Chapter 4

NATURAL MATRICES

This chapter deals with natural matrices. The direct correspondence between graphs and square natural matrices enables the study of graphs as natural matrices. Properties of graphs are discussed in terms of the natural matrix representation. First a brief introduction to the fundamental concepts of graphs is given [9,14,20].

4.1 Basic Concepts

In this section the terminology of this chapter is explained along with examples wherever necessary.

- A matrix having natural numbers as its elements is a natural matrix.

\[
A = \begin{pmatrix}
1 & 0 \\
5 & 8
\end{pmatrix}
\]

is an example of a natural matrix. Replacing every non-zero element in a natural matrix by 1 results in a boolean matrix with zero and one as its elements. \[
B = \begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
\]
is the boolean matrix corresponding to the natural matrix \( A \).
• A graph $G$ is defined as the pair $G = (V, E)$ where $V$ is the set of nodes, $|V| = n$ and $E$ is the set of edges, $|E| = q$. An edge $e_1$ is a link between two nodes say $v_1, v_2$ and hence is also denoted by the node pair $(v_1, v_2)$. When the node pair is not ordered the graph is undirected and $e_1 = \{v_1, v_2\} = \{v_2, v_1\}$. Edge $e_1$ is said to be incident upon vertices $v_1$ and $v_2$, and $v_1,v_2$ are called the end points of $e_1$.

When both the nodes in the node pair are same, as in $(v_1, v_1)$, the edge is called a loop. Edges having the same endpoints are called parallel edges. Edges having a common end point are adjacent to each other. Nodes are said to be adjacent if there is an edge connecting them. We shall normally be dealing with labeled graphs. It is conventional to use lower case letters of English alphabet as labels of edges and digits ($1 \cdots 9$) or upper case letters as labels of nodes.

• A directed graph or simply digraph is a graph in which an edge $e_1$ is denoted by an ordered node pair $(v_1, v_2)$. The edge $e_1$ is said to be incident out of $v_1$ and into $v_2$.

In general, by graphs we mean digraphs unless otherwise specified.

• A bipartite graph or simply bigraph is a graph $G = (V, E)$ where the node set $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, $|V_1| = m$, $|V_2| = n$ and $E = \{(v_1, v_2) | (v_1 \in V_1 \text{ and } v_2 \in V_2)\}$. In short, the node set of a bigraph can be partitioned into two disjoint sets $V_1, V_2$ such that all edges are links between nodes in $V_1$ and those in $V_2$. Wherever arrows are omitted, the direction of edges is assumed to be from nodes in $V_1$ to nodes in $V_2$. Also it is conventional to take $V_1, V_2$ such that $m \leq n$.

• A path is a sequence of distinct edges adjacent to each other. The vertices on a path are distinct. Such a path has sometimes been called an
elementary path [3]. A cycle is a path which begins and ends at the same node. We define the disjoint union of paths as a route and a disjoint union of cycles as a circuit. A circuit including all the nodes of the graph is a spanning circuit.

Note that by our definition, a circuit need not be connected, as is usually required by the conventional definition. A single connected path is a basic route and a cycle is a basic circuit. Note that a connected spanning circuit is nothing but a Hamiltonian circuit.

![Graph Image](image-url)

Fig. 4.1

In the undirected graph of Fig. 4.1, \(bade\) is a path, \(bae\) is a route, \(abc\) and \(ef\) are cycles, and \(abce\) is a circuit. If the adjacent edges in a path can be traversed moving along the direction of arrows in a digraph, a directed path is obtained. A directed cycle is similarly defined. Usually the context makes it clear whether directed or undirected path (or cycle) is implied, hence explicit mention is often not required.

- A connected graph without any circuits is a tree. A graph having trees as its components is a forest.

### 4.1.1 Divergence and Convergence

We now define two new terms, divergence and convergence. It will be seen later that nonzero minors in the natural adjacency and boolean incidence matrices are
deeply related to divergences and convergences.

- We define a graph to be a *divergence* if all the edges emerge from distinct nodes. Every node in a divergence has an outdegree of 0 or 1. A node with outdegree 0 is called a *sink*. Similarly we define a graph to be a *convergence* if all the edges go into distinct nodes. Every node in a convergence has an indegree of 0 or 1. A node with indegree 0 is called a *source*.

For example the graphs in Fig. 4.2.a are divergences with no sink.

![Fig. 4.2.a](image)

From Fig. 4.2.a it is obvious that in general, a divergence with no sinks is a disjoint union of cycles with incoming heads, if any. Similarly, a convergence with no sources is a disjoint union of cycles with outgoing tails, if any.

- We define a *lasso* as a cycle with incoming heads and a *noose* as a cycle with outgoing tails.

Hence, a divergence with no sinks is a disjoint union of cycles and lasso, and a convergence with no sources is a disjoint union of nooses and cycles.
The graphs in Fig. 4.2.b are convergences with no source.

Fig. 4.2.b

4.1.2 Confluence and Arborescence

- We define a confluence as a rooted tree in which every node other than the root has an outdegree of one. It is basically a divergence with one sink, the root.

Fig. 4.3

For example, the graph in Fig. 4.3 is a confluence. A single node by definition is a confluence. A divergence with one sink consists of lassos, cycles and one confluence. Note that a confluence must necessarily be present in a divergence with one sink.
A divergence with \( k \) sinks contains exactly \( k \) confluences.

This is obvious from the fact that the addition of every confluence adds exactly one sink to the divergence.

The graphs in Fig. 4.4 are divergences with one sink.

- An *arborescence* [9] is a rooted tree in which every node other than the root has an indegree of one. It is a convergence with a source.

The graph in Fig. 4.5 is an example of arborescence. A single node, by definition, is an arborescence. A convergence with one source consists of nooses, cycles and one arborescence. We have a result similar to the one for divergence with \( k \) sinks.
A convergence with \( k \) sources consists of \( k \) arborescences.

This is due to the fact that the addition of every arborescence adds exactly one source.

\[ \text{Source} \]

\[ \text{Source} \]

\[ \text{Source} \]

Fig. 4.6

The graphs in Fig. 4.6 are convergences with one source.

### 4.2 Adjacency Matrix

A graph of \( n \) nodes is represented by an adjacency matrix \( A = [a_{ij}]_{n \times n} \) such that \( a_{ij} \) is the number of edges from node \( i \) to node \( j \). The adjacency matrix is a natural matrix since \( a_{ij} \) is always a natural number. Hence a square natural matrix represents a graph and study of graphs is nothing but the study of square natural matrices.

The adjacency matrix with natural elements we call as the \( A \) matrix of the graph. It is sometimes convenient to write the adjacency matrix with labels of edges, treated as boolean literals, as its elements. We then call it the \( C \) matrix. If \( a \) and \( b \) are parallel edges between nodes 1 and 2 then \( C[1, 2] = a + b \). The \( A \) and \( C \) matrix of an undirected graph are symmetric. For example consider the graph in Fig. 4.7.
Its A matrix and C matrix are,

\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & 0 & 1 & 1 \\
2 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 0 \\
4 & 0 & 2 & 1 & 0 \\
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & 0 & a & d \\
2 & 0 & 0 & b & 0 \\
3 & 0 & 0 & 0 & 0 \\
4 & 0 & e + f & c & 0 \\
\end{pmatrix}
\]

We consider now the adjacency matrices of bigraphs.
For the bigraph in Fig. 4.8, the corresponding adjacency matrices are,

\[
A = \begin{pmatrix}
1' & 2' & 3' \\
1 & 0 & 0 \\
2 & 0 & 1 & 1
\end{pmatrix}, \quad C = \begin{pmatrix}
1' & 2' & 3' \\
a & 0 & 0 \\
2 & 0 & b & c
\end{pmatrix}
\]

When \( m = n \), adjacency matrix of the bigraph becomes a square natural matrix and hence represents a graph of \( n \) nodes. For example, consider the bigraph in Fig. 4.9.a and the graph in Fig. 4.9.b.

Note that the \( A \) and \( C \) matrices for the bigraph are,

\[
A = \begin{pmatrix}
1' & 2' & 3' \\
1 & 1 & 0 \\
2 & 0 & 1 & 0 \\
3 & 1 & 0 & 0
\end{pmatrix}, \quad C = \begin{pmatrix}
1' & 2' & 3' \\
a & d & 0 \\
2 & 0 & b & 0 \\
3 & c & 0 & 0
\end{pmatrix}
\]

In the above matrices if \( 1', 2', 3' \) are replaced with 1,2,3 respectively, then these become the adjacency matrices for the graph in Fig. 4.9.b also. Hence the bigraph in Fig. 4.9.a represents the graph in Fig. 4.9.b. We call such bigraphs as \textit{indexed bigraphs}. What we mean by the term \textit{indexed} is that the nodes in \( V_1 \) and \( V_2 \) have a one to one correspondence and can be considered as having the same indices.
Every graph of \( n \) nodes can be represented as an indexed bigraph of \( 2n \) nodes. The bigraph representation of a graph \( G = (V, E) \) is a bigraph \( G' = (V', E') \) where \( V' = V_1 \cup V_2, V_1 = V, V_2 = \{v'|v \in V\} \) and \( E' = \{(v_1, v'_2)|(v_1, v_2) \in E\} \).

Consider the graphs in Fig. 4.10.a and Fig. 4.10.b and their \( A \) and \( C \) matrices.

\[
\begin{align*}
\begin{array}{c|ccc}
& 1 & 2 & 3 \\
1 & 1 & 1 & 1 \\
2 & 1 & 0 & 1 \\
3 & 1 & 1 & 0 \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c|ccc}
& 1 & 2 & 3 \\
1 & a & b & d \\
2 & b & 0 & c \\
3 & d & c & 0 \\
\end{array}
\end{align*}
\]

Note that \( A \) and \( C \) matrices are symmetric. Also that the symmetric bigraph in Fig. 4.10.a represents the undirected graph in Fig. 4.10.b.

An undirected graph is represented by a symmetric bigraph. For every edge \((v_1, v_2)\) in the undirected graph there are two edges \((v_1, v'_2)\) and \((v_2, v'_1)\) in the bigraph. Loops are represented by single edges.

The permanents and matchants in \( A \) and \( C \) matrices are deeply related to the properties of graphs and will be discussed in detail later. In the next section matchings in bigraphs and ordinary graphs are explained.
4.3 Matchings

We often need to map one set of elements with another set such that the mapping is one-one. The two sets need not be distinct, in which case an element can also be mapped onto itself. When the number of elements in the two sets is same we look for a one-one onto mapping so that every element in a set is paired with an element in the other set. This problem corresponds directly to the problem of finding matchings in bigraphs (when the sets are distinct) or in ordinary graphs (when a set is mapped with itself). We will discuss matching in bigraphs and ordinary graphs separately in the following two subsections.

4.3.1 Matchings in Bigraphs

- A matching in a bigraph is a subset of edges in which no two edges are adjacent. An edge by itself is a matching.

![Diagram of a bigraph with matchings](image)

For example, $b$, $bd$, $acd$ are matchings in the bigraph of Fig. 4.11.

- A maximal matching is a set of edges in which no more edges can be added without destroying its matching property.

No set containing a matching can be a maximal matching. In the bigraph of
Fig. 4.11, *abd* and *acd* are maximal matchings. In a bigraph of *n* nodes, a maximal matching can have a maximum of \( \lfloor \frac{n}{2} \rfloor \) edges.

- **Degree** of a matching is the number of edges in it. **Matching number** of a bigraph is the degree of its largest matching.

It should be obvious that the largest matching has to be maximal. Matching number of the bigraph in Fig. 4.11 is 3.

- A matching that includes all the vertices of *V₁* in a bigraph is a **complete matching**. The degree of a complete matching is *m*.

In other words, the matching number of a bigraph having a complete matching is *m*. In the bigraph of Fig. 4.9 *abd* and *acd* are complete matchings. A complete matching has to be maximal although the converse is not always true.

- **Index matching** in an indexed bigraph is such that vertices with same indices are matched, not necessarily to each other, but collectively.

![Fig. 4.12.a](image1)

![Fig. 4.12.b](image2)

![Fig. 4.12.c](image3)

In the indexed bigraph of Fig. 4.12.a, *ab* and *cde* are index matchings. The bigraphs representing these matchings are given in Fig. 4.12.b and Fig. 4.12.c respectively.

Indexed bigraphs have digraphs corresponding to them hence matchings in indexed bigraphs hold a special significance. The digraph corresponding to the in-
dexed bigraph of Fig. 4.12.a is given in Fig. 4.13.a and the subgraphs corresponding to the index matchings are given in Fig. 4.13.b and Fig. 4.13.c respectively.

* Index matchings in an indexed bigraph correspond to circuits in the corresponding graph.

Now consider the matching $acd$ and its corresponding digraph.

* Let $V_{m_1}$ and $V_{m_2}$ be the set of nodes in $V_1$ and $V_2$, respectively, included in the matching. The subgraph corresponding to the matching is a divergence with nodes in the set $V_1 - V_{m_1}$ as sinks, and convergence with nodes in the set $V_2 - V_{m_2}$ as sources.

The graph is a divergence and a convergence because the indegree and outde-
gree of every node is exactly one. The nodes in $V_1$ not included in the matching are those with outdegree 0, hence are the sinks of the divergence. The nodes in $V_2$ not included in the matching have an indegree 0 and hence are the sources of the convergence.

### 4.3.2 Matchings in Ordinary Graphs

Matchings in a digraph are same as those in the corresponding undirected graphs hence the discussion shall be restricted to undirected graphs only. All the terms defined in the previous section hold good here. The only difference is that a complete matching is one which includes all the nodes of the graph. Consider the graph in Fig. 4.15.

![Graph Image](image)

Fig. 4.15

$a$, $ac$, $bd$ are matchings in the graph of Fig. 4.15. It is obvious that $ac$ and $bd$ are maximal matchings. Matching number of the graph is 2. $ac$, $bd$ are complete matchings also. Note that a complete matching for a graph of odd number of nodes is possible only if a loop is included in the matching. In a graph of $n$ nodes, a maximal matching can have a maximum of $n$ edges if loops are included, otherwise $\lfloor \frac{n}{2} \rfloor$ edges.
The symmetric bigraphs corresponding to the graph in Fig. 4.15 and its matchings \( ac, bd \) are given in Figs. 4.16.a-c respectively. Note that the matchings also give symmetric bigraphs. Matchings in ordinary graphs correspond to symmetric bigraphs and hence are called *symmetric matchings*.

Thus the bigraph representation of ordinary graphs enables us to obtain the matchings in an ordinary graph as the symmetric matchings in the corresponding bigraph.

### 4.4 Coverings

In this section we deal with another edge property of graphs, its *covering* set of edges. The coverings in a graph depend upon the number of edges between nodes, hence it is appropriate to discuss them in this chapter.

- A *covering* of an undirected graph is the set of edges such that every vertex of the graph is incident on at least one edge in the set. A *minimal covering* is one from which no edges can be removed without destroying its covering property.

Informally speaking, when a covering is removed from a graph, all the nodes get pulled off. For the graph given in Fig. 4.17, \( ae \) is a covering since all nodes are covered by it.
We give below an algorithm to compute the minimal coverings in a graph.

**Algorithm:**

- For each node write the boolean sum of labels of all edges incident on it.
- Take the boolean product of the above sums.
- When the above function is written as a sum of products, each term gives a minimal covering.

For the graph in Fig. 4.17, the boolean function is

\[
f = (a + b + f)(a + c + d + f)(b + c + e)(d + e) = bd + ae + acd + bce + ef + cdf
\]

Extending the concept to digraphs we define the following two sets.

- An **outcovering** is a set such that every vertex of the graph has at least one outgoing edge in the set. An **incovering** is a set such that every vertex of the graph has at least one incoming edge in the set.

It is obvious that the size of an outcovering and incovering is \( n \), equal to the number of nodes in the graph. Consider the graph in Fig. 4.18.

![Fig. 4.18](image)

In this graph, \( acde \) is an outcovering and \( bcde \) is an incovering. The algorithm for computing outcovering and incovering in digraphs is same as the algorithm for undirected graphs except that it considers only the outgoing edges for outcovering,
and incoming edges for incovering, when writing the boolean sum. For the graph in Fig. 4.18 the boolean function for outcovering is,

\[ f = a(b + c)(d + f)e = abde + acde + abef + acef \]

and for incovering it is,

\[ f = (b + f)(a + c)de = abde + bcde + adef + cdef \]

Note that an outcovering is basically a divergence with no sink and an incovering is a convergence with no source.

In the next section we give theorems relating the adjacency matrix and matchings in a graph, hence giving matrix algorithms to compute the same. A matrix algorithm to compute the coverings in a graph using boolean matrices will be given in the next chapter.

4.5 Adjacency Matrix and Matchings

In this section first we explain the significance of permanents and matchants of the adjacency matrices of a given graph. Then the permanent and matchant compounds of adjacency matrices are discussed. The significance of Frobenius König theorem with reference to matchings in graphs is also given.

4.5.1 Permanents and Permanent Compounds

Consider the bigraph in Fig. 4.19 and its A and C matrices.

\[ \begin{align*}
\text{1} & \quad \text{a} & \quad \text{1}' \\
\text{2} & \quad \text{e} & \quad \text{b} & \quad \text{2}' \\
\text{3} & \quad \text{c} & \quad \text{3}' \\
\text{4} & \quad \text{d} & \quad \text{4}'
\end{align*} \]

Fig. 4.19
In $A$ matrix the permanent, $\text{per}(1\cdot2, 1'\cdot3') = 1$, and in $C$ matrix $\text{per}(1\cdot2, 1'\cdot3') = ac$. This gives one matching $ac$ of degree 2 between nodes 1,2 and 1',3' in the bigraph. Again, in $A$ matrix $\text{per}(1\cdot2\cdot3, 1'\cdot2'\cdot4') = 1$ and $\text{per}(1\cdot2\cdot3, 2'\cdot3'\cdot4') = 1$, and in $C$ matrix $\text{per}(1\cdot2\cdot3, 1'\cdot2'\cdot4') = abd$ and $\text{per}(1\cdot2\cdot3, 2'\cdot3'\cdot4') = ebd$. The matchings represented by the above three permanents are given in Figs. 4.20.a-c.

Nonzero permanents of order $k$ in the $A$ matrix give the number of matchings of degree $k$ between the $k$ row nodes (nodes of $V_1$) and $k$ column nodes (nodes of $V_2$). A $k \times k$ nonzero permanent in the $C$ matrix lists out the edges in matchings of degree $k$.

First the notations and definitions are explained. Then theorems relating the compounds to matchings are given.

$A^{(k)}$: the $k^{th}$ compound of $A$. It is the ordinary compound (recall the definition of compound matrices from section 2.2) except that permanents, and not determinants, of order $k$ form the elements of $A^{(k)}$. Its order is $\binom{m}{k} \times \binom{n}{k}$.

$C^{(k)}$: the $k^{th}$ compound of $C$, defined in the same fashion as $A^{(k)}$. 

\[ A = \begin{pmatrix} 1' & 2' & 3' & 4' \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1' & 2' & 3' & 4' \\ a & 0 & e & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \]
Rank of $A$ : is $r$ if $A^{(r)} \neq 0$ and $A^{(r+1)} = 0$.

Rank of $C$ : is $r$ if $C^{(r)} \neq 0$ and $C^{(r+1)} = 0$.

Content of $A^{(k)}$ : is the sum of nonzero elements of $A^{(k)}$.

Content of $C^{(k)}$ : is the boolean sum of nonzero elements of $C^{(k)}$.

Permanent degree of $A$ : is the largest $r$ such that the trace of $A^{(r)}$ is nonzero.

Permanent degree of $C$ is similarly defined.

**Theorem:** The matching number of a bigraph is equal to the rank of $A$, which is the same as rank of $C$.

**Proof:** Let $r$ be the matching number of the bigraph. This means that there are no matchings of degree greater than $r$, i.e., all permanents of order higher than $r$ in the adjacency matrices $A$ and $C$ are zero. Hence, $A^{(r+1)} = A^{(r+2)} = \cdots = 0$ and $C^{(r+1)} = C^{(r+2)} = \cdots = 0$. This implies, rank of $A = \text{rank of } C = r$.

Now let the rank of $A$ (or $C$) be $r$. This implies no nonzero permanent of order higher than $r$ exists. Hence there is no matching of degree higher than $r$ and the matching number of the bigraph is $r$.

**Corollary:** Number of largest matchings in a bigraph is given by the content of $A^{(r)}$, where $r$ is the rank of $A$.

**Corollary:** Largest matchings in a bigraph are given by the content of $C^{(r)}$, where $r$ is the rank of $C$.

**Corollary:** A complete matching exists if and only if the rank of $A$ (or $C$) is $m$.

For example consider the bigraph given in Fig. 4.21.

![Fig. 4.21]
The adjacency matrices and their permanent compounds are,

\[
\begin{align*}
A &= \begin{pmatrix}
1 & 0 & 1 & 0 & 0 \\
2 & 1 & 1 & 1 & 1 \\
3 & 0 & 1 & 0 & 0 \\
\end{pmatrix} \\
C &= \begin{pmatrix}
1 & 0 & a & 0 & 0 \\
2 & b & c & d & e \\
3 & 0 & f & 0 & 0 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{pmatrix}
1'2' & 1'3' & 1'4' & 2'3' & 2'4' & 3'4' \\
12 & 1 & 0 & 0 & 1 & 1 & 0 \\
13 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 1 & 0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1'2' & 1'3' & 1'4' & 2'3' & 2'4' & 3'4' \\
12 & ab & 0 & 0 & ad & ae & 0 \\
13 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & bf & 0 & 0 & df & ef & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1'2'3' & 1'2'4' & 1'3'4' & 2'3'4' \\
A^{(3)} &= C^{(3)} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\end{pmatrix}
\end{pmatrix}
\]

Rank of \( A = 2 \), \( \Rightarrow \) matching number = 2.

Total number of largest matchings = content of \( A^{(2)} = 6 \).

Largest matchings = content of \( C^{(2)} = ab + ad + ae + bf + df + ef \).

For indexed bigraphs, permanents and compounds of \( A \) matrix relate to the cyclic properties of the graph. Nonzero, nonprincipal \( k \times k \) permanents in \( A \) give the number of matchings in the indexed bigraph. This is also same as the number of divergences of \( k \) row nodes and convergences of \( k \) column nodes in the graph represented by the indexed bigraph. Nonzero principal \( k \times k \) permanents in \( A \) give the number of index matchings in the indexed bigraph, which is same as the number of circuits in the corresponding graph. This leads to the following theorem.

**Theorem:** The degree of the largest index matching in an indexed bigraph, and hence the length of the largest circuit in the corresponding
graph, is the permanent degree of $A$, which is the same as permanent degree of $C$.

**Corollary:** The number of largest index matchings is given by the sum of nonzero diagonal elements of $A^{(r)}$, where $r$ is the permanent degree of $A$.

**Corollary:** The largest index matchings are given by the boolean sum of nonzero diagonal elements of $C^{(r)}$, where $r$ is the permanent degree of $C$.

Consider for example the following bigraph.

![Fig. 4.22](image)

\[
A = \begin{pmatrix}
1 & 2 & 3' & 4'
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 1 & 0 & 0
2 & 1 & 0 & 1 & 0
3 & 0 & 0 & 0 & 1
4 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
1' & 2' & 3' & 4'
\end{pmatrix} = \begin{pmatrix}
0 & a & 0 & 0
b & 0 & c & 0
0 & 0 & 0 & d
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
A^{(2)} = \begin{pmatrix}
1'2' & 1'3' & 1'4' & 2'3' & 2'4' & 3'4'
\end{pmatrix} = \begin{pmatrix}
12 & 1 & 0 & 0 & 1 & 0 & 0
13 & 0 & 0 & 0 & 0 & 1 & 0
14 & 0 & 0 & 0 & 0 & 0 & 0
23 & 0 & 0 & 1 & 0 & 0 & 1
24 & 0 & 0 & 0 & 0 & 0 & 0
34 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
\[
C^{(2)} = \begin{pmatrix}
12 & ab & 0 & 0 & ac & 0 & 0 \\
13 & 0 & 0 & 0 & 0 & ad & 0 \\
14 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 0 & 0 & bd & 0 & 0 & cd \\
24 & 0 & 0 & 0 & 0 & 0 & 0 \\
34 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
A^{(3)} = \begin{pmatrix}
123 & 0 & 0 & 0 & 1 \\
124 & 0 & 0 & 0 & 0 \\
134 & 0 & 0 & 0 & 0 \\
234 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
C^{(3)} = \begin{pmatrix}
123 & 0 & 0 & 0 & 0 & acd \\
124 & 0 & 0 & 0 & 0 \\
134 & 0 & 0 & 0 & 0 \\
234 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
A^{(4)} = C^{(4)} = (0)
\]

Rank of $A = 3$, and the largest matching = $acd$. Permanent degree of $A$ the largest index matching = $ab$.

### 4.5.2 Frobenius–König Theorem: An Interpretation

Consider the Frobenius–König theorem mentioned in the previous chapter. If there is a zero $s \times t$ submatrix in $A$ such that $s + t = n + 1$, it means that one node in the node set $V_1$ of the corresponding bigraph is not connected to any of the $n$ nodes in $V_2$, or two nodes in $V_1$ are connected to at most one node in $V_2$, or
in general \( k \) nodes in \( V_1 \) are connected to at most \((k - 1)\) nodes in \( V_2 \). In either of these cases there is no complete matching in the bigraph. And also, by Frobenius König theorem \( \text{Per}(A) \) is zero.

Similarly if there is no complete matching in a bigraph it means all the \( m \times m \) permanents in the corresponding adjacency matrix are zero. Also that some \( k \) nodes in \( V_1 \) are connected to at most \((k - 1)\) nodes in \( V_2 \). This implies there is a zero \( s \times t \) submatrix in \( A \) such that \( s + t = n + 1 \). Hence \( \text{Per}(A) = 0 \). The Frobenius König theorem thus leads to the following result.

\[ \begin{align*}
* \, \text{The permanent of a matrix is zero if and only if there are no complete matchings in the corresponding bigraph.}
\end{align*} \]

Now recall from the chapter on real matrices, the definition of permanent of \( A \). For a \( 3 \times 4 \) matrix \( A \),

\[
\text{Per}(A) = a_{11}a_{22}a_{33} + a_{11}a_{22}a_{34} + a_{11}a_{23}a_{32} + a_{11}a_{23}a_{34} + a_{11}a_{24}a_{32} + a_{11}a_{24}a_{33} \\
+ a_{12}a_{21}a_{33} + a_{12}a_{21}a_{34} + a_{12}a_{23}a_{31} + a_{12}a_{23}a_{34} + a_{12}a_{24}a_{31} + a_{12}a_{24}a_{33} \\
+ a_{13}a_{21}a_{32} + a_{13}a_{21}a_{34} + a_{13}a_{22}a_{31} + a_{13}a_{22}a_{34} + a_{13}a_{24}a_{31} + a_{13}a_{24}a_{32} \\
+ a_{14}a_{21}a_{32} + a_{14}a_{21}a_{33} + a_{14}a_{22}a_{31} + a_{14}a_{22}a_{33} + a_{14}a_{23}a_{31} + a_{14}a_{23}a_{32}
\]

If \( A \) is the adjacency matrix of a bigraph, each nonzero term in \( \text{Per}(A) \) gives a complete matching in the bigraph. The terms in \( \text{Per}(C) \) list out edges in the complete matchings of the bigraph. For the adjacency matrix \( A \) of bigraph in Fig. 4.19,

\[
\text{Per}(A) = 3
\]

\[
\text{Per}(C) = abd + acd + ebd
\]

Note that these are the only complete matchings in the bigraph. Thus,

\[ \begin{align*}
* \, \text{the permanent of } A \text{ gives the number of complete matchings in the bigraph, and that of } C \text{ lists them out.}
\end{align*} \]
4.5.3 Matchants and Matchant Compounds

Recall from section 2.1.1 the definition of matchants for symmetric matrices. It will be shown here that the matchants of C matrix of an undirected graph are related to matchings in it. Consider the graph in Fig. 4.23 and its C matrix.

![Fig. 4.23](image)

\[
C = \begin{pmatrix}
0 & a & 0 & b + g \\
 a & f & d & c \\
0 & d & 0 & e \\
b + g & c & e & 0
\end{pmatrix}
\]

\[
match(1\cdot2\cdot4) = \begin{pmatrix}
0 & a & b + g \\
0 & f & c \\
b + g & c & 0
\end{pmatrix}
= bf + gf
\]

Note that \(match(1\cdot2\cdot4)\) is the complete matching for the subgraph formed by the node set \(\{1, 2, 4\}\). Similarly \(match(2\cdot3) = d\) is a complete matching for nodes 2, 3.

* A matchant of order \(k\) in the C matrix gives the complete matchings of the subgraphs formed by the given \(k\) nodes.

We define some notations to state our results in a more precise manner.

\(C^{(k)}\): Matchant compound \(C^{(k)}\) of a symmetric matrix \(C\), obtained by calculating the matchants on the diagonal. It is a diagonal matrix of order \(\binom{n}{k}\).
Rank of $C$ : is $r$ if $C^{(r)} \neq 0$ and $C^{(r+1)} = C^{(r+2)} = 0$.

It is necessary to check two higher compounds of $C$ to decide the rank. This is due to the fact that complete matchings for larger subgraphs may exist even if those for smaller ones do not. For example the absence of loops will make all odd order compounds zero, even though higher even order compounds may be nonzero.

* Rank of $C$ gives the matching number of the symmetric bigraph corresponding to the graph. This is equal to the number of nodes of the graph included in the largest matching. The largest matchings are obtained by the content of $C^{(r)}$, where $r$ is the rank of $C$.

The above results lead to the following theorem.

**Theorem**: Complete matchings in a graph exist if and only if the rank of $C$ is $n$, and they are obtained from the content of $C^{(n)}$.

**Corollary**: A nonzero element in $C^{(k)}$ gives the complete matchings in the subgraph formed by the given $k$ nodes.

Compounds of the $C$ matrix of the graph in Fig. 4.23 are as follows.

\[
C(2) = \begin{pmatrix}
12 & 13 & 14 & 23 & 24 & 34 \\
12 & a & 0 & 0 & 0 & 0 \\
13 & 0 & 0 & 0 & 0 & 0 \\
14 & 0 & 0 & b+g & 0 & 0 \\
23 & 0 & 0 & d & 0 & 0 \\
24 & 0 & 0 & 0 & c & 0 \\
34 & 0 & 0 & 0 & 0 & e
\end{pmatrix}
\]
\[ C^{(3)} = \begin{pmatrix} 123 & 124 & 134 & 234 \\ 123 & 0 & 0 & 0 \\ 124 & 0 & b f + g f & 0 \\ 134 & 0 & 0 & 0 \\ 234 & 0 & 0 & 0 & e f \end{pmatrix} \]

\[ C^{(4)} = 1234 \begin{pmatrix} a e + b d + g d \end{pmatrix} \]

Rank of \( C = 4 = n \). The complete matchings in the graph are \( a e, b d, \) and \( g d \) which is the same as content of \( C^{(4)} \).

Similarly for the graph in Fig. 4.15,

\[ C = \begin{pmatrix} a & b & 0 \\ b & 0 & c \\ 0 & c & d \end{pmatrix} \]

\[ C^{(2)} = \begin{pmatrix} 12 & 13 & 23 \\ 12 & b & 0 & 0 \\ 13 & 0 & a d & 0 \\ 23 & 0 & 0 & c \end{pmatrix} \]

\[ C^{(3)} = 123 \begin{pmatrix} a c + b d \end{pmatrix} \]

Rank of \( C = 3 = n \), and the complete matchings in the graph are \( a c \) and \( b d \).

Matchants of \( C \) matrix also give the *maximal* matchings in a graph if \( C \) matrix is written in a special way. For the graph in Fig. 4.21 the \( C \) matrix is written as,

\[ C = \begin{pmatrix} 1 & a & 0 & b + g \\ a & f & d & c \\ 0 & d & 1 & e \\ b + g & c & e & 1 \end{pmatrix} \]
Note that we have written 1 on the diagonal. Here 1 is a dummy variable which is absorbed in both the operations + and ·, that is,

\[ 1 + a = a + 1 = a \quad \text{and} \quad 1 \cdot a = a \cdot 1 = a \]

This is why \( C[2,2] = 1 + f = f \).

\[
\text{match}(1\cdot2\cdot4) = \text{match} \begin{pmatrix} 1 & a & b + g \\ a & f & c \\ b + g & c & 1 \end{pmatrix} = 1 \cdot f + 1 \cdot c + a \cdot 1 + (b + g) \cdot f \\
= a + c + bf + gf
\]

Verify that \( \text{match}(1\cdot2\cdot4) \) gives the maximal matchings in the subgraph formed by nodes 1,2,4. Again,

\[
\text{match}(C) = \text{match} \begin{pmatrix} f & d & c \\ d & 1 & e \\ c & e & 1 \end{pmatrix} + a \text{ match} \begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix} + (b + g) \text{ match} \begin{pmatrix} f & d \\ d & 1 \end{pmatrix} \\
= f + ef + d + c + ae + bf + bd + gf + gd \\
= c + ae + bd + bf + ef + gf + gd
\]

These are the maximal matchings in the graph. On comparing with the complete matchings obtained earlier it is seen that while all complete matchings are maximal, the converse is not true.

* A matchant of order \( k \) in the symmetric \( C \) matrix, written in a special form, gives the maximal matchings in the corresponding subgraph.

**Theorem:** Maximal matchings in a graph are given by the content of \( C^{(n)} \), which is the same as \( \text{match}(C) \).

**Corollary:** A nonzero element in \( C^{(k)} \) gives the maximal matchings in the subgraph formed by the given \( k \) nodes.
Proof of the above theorem is obvious from the above discussion and example. For the graph in Fig. 4.15, C matrix and its matchant compounds are,

\[ C = \begin{pmatrix} a & b & 0 \\ b & 1 & c \\ 0 & c & d \end{pmatrix} \]

\[ C^{(2)} = \begin{pmatrix} 12 & 13 & 23 \\ 12 & a+b & 0 & 0 \\ 13 & 0 & ad & 0 \\ 23 & 0 & 0 & d+c \end{pmatrix} \]

\[ C^{(3)} = 123 \left( ad + ac + bd \right) \]

*Content of \( C^{(3)} = ad + ac + bd \). Note that these are the only maximal matchings in the graph.*

Consider for example another graph.

The C matrix of the graph in Fig. 4.24 is,

\[ C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & a+b & c & 0 \\ 2 & a+b & 1 & d & 0 \\ 3 & c & d & e & f \\ 4 & 0 & 0 & f & 1 \end{pmatrix} \]
The maximal matchings in the graph are given by the content of $C^{(4)}$.

4.6 Polynomials of a Graph

The intrinsic and circuit polynomials of a matrix have been defined earlier. Here we discuss these polynomials with reference to the adjacency matrix of a graph, and refer to them as polynomials of the graph itself.

4.6.1 Intrinsic Polynomial

The intrinsic polynomial of a graph is closely related to the circuits in a graph and its coefficients can be obtained from the cyclic structure of the graph.

First the significance of minors in the adjacency matrix is discussed. Consider for example the graph in Fig. 4.25 and its adjacency matrix.
The $3 \times 3$ minor formed of rows and columns 1, 2, 3 is 1. Now look at the subgraph corresponding to nodes 1, 2, 3. There is a circuit in it apart from the loop at node 1. Note also that the edges contributing to the nonzero minor are those forming the circuit. This becomes clear when $C$ matrix and its corresponding minor is considered.

The minor formed of rows and columns 1, 2, 3 is $abe$ and these obviously form a circuit. The minor corresponding to rows and columns 1, 3, 4 is also nonzero in $A$ and equal to $cde$ in $C$.

Now consider the $3 \times 3$ minor formed of rows 1, 3, 4 and columns 1, 2, 3. It is -1 in $A$ matrix and $-ace$ in $C$ matrix. The set of edges $ace$ form a divergence of the row nodes 1, 3, 4 with node 2 as its sink and a convergence of column nodes 1, 2, 3, with node 4 as its source. Also note that $ace$ is a path from node 4 to node 2.

Consider now another graph given in Fig. 4.26 and its adjacency matrix.
Fig. 4.26

\[ A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \]

The minor corresponding to rows and columns 1, 2 is nonzero, and corresponds to a circuit. But the determinant of \( A \), corresponding to rows and columns 1, 2, 3 is 0 even though circuits of length 3, \( acd \) and \( abe \) exist in the graph. This comes also from the determinant of \( C \), which is equal to \( acd - abe \). It is also obvious now how \( |A| \) became 0. Thus, although nonzero principal minors in \( A \) correspond to circuits in the graph, all the circuits in the graph do not necessarily give nonzero minors in \( A \). Recall that,

\[ |I - sA| = 1 - a_1 s + a_2 s^2 + \cdots + (-1)^n a_n s^n \]

where \( A^{(k)} \) is the determinant compound and \( a_k \) is the trace of \( A^{(k)} \).

Hence, coefficient of \( s^k \) in \( |I - sA| \) gives circuits of length \( k \) in the graph though not all of them. For example, the intrinsic polynomial of the graph in Fig. 4.25 is,

\[ |I - sA| = \begin{pmatrix} 1 - s & -s & 0 & -s \\ 0 & 1 & -s & 0 \\ -s & 0 & 1 & 0 \\ 0 & 0 & -s & 1 \end{pmatrix} = 1 - s - 2s^3 \]

Note that there is one circuit of length 1 and two circuits of length 3 in the graph.
The intrinsic polynomial of the graph in Fig. 4.26 is,

\[
|I - sA| = \begin{pmatrix}
1 & -s & 0 \\
-s & 1 & -s \\
-s & 0 & 1 - s
\end{pmatrix}
= 1 - s - s^2
\]

This does not give the circuit of length 3 present in the graph. The coefficient of \(s^k\) in the intrinsic polynomial gives what we have defined later as the excess circuits of length \(k\). To obtain all the circuits in a graph from a polynomial of \(A\) matrix, we consider a modified form of intrinsic polynomial, called the circuit polynomial defined in the chapter on nonnegative matrices. It is obvious that,

* degree of the intrinsic polynomial of a graph having no spanning circuits is less than the number of nodes in the graph, \(n\). Such a graph has zero eigenvalues.

This is due to the fact that the adjacency matrix of such a graph has a zero determinant, hence coefficient of \(s^n\) is 0. Hence degree of \(|I - sA|\) is definitely less than \(n\). Note also that in such a case, \(\lambda = 0\) is a root of the characteristic polynomial. When the adjacency matrix of a graph has no zero eigenvalues, the degree of its intrinsic polynomial is \(n\). Consider the graph given in Fig. 4.27.

![Fig. 4.27](image)

The adjacency matrix and intrinsic polynomial of the graph in Fig. 4.27 are,

\[
A = \begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\]

\[
|I - sA| = \begin{pmatrix}
1 - s & -s \\
-s & 1
\end{pmatrix} = 1 - s - s^2
\]
Graphs whose adjacency matrix have zero eigenvalues have fewer than \( n \) intrinsic values. The degree of the intrinsic polynomial of such graphs is less than \( n \). Consider the following graph.

![Graph](image)

**Fig. 4.28**

The adjacency matrix and intrinsic polynomial are,

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
|I - sA| = 0 \quad \Rightarrow \quad 1 - s^2 = 0
\]

* For an acyclic graph, \( |I - sA| = 1 \).

This is due to the fact that the trace of at least one of \( A, A^{(2)}, \ldots, A^{(n)} \) for a cyclic graph will be nonzero. If all of them are zero then the graph is acyclic.

* For a complete graph of \( n \) nodes,

\[
|I - sA| = [1 - (n - 1)s] (1 + s)^{n-1}
\]

Consider the complete graph of four nodes.

\[
|I - sA| = \begin{vmatrix}
1 & -s & -s & -s \\
-s & 1 & -s & -s \\
-s & -s & 1 & -s \\
-s & -s & -s & 1
\end{vmatrix}
\]

On adding all rows to the first row we obtain,

\[
|I - sA| = \begin{vmatrix}
1 - 3s & 1 - 3s & 1 - 3s & 1 - 3s \\
-s & 1 & -s & -s \\
-s & -s & 1 & -s \\
-s & -s & -s & 1
\end{vmatrix}
\]
\[ |I - sA| = (1 - 3s) \]

\[
\begin{vmatrix}
1 & 1 & 1 & 1 \\
-s & 1 & -s & -s \\
-s & -s & 1 & -s \\
-s & -s & -s & 1 \\
\end{vmatrix}
\]

\[ |I - sA| = (1 - 3s) \]

\[
\begin{vmatrix}
1 & 0 & 0 & 0 \\
-s & 1 + s & 0 & 0 \\
-s & 0 & 1 + s & 0 \\
-s & 0 & 0 & 1 + s \\
\end{vmatrix}
\]

\[ |I - sA| = (1 - 3s)(1 + s)^3 \]

Note that the above procedure can be carried out with any \( n \times n \) matrix to obtain the general form of \( |I - sA| \).

### 4.6.2 Circuit Polynomial

A cycle consisting of even number of edges is an *even cycle*, and that with odd number of edges is an *odd cycle*. Similarly, circuits are also called even or odd based on the number of edges in them.

- An even circuit is a *positive circuit* if it contains an even number of even cycles in it. The number of odd cycles in an even circuit is always even. An odd circuit is a positive circuit if it contains an odd number of even cycles in it. The number of odd cycles in an odd circuit is always odd.

![Fig. 4.29](image-url)
In the graph of Fig. 4.29 \( abef \), \( acdg \), and \( abg \) are examples of positive circuits.

- An even circuit is a *negative circuit* if it contains an odd number of even cycles in it. An odd circuit is negative if it contains an even number of even cycles in it.

![Graph](image)

Fig. 4.30

In the graph of Fig. 4.30 \( ef \), \( bcd \), and \( aefgh \) are examples of negative circuits.

- The difference in the number of positive circuits and negative circuits of length \( k \) is the number of *excess circuits* of length \( k \).

Note that, a basic circuit is always negative. This follows from the fact that a basic even circuit contains one *(odd number)* even cycle, and a basic odd circuit contains zero *(even number)* even cycle. The circuit polynomial of the adjacency matrix \( A \) as defined in the previous chapter is the cyclant of \( (I + \kappa sA) \) and is a polynomial in \( s \), of the form,

\[
\Gamma(I + \kappa sA) = \Delta_1(s) + \kappa \Delta_2(s).
\]

\[
= 1 + P_1(s) + P_2(s)
\]

Similarly, the circuit polynomial of \( C \) matrix can also be split into two parts – one with, and the other without, \( \kappa \). As shown below, the circuit polynomials of adjacency matrices behave in an interesting manner and separate out the positive
and negative circuits. The circuit polynomial of $A$ is denoted as $\Gamma(I + \kappa sA)$ and that of $C$ matrix as $\Gamma(I + \kappa sC)$. Consider the graph given below.

\[
\begin{align*}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 0 & 1 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 1 \\
4 & 0 & 0 & 1 & 0 \\
\end{array}
\end{align*}
\]

\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & 1 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 1 \\
4 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

\[
I + \kappa sA = \begin{pmatrix}
1 & \kappa s & 0 & \kappa s \\
\kappa s & 1 & 0 & 0 \\
0 & \kappa s & 1 & \kappa s \\
0 & 0 & \kappa s & 1 \\
\end{pmatrix}
\]

\[
\Gamma(I + \kappa sA) = 1 + 2\kappa s^2 + (1 + \kappa)s^4
\]

Notice that there is one positive and one negative circuit of length 4, and two negative circuits of length 2. Also the length of the longest circuit is 4, which is the degree of the circuit polynomial.
Consider another example.

\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}
\]

\[
I + \kappa s A = \begin{pmatrix} 1 & \kappa s & 0 \\ \kappa s & 1 & \kappa s \\ 0 & \kappa s & 1 + \kappa s \end{pmatrix}
\]

\[
\Gamma(I + \kappa s A) = 1 + s^3 + \kappa(s + 2s^2)
\]

The graph contains one positive circuit of length 3 and two negative circuits of length 1 and 2 respectively. Length of the longest circuit is 3.
Write the \( C \) matrix for the graph in Fig. 4.31.

\[
C = \begin{pmatrix}
0 & b & 0 & c \\
a & 0 & 0 & 0 \\
0 & d & 0 & e \\
0 & 0 & f & 0
\end{pmatrix}
\]

\[
I + \kappa sC = \begin{pmatrix}
1 & \kappa sb & 0 & \kappa sc \\
\kappa sa & 1 & 0 & 0 \\
0 & \kappa sd & 1 & \kappa se \\
0 & 0 & \kappa sf & 1
\end{pmatrix}
\]

\[
\Gamma(I + \kappa sC) = 1 + \kappa(ef + ab)s^2 + (abef + \kappa acdf)s^4
\]

\[
= 1 + abef s^4 + \kappa[(ef + ab)s^2 + acdf s^4]
\]

The edges in the circuits are thus listed out by the circuit polynomial of \( C \) matrix. This leads to the following results.

* Coefficient of \( s^m \) in the polynomial \( P_1(s) \) of \( \Gamma(I + \kappa sA) \) gives the number of positive circuits of length \( m \) in the graph. Coefficient of \( s^m \) in the polynomial \( P_2(s) \) of \( \Gamma(I + \kappa sA) \) gives the number of negative circuits of length \( m \) in the graph.

* Coefficient of \( s^m \) in \( P_1(s) \) and \( P_2(s) \) of \( \Gamma(I + \kappa sC) \) lists out the edges in the positive and negative circuits, respectively, of length \( m \) in the graph.

* The total number of circuits of all lengths in the graph is equal to,

\[
\left. \left( \frac{\Gamma(I + \kappa sA)}{s} \right) \right|_{k=1, s=1} - 1
\]

* The degree of circuit polynomial gives the length of the longest circuit in the graph. This is equal to the permanent degree of the adjacency matrix.

Consider the graph given below.
Degree of the circuit polynomial is 2, equal to the length of the longest circuit. Note that the indexed bigraph of the above graph was given in Fig. 4.22 and the permanent degree of $A$ was found to be 2. This is equal to the degree of the largest index matching.

### 4.7 Path Matrix

The paths in a graph can be obtained from its path matrix, a term which we define in this section. Consider the matrix $(I - sA)$. Recall that,

$$|I - sA| = 1 - a_1 s + a_2 s^2 + \cdots + (-1)^n a_n s^n$$
This implies that the determinant of \( (I - sA) \) is always nonzero and hence it is a nonsingular matrix.

- We define the matrix \( D = (I - sA)^{-1} \) as the path matrix of the graph corresponding to the adjacency matrix \( A \).

The variable \( s \) here acts as a counting variable to pick paths of a given length.

As we have seen,

\[
A = MAM^{-1}
\]

Hence,

\[
I - sA = I - sMAM^{-1}
\]

\[
= MM^{-1} - MsAM^{-1}
\]

\[
= M(I - sA)M^{-1}
\]

Thus for a \( 3 \times 3 \) matrix,

\[
(I - sA)^{-1} = M(I - sA)^{-1}M^{-1}
\]

\[
= M\begin{pmatrix}
\frac{1}{1-\lambda_1 s} & 0 & 0 \\
0 & \frac{1}{1-\lambda_2 s} & 0 \\
0 & 0 & \frac{1}{1-\lambda_3 s}
\end{pmatrix}M^{-1}
\]

\[
= M\begin{pmatrix}
1 + \sum_{i=1}^{\infty} \lambda_1^i s^i & 0 & 0 \\
0 & 1 + \sum_{i=1}^{\infty} \lambda_2^i s^i & 0 \\
0 & 0 & 1 + \sum_{i=1}^{\infty} \lambda_3^i s^i
\end{pmatrix}M^{-1}
\]

\[
= I + sMAM^{-1} + + \cdots + s^kMAM^kM^{-1} + \cdots
\]

\[
= I + sA + s^2A^2 + \cdots
\]

* The coefficient of \( s^k \) in the expansion of \( (I - sA)^{-1} \) is \( A^k \) which is the same as \( MAM^kM^{-1} \).
For the graph in Fig. 4.27,

\[ \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ (\mathbf{I} - s\mathbf{A}) = \begin{pmatrix} 1 - s & -s \\ -s & 1 \end{pmatrix} \]

\[ (\mathbf{I} - s\mathbf{A})^{-1} = \begin{bmatrix} 1 & s \\ 1 - s - s^2 & 1 - s - s^2 \\ 1 - s - s^2 & 1 - s - s^2 \end{bmatrix} \]

\[ = \begin{bmatrix} 1 + s + 2s^2 + 3s^3 + 5s^4 + \cdots \\ s + s^2 + 2s^3 + 3s^4 + 5s^5 + \cdots \\ s + s^2 + 2s^3 + 3s^4 + 5s^5 + \cdots \\ 1 + s^2 + s^3 + 2s^4 + 3s^5 + \cdots \end{bmatrix} \]

\[ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + s \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + s^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + s^3 \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} + \cdots \]

\[ = \mathbf{I} + s\mathbf{A} + s^2\mathbf{A}^2 + s^3\mathbf{A}^3 + s^4\mathbf{A}^4 + \cdots. \]

* The coefficient of \( s^k \) in \((\mathbf{I} - s\mathbf{A})^{-1}\) gives the number of paths of length \( k \) between all nodes of the graph.

As seen above,

\[ \mathbf{D} = (\mathbf{I} - s\mathbf{A})^{-1} = \mathbf{I} + s\mathbf{A} + (s\mathbf{A})^2 + (s\mathbf{A})^3 + \cdots \]

The coefficient of \( s^k \) in \((\mathbf{I} - s\mathbf{A})^{-1}\) is the matrix \( \mathbf{A}^k \) which is a natural matrix since \( \mathbf{A} \) is a natural matrix. The element \( \mathbf{A}^k[i,j] \) gives the number of paths of length \( k \) from node \( i \) to node \( j \). Hence the coefficient of \( s^k \) in \((\mathbf{I} - s\mathbf{A})^{-1}\) gives the number of paths of length \( k \) between the nodes. When \((\mathbf{I} - s\mathbf{C})^{-1}\) is considered, edges in the paths are listed out. Note that,

\[ \left( \frac{1}{k!} \right) \frac{d\mathbf{D}}{ds} \bigg|_{s=0} = \mathbf{A}^k \]
Hence the $k^{th}$ derivative, with respect to $s$, of path matrix gives all paths of length $k$ in the graph. Consider the graph in Fig. 4.36.

\[ 1 \rightarrow a \rightarrow 2 \rightarrow d \rightarrow 4 \]

**Fig. 4.36**

\[
\begin{align*}
A &= \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \\
C &= \begin{pmatrix}
0 & a & 0 & 0 \\
0 & 0 & b & d \\
0 & c & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\end{align*}
\]

\[
I - sA = \begin{pmatrix}
1 - s & 0 & 0 \\
0 & 1 - s & -s \\
0 & -s & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
(1 - sA)^{-1} =
\begin{bmatrix}
1 & \frac{s}{1 - s^2} & \frac{s^2}{1 - s^2} & \frac{s^2}{1 - s^2} \\
0 & 1 & \frac{s}{1 - s^2} & \frac{s}{1 - s^2} \\
0 & 0 & 1 & \frac{s^2}{1 - s^2} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & s(1 + s^2 + s^4 + \cdots) & s^2(1 + s^2 + s^4 + \cdots) & s^2(1 + s^2 + s^4 + \cdots) \\
0 & 1 + s^2 + s^4 + \cdots & s(1 + s^2 + s^4 + \cdots) & s(1 + s^2 + s^4 + \cdots) \\
0 & 0 & 1 + s^2 + s^4 + \cdots & s^2(1 + s^2 + s^4 + \cdots) \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Note from the above example that every element in $(I - sA)^{-1}$ is a rational function which on expanding as a power series in $s$ has all natural number coefficients. Such functions are called natural functions. Note that for a natural function $F(s)$, and a natural number $n,$

\[
\left( \frac{1}{n!} \right) \frac{dF}{ds} \bigg|_{s=0}
\]

is a natural number for all values of $n$.

Note that the adjacency matrix of a graph can be written as a boolean matrix with 0 and 1 as its elements by replacing every nonzero element of the natural matrix $A$ by 1. For studying those properties of graphs that depend only upon node adjacency and not on the actual number of edges, it suffices to write $A$ as a boolean matrix. The following chapter deals with matrices having boolean elements.