and output $y(t)$ signals, respectively.

![Block diagram of a feedback system](Fig. 1.13: Block diagram of a feedback system.)
The relation between Laplace transforms of input $X(s)$ and output $Y(s)$ functions can be written as

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)} \quad (1.24)$$

The product $G(s)H(s)$ is named the loop gain as it is the product of the gains around the feedback loop.

The characteristic equation is given by

$$1 + G(s)H(s)|_{s=j\omega} = 0 \quad (1.25)$$

If the roots of the characteristic equation have negative real parts, the system will be stable, but if the roots of the characteristic equation have one or more than one a positive real part, the system will not be stable.

1.12 Stability Test:

There are two methods to test the stability of a transfer function of a filter, one is called Routh–Hurwitz test (Routh stability criterion) and the other is called Jury-Marden test (Jury stability criterion). Here we will choose Jury-Marden test because it has some similarity with the Routh–Hurwitz test, and it is easier than the Routh–Hurwitz test.

1.12.1 Jury-Marden Test:

To determine, whether the poles are inside or on the unit circle in the $s$-plane, we have to use the Jury-Marden test as a following [61].

Let us assume that, the polynomial of denominator of a transfer function of a filter is given by:

$$D(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + \cdots + a_N \quad (1.26)$$

Where $a_0 > 0$.

The first row of the Jury-Marden array are the coefficients of the polynomial $a_0, a_1, a_2, \ldots, a_{N-2}, a_{N-1}, a_N$, and the coefficients of the second row are the coefficients of the first row in the reverse order, as shown below.

$$a_N, a_{N-1}, a_{N-2}, \ldots, a_2, a_1, a_0$$

So, we have two rows with coefficients chosen directly from the given polynomial as follows;

$$a_0, a_1, a_2, \ldots, a_{N-2}, a_{N-1}, a_N$$
The coefficients of the third row are calculated according to the following rule:

\[ c_i = \left| \begin{array}{cc} a_0 & a_{N-i} \\ a_N & a_i \end{array} \right| \quad \text{for } i = 0, 1, 2, \ldots, (n - 1) \quad (1.27) \]

For example:

\[ c_0 = \left| \begin{array}{cc} a_0 & a_N \\ a_N & a_0 \end{array} \right|, c_1 = \left| \begin{array}{cc} a_0 & a_{N-1} \\ a_N & a_1 \end{array} \right|, c_2 = \left| \begin{array}{cc} a_0 & a_{N-2} \\ a_N & a_2 \end{array} \right|, \text{and so on.} \]

The coefficients of the fourth row are the coefficients of the third row in reverse order, as shown in table 1.1.

The coefficients of the fifth row are calculated according to the following rule:

\[ d_i = \left| \begin{array}{cc} c_0 & c_{N-1-i} \\ c_{N-1} & c_i \end{array} \right| \quad \text{for } i = 0, 1, 2, \ldots, (n - 1) \quad (1.28) \]

For example:

\[ d_0 = \left| \begin{array}{cc} c_0 & c_{N-1} \\ c_{N-1} & c_0 \end{array} \right|, d_1 = \left| \begin{array}{cc} c_0 & c_{N-2} \\ c_{N-1} & c_1 \end{array} \right|, d_2 = \left| \begin{array}{cc} c_0 & c_{N-3} \\ c_{N-1} & c_2 \end{array} \right|, \text{and so on.} \]

And the coefficients of the sixth row are the coefficients of the fifth row in reverse order, as shown in the table 1.1. Remember that, the number of coefficients in these rows are one less than those in the two rows above. As we continue this procedure, the number of coefficients in each successive pair of rows decreases by one, until we construct \((2N-3)\) rows and end up with the last row having three coefficients.

Table 1.1: The Jury-Marden array of coefficients.

<table>
<thead>
<tr>
<th>Row</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(a_0)</td>
</tr>
<tr>
<td>2</td>
<td>(a_N)</td>
</tr>
<tr>
<td>3</td>
<td>(c_0)</td>
</tr>
<tr>
<td>4</td>
<td>(c_{N-1})</td>
</tr>
<tr>
<td>5</td>
<td>(d_0)</td>
</tr>
<tr>
<td>6</td>
<td>(d_{N-2})</td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>(\ldots)</td>
<td></td>
</tr>
<tr>
<td>2N-3</td>
<td>(r_0)</td>
</tr>
</tbody>
</table>
1.13 Poles and Zeros of Transfer Functions:

In general, a transfer function is defined as the ratio of the Laplace transform of
the output function to the Laplace transform of the input function.

The transfer function \( G(s) \) can be expressed as the ratio of two polynomials
\( N(s) \) and \( D(s) \) as given below:

\[
G(s) = \frac{N(s)}{D(s)} = \frac{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}
\]

(1.29)

Where \( a \)'s and \( b \)'s are coefficients of the complex frequency variable \( s \).

If the numerator \( N(s) = 0 \) has \( n \) roots (roots of numerator are zeros of a
system), and similarly denominator \( D(s) = 0 \) has \( m \) roots (roots of denominator are
poles of a system). Both \( N(s) \) and \( D(s) \) may be written as a product of linear factors
include these roots as given below:

\[
G(s) = H \frac{(s - z_1)(s - z_2)(s - z_3) \ldots (s - z_n)}{(s - p_1)(s - p_2)(s - p_3) \ldots (s - p_m)}
\]

(1.30)

Where \( H = \frac{a_0}{b_0} \) is a constant known as the scale sector \( z_1, z_2, z_3, \ldots \):

\( p_1, p_2, p_3, \ldots \) are complex frequencies, factors \( (s - z_i) \) are called zero factors, and
factors \( (s - p_i) \) are called pole factors. Zeros are called also transfer-function zeros or
transmission zeros, and poles are called also transfer-function poles or natural modes
of a network. Transfer functions always have the same number of poles and zeros, but
some exist at infinity. It is noted that:

I. When the complex frequency variable \( S \) has one of the values
\( z_1, z_2, \ldots, z_n \), the network function vanishes.

II. When it has one of the values \( p_1, p_2, \ldots, p_m \), the network function
becomes infinite.

The poles and zeros of a network function can be represented by a pole/zero
map (pole-zero plot) as shown in fig. 1.14.
Fig. 1. 14: Pole-zero plot.

Where a zero is conventionally symbolized by O and a pole symbolized by $\times$.

A network function (system) is asymptotically stable if all its poles are lying within the left-half of the s-plane (have negative real parts).

### 1.14 Step Response:

The performance of the closed-loop system will be assessed by examining the response to step inputs [62]. Specifically, with reference to Fig. 1.15 we will look at the rise time ($t_r$), settling time ($t_s$) and percentage overshoot. Rise time refers to the time it takes for the step response to rise from 10% of the final value to 90% of the final value. Settling time refers to the time it takes for the step response to remain in a band of $\pm 5\%$ of the final value. The percentage overshoot ($OSH$) is determined by the following formula:

$$Overshoot(OSH) = \frac{B - A}{A} \times 100\%$$  \hspace{1cm} (1.31)
There are four types of motion that are possible in the transfer function of a system.

- If the poles of a system are real and unequal, the response is over-damped with no overshoot.
- If the poles of a system are real and equal, the response is critically damped with no overshoot.
- If the poles of a system are complex the system is under-damped with an overshoot.
- If the poles of a system are purely imaginary, then the system has no damping and the response will be oscillatory with no reduction in amplitude (harmonic).

1.15 Sensitivity Function:

Active filters are designed to perform certain functions such as wave shaping or signal processing.
The cause-and-effect relationship between the network element variations and the resulting changes in the network transfer function is known as the sensitivity. So, that sensitivity is a measure of the change of the overall network function to the change of a particular parameter in the network. To minimize this change or to reduce the sensitivity is to choose components with small manufacturing tolerances, low temperature, aging, and humidity coefficients [1].

1.15.1 Definition:

The sensitivity function is defined as the ratio of the fractional change in a network function to the fractional change in an element of interest.

The sensitivity function, can be defined by the formula:

\[
S^H_x = \lim_{\Delta x \to \infty} \left( \frac{\Delta H}{\Delta x} \right) = \frac{x}{H} \frac{\partial H}{\partial x}
\]  

(1.32)

Where \( S^H_x \) is the sensitivity function, \( H \) is the network function and \( x \) is the element of interest.

1.15.2 Properties of Sensitivity Function:

The properties of sensitivity function are summarized in table 1.2.

Table 1.2: The properties of sensitivity function.

<table>
<thead>
<tr>
<th>Property No.</th>
<th>Relation</th>
<th>Property No.</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( S^{ky}_x = S^{y}_k ) = ( S^{y}_x )</td>
<td>6</td>
<td>( S^y_x = S^{y}_k ) = ( \frac{1}{n} S^{y}_x ) = ( \frac{1}{n} )</td>
</tr>
<tr>
<td>2</td>
<td>( S^x_x = S^{kx}_x = S^{x}_x = 1 )</td>
<td>7</td>
<td>( S^{y_1/y_2}_x = S^{y_1}_x + S^{y_2}_x )</td>
</tr>
<tr>
<td>3</td>
<td>( S^{y/y}_x = S^{y/y}_x = -S^{y}_x )</td>
<td>8</td>
<td>( S^{y_1/y_2}_x = S^{y_1}_x - S^{y_2}_x )</td>
</tr>
<tr>
<td>4</td>
<td>( S^{y^n}_x = n S^{y}_x )</td>
<td>9</td>
<td>( S^{x_1}_x = S^{x_2}_x S^{x_2}_x )</td>
</tr>
<tr>
<td>5</td>
<td>( S^{y^n}_x = n S^{kx/n}_x = n )</td>
<td>10</td>
<td>( S^{y_1/y_2}_x = \frac{y_1 S^{y_1}_x + y_2 S^{y_2}_x}{y_1 + y_2} )</td>
</tr>
</tbody>
</table>

In this relation, \( y \) is a complex quantity and \( x \) is a real quantity.

1.15.3 Coefficient Sensitivity:

In general, a network function \( F(s) \) for any active or passive lumped network is a ratio of polynomials having the form [8]:

\[
F(s) = \frac{P(s)}{Q(s)}
\]
\[ F(s) = \frac{N(s)}{D(s)} = \frac{a_0 + a_1s + a_2s^2 + \cdots + a_ns^n}{d_0 + d_1s + d_2s^2 + \cdots + d_ns^n} \]  \hspace{1cm} (1.33)

Here the coefficients of \( a_i \) and \( d_i \) are real and can be functions of an arbitrary filter element \( x \). For such an element \( x \) one may define the relative coefficient sensitivities as follows:

\[ S_x(a_i) = \frac{x}{a_i} \frac{\partial a_i}{\partial x} \]  \hspace{1cm} (1.34)

\[ S_x(d_i) = \frac{x}{a_i} \frac{\partial d_i}{\partial x} \]  \hspace{1cm} (1.35)

or the semi-relative coefficient sensitivities (they are even more useful)

\[ S_x(a_i) = x \frac{\partial a_i}{\partial x} \]  \hspace{1cm} (1.36)

\[ S_x(d_i) = x \frac{\partial d_i}{\partial x} \]  \hspace{1cm} (1.37)

1.15.4 Root Sensitivities:

A filter function can also be represented as

\[ F(s) = \frac{a_i \prod_{i=0}^{m} (S - Z_i)}{d_i \prod_{i=0}^{n} (S - P_i)} \]  \hspace{1cm} (1.38)

Where \( Z_i \) are zeros and \( P_i \) are poles. If \( F(s) \) is also a function of the filter element \( x \), the location of these poles and zeros will depend on this element. This dependence is described by the semi-relative root sensitivities

\[ S_x(Z_i) = x \frac{\partial Z_i}{\partial x} \]  \hspace{1cm} (1.39)

\[ S_x(P_i) = x \frac{\partial P_i}{\partial x} \]  \hspace{1cm} (1.40)

1.16 Basic Switching Operation:

The principle of the switched-capacitor is the use of capacitors and analog switches to perform the same function as a resistor. The switching function of the MOSFET produces a discrete response rather than a continuous response from the filter [63]. The operation of switched-capacitor can be explained with the help of following circuit diagram.
Fig. 1. 16: Circuit diagram of operation of the switched-capacitor.

Since the charge $\tilde{q}$ on a capacitor $c_1$ is given by

$$\tilde{q} = c_1 v$$  \hspace{1cm} (1.41)

Where $v$ is the voltage across the capacitor $c_1$.

Therefore, when $S_2$ closes with $S_1$ opens, then $S_1$ closes with $S_2$ opens a charge $\tilde{q}$ is transferred from $v_2$ to $v_1$ with:

$$\Delta \tilde{q} = c_1 (v_2 - v_1)$$  \hspace{1cm} (1.42)

If this switching process is repeated ($N$) times in time ($t$), then the amount of charge transferred per unit time ($\Delta \tilde{q}/\Delta t$) is given by:

$$\frac{\Delta \tilde{q}}{\Delta t} = c_1 (v_2 - v_1) \frac{N}{\Delta t}$$  \hspace{1cm} (1.43)

L.H.S. i.e. ($\Delta q/\Delta t$) is current and number of cycles per unit time ($N/\Delta t$) is switching frequency.

$$\therefore i = c_1 (v_2 - v_1) f_{clk}$$  \hspace{1cm} (1.44)

$$\therefore \frac{(v_2 - v_1)}{i} = \frac{1}{c_1 f_{clk}} = R$$  \hspace{1cm} (1.45)

Thus, the switched-capacitor is equivalent a resistor.

1.17 Concluding Remarks:

A filter is basically a device designed to isolate, suppress or pass a group of signals from a mixture of signals. Active filters can be categorized at many stages based on the type of signals processed into analog and digital filters, based on the type of elements used in their construction into passive and active filters, and based on the type of function achieved into low pass, band pass, high pass and band stop (reject) filters.

The cause-and-effect relationship between the network element variations and the resulting changes in the network transfer function is known as the sensitivity. Consequently, the sensitivity is a measure of the change of the overall network function to the change of a particular parameter in the network.
It is necessary that any filter designed by an engineer has to be extremely stable in practical use. The gain and phase margins are one way to examine the stability of a feedback system.

There are two methods to test the stability of a transfer function of a filter, one is called Routh–Hurwitz test (Routh stability criterion) and the other is called Jury-Marden test (Jury stability criterion).

Active filters are most extensively used in the field of instrumentation system, communication system, audio system, biomedical system, and signal processing.