CHAPTER 5

COMMON FIXED POINT THEOREMS UNDER
CONTRACTIVE AND NONCONTRACTIVE
CONDITIONS

Present chapter is aimed at obtaining some further consequences of the concept of weak reciprocal continuity together with absorbing maps. Further we deduce some well known fixed point theorems due to Banach [6], Kannan [50], Chatterjea [15] and Hardy and Rogers [28] as particular cases of our theorems. While proving our main results we use minimal commutativity condition, i.e., pointwise $R$-weakly commuting maps.

Preliminaries:

It is well known that pointwise $R$-weak commutativity is equivalent to commutativity at coincidence points and in the setting of metric spaces these notions are equivalent to weak compatibility. On the other hand, Pointwise $R$-weak commutativity is more useful in establishing common fixed point theorems since it not only imply commutativity at coincidence points but may also help in the determination of coincidence points.

It is pertinent to mention here that the absorbing maps are neither a subclass of compatible maps nor a subclass of noncompatible maps [28]. Weak reciprocal continuity provides a powerful tool to study common fixed point theorems under contractive conditions or possibly Lipschitz type mapping pair
and extends the scope of the study of common fixed point theorems from the class of compatible continuous mappings to a wider class of mappings which also includes noncompatible and discontinuous mappings [72].

**Main Results:**

**Theorem 5.1:** Let \( f \) and \( g \) be weakly reciprocally continuous of weakly compatible self-mappings of a complete metric space \((X, d)\) such that

1. \( fX \subseteq gX \)
2. \( d(fx, fy) \leq a_1d(gx, gy) + a_2[d(fx, gx) + d(fy, gy)] + a_3[d(fy, gx) + d(fx, gy)] \)

where \( a_1, a_2, a_3 \geq 0, 0 \leq a_1 + 2a_2 + 2a_3 < 1 \). If \( g \) is \( f \)-absorbing or \( f \) is \( g \)-absorbing then \( f \) and \( g \) have a unique common fixed point.

**Proof:** Let \( x_0 \) be any point in \( X \). Define sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) by

\[
y_n = fx_n = gx_{n+1} \quad \ldots (5.1)
\]

We claim that \( \{y_n\} \) is a Cauchy sequence. Using (ii) we obtain

\[
d(y_n, y_{n+1}) = d(fx_n, fx_{n+1}) \leq a_1d(gx_n, gx_{n+1}) + a_2[d(fx_n, gx_n) + d(fx_{n+1}, gx_{n+1})] + a_3[d(fx_n, gx_{n+1}) + d(fx_{n+1}, gx_n)],
\]

i.e.,

\[
d(y_n, y_{n+1}) \leq a_1d(y_{n-1}, y_n) + a_2[d(y_n, y_{n-1}) + d(y_{n+1}, y_n)] + a_3[d(y_n, y_{n-1}) + d(y_{n+1}, y_{n-1})]
\]

\[
d(y_n, y_{n+1}) \leq a_1d(y_{n-1}, y_n) + a_2[d(y_n, y_{n-1}) + d(y_{n+1}, y_n)] + a_3[d(y_n, y_{n-1}) + d(y_{n+1}, y_{n-1})]
\]

\[
d(y_n, y_{n+1}) = \left(\frac{a_1 + a_2 + a_3}{1 - a_2 - a_3}\right)d(y_{n-1}, y_n) = kd(y_{n-1}, y_n), \text{ since } \left(\frac{a_1 + a_2 + a_3}{1 - a_2 - a_3}\right) < 1.
\]

Moreover, for every integer \( p > 0 \), we get

\[
d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{n+p-1}, y_{n+p})
\]
\[ \leq d(y_n, y_{n+1}) + k d(y_n, y_{n+1}) + \ldots + k^{p-1} d(y_n, y_{n+1}) \]

\[ = (1 + k + k^2 + \ldots + k^{p-1}) d(y_n, y_{n+1}) \]

\[ \leq \left( \frac{1}{1-k} \right) d(y_n, y_{n+1}) \leq \left( \frac{k^n}{1-k} \right) d(y_0, y_1). \]

This means that \( d(y_n, y_{n+p}) \to 0 \) as \( n \to \infty \). Therefore \( \{y_n\} \) is a Cauchy sequence. Since \( X \) is complete, there exists a point \( t \) in \( X \) such that \( y_n \to t \). Moreover, \( y_n = f x_n = g x_{n+1} \to t \).

Suppose that \( g \) is \( f \)-absorbing. Now, weak reciprocal continuity of \( f \) and \( g \) implies that \( f g x_n \to ft \) or \( g f x_n \to gt \). Let \( f g x_n \to gt \). By virtue of (5.1) this also yields \( g g x_{n+1} = g f x_n \to gt \). Since \( g \) is \( f \)-absorbing, \( d(f x_n, g f x_n) \leq R d(f x_n, g x_n) \). On letting \( n \to \infty \), we obtain \( f g x_n \to t \). Using (ii) we get \( d(ft, f g x_n) \leq a_1 d(gt, gg x_n) + a_2 [d(f t, g t) + d(g f x_n, gg x_n)] + a_3 [d(f t, g g x_n) + d(f x_n, g t)] \). On making \( n \to \infty \) we get \( f g x_n \to ft \). Hence \( t = ft \). Since \( f X \subseteq g X \), there exists \( u \) in \( X \) such that \( t = ft = gu \). Now using (ii), we obtain \( d(f x_n, f u) \leq a_1 [d(g x_n, gu)] + a_2 [d(f x_n, g x_n)] + a_3 [d(f x_n, g u) + d(f u, g x_n)] \). On letting \( n \to \infty \), we get \( f u = gu \). Since \( f \) and \( g \) are weakly compatible, hence \( f g u = g f u \) and \( g f u = g g u = gg u = ff u \). Finally using (ii), we obtain \( d(f u, f f u) \leq a_1 [d(g u, g f u)] + a_2 [d(f u, g u)] + a_3 [d(f u, g f u) + d(f f u, g f u)] \), that is, \( (1 - a_1 - 2a_3) d(f u, f f u) = 0 \). Hence \( f u = ff u = g f u \) and \( f u \) is a common fixed point of \( f \) and \( g \).

Next suppose that \( f g x_n \to ft \). Since \( g \) is \( f \)-absorbing, \( d(f x_n, f g x_n) \leq R d(f x_n, g x_n) \). On letting \( n \to \infty \), we get \( t = ft \). Since \( f X \subseteq g X \), there exists \( u \) in \( X \) such that \( t = ft = gu \). Now using (ii), we obtain \( d(f x_n, f u) \leq a_1 [d(g x_n, gu)] + a_2 [d(f x_n, g x_n)] + a_3 [d(f x_n, g u) + d(f u, g x_n)] \). On letting \( n \to \infty \), we get \( f u = gu \). Since \( f \) and \( g \) are weakly compatible, hence \( f g u = g f u \) and \( g f u = g g u = ff u \). Finally using (ii), we obtain \( d(f u, f f u) \leq a_1 [d(g u, g f u)] + a_2 [d(f u, g u)] + a_3 [d(f u, g f u) + d(f f u, g f u)] \), that is, \( (1 - a_1 - 2a_3) d(f u, f f u) = 0 \). Hence \( f u = ff u = g f u \) and \( f u \) is a common fixed point of \( f \) and \( g \).
Finally suppose that \( f \) is \( g \)-absorbing. Now, weak reciprocal continuity of \( f \) and \( g \) implies that \( fgx_n \to ft \) or \( gfx_n \to gt \). Let us first assume that \( gfx_n \to gt \). Since \( f \) is \( g \)-absorbing, \( d(gx_n, gfx_n) \leq Rd(fx_n, gx_n) \). On making \( n \to \infty \), we get \( t = gt \). Using (ii) we get
\[
d(fx_n, ft) \leq a_1 d(gx_n, gt) + a_2 [d(fx_n, gfx_n) + d(ft, gt)] + a_3 [d(fx_n, gt) + d(ft, gfx_n)].
\]
On letting \( n \to \infty \), we get \( fx_n \to ft \). Hence \( t = ft = gt \) and \( t \) is a common fixed point of \( f \) and \( g \).

Next suppose that \( fgx_n \to ft \). Then \( fX \subseteq gX \) implies that \( ft = gu \) for some \( u \in X \). By virtue of (5.1) this also yields \( ffx_{n-1} \to gu \). Since \( f \) is \( g \)-absorbing, \( d(gx_n,gfx_n) \leq Rd(fx_n, gx_n) \). On letting \( n \to \infty \), we get \( gfx_n \to t \). Now, using (ii), we get
\[
d(fx_n, fu) \leq a_1 d(gx_n, gu) + a_2 [d(fx_n, gfx_n) + d(fu, gu)] + a_3 [d(fx_n, gu) + d(fu, gfx_n)].
\]
On making \( n \to \infty \), we obtain \( t = gu \). Again, by virtue of (ii),
\[
d(fx_n, gu) \leq a_1 d(gx_n, gu) + a_2 [d(fx_n, gfx_n) + d(fu, gu)] + a_3 [d(fx_n, gu) + d(fu, gfx_n)].
\]
On letting \( n \to \infty \), we get \( fu = t \). Thus \( fu = gu \). Since \( f \) and \( g \) are weakly compatible, hence \( fgu = gfu \) and \( fgu = gfu = gfu = ffu \). Finally using (ii), we obtain
\[
d(fu, ffu) \leq a_1 d(gu, gfu) + a_2 [d(fu, gu) + d(ffff, gfu)] + a_3 [d(fu, gfu) + d(ffff, gu)],
\]
that is, \( (1-a_1-2a_3)d(fu, ffu) = 0 \). Hence \( fu = ffu = gfu \) and \( fu \) is a common fixed point of \( f \) and \( g \). Uniqueness of the common fixed point theorem follows easily in each of the two cases.

We now give an example to illustrate the above theorem.

**Example 5.1** [67, 8]: Let \( X = [2, 20] \) and \( d \) be the usual metric on \( X \). Define \( f, g : X \to X \) as follows

\[
fx = 2 \text{ if } x = 2 \text{ or } x > 5, \quad fx = 6 \text{ if } 2 < x \leq 5,
\]

\[
gx = x + 1 \text{ if } 2 < x \leq 5, \quad gx = \frac{x + 1}{3} \text{ if } x > 5.
\]

Then \( f \) and \( g \) satisfy all the conditions of Theorem 5.1 and have a unique common fixed point at \( x = 2 \). It can be verified in this example that \( f \) and \( g \) satisfy
the contraction condition (ii) for \( a_1 = \frac{4}{5} \), \( a_2 = \frac{1}{12} \), \( a_3 = \frac{1}{18} \). The mappings f and g are weakly compatible maps as they commute at their only coincidence point \( x = 2 \). Furthermore, f is g-absorbing with \( R = \frac{29}{18} \). It can also be noted that f and g are weakly reciprocally continuous. To see this, let \( \{x_n\} \) be a sequence in X such that \( f x_n \to t \), \( g x_n \to t \) for some t. Then t = 2 and either \( x_n = 2 \) for each n from some place onwards or \( x_n = 5 + \varepsilon_n \) where \( \varepsilon_n \to 0 \) as \( n \to \infty \). If \( x_n = 2 \) for each n from some place onwards, \( fg x_n \to 2 = f 2 \) and \( gf x_n \to 2 = g 2 \). If \( x_n = 5 + \varepsilon_n \) then \( fx_n \to 2 \), \( gx_n = (2 + \frac{\varepsilon_n}{3}) \to 2 \), \( fg x_n = f (2 + \frac{\varepsilon_n}{3}) \to 6 \neq f 2 \), and \( gfx_n \to g 2 = 2 \). Thus \( \lim_{n \to \infty} gfx_n = g 2 \) but \( \lim_{n \to \infty} fg x_n \neq f 2 \). Hence f and g are weakly reciprocally continuous. It is also obvious that f and g are not reciprocally continuous mappings.

Putting \( a_3 = 0 \), we get the following result:

**Corollary 5.1:** Let f and g be weakly reciprocally continuous of weakly compatible self-mappings of a complete metric space \((X, d)\) such that

(i) \( f X \subseteq g X \)

(ii) \( d(f x, f y) \leq a_1 d(g x, g y) + a_2 [d(f x, g x) + d(f y, g y)] \),

where \( a_1, a_2 \geq 0, 0 \leq a_1 + 2a_2 < 1 \). If g is f-absorbing or f- is g- absorbing then f and g have a unique common fixed point.

Putting \( a_2 = 0 \), and we get the following result:

**Corollary 5.2:** Let f and g be weakly reciprocally continuous of weakly compatible self-mappings of a complete metric space \((X, d)\) such that

(i) \( f X \subseteq g X \)

(ii) \( d(f x, f y) \leq a_1 d(g x, g y) + a_3 [d(f x, g y) + d(f y, g x)] \),

where \( a_1, a_3 \geq 0, 0 \leq a_1 + 2a_3 < 1 \). If g is f-absorbing or f- is g- absorbing then f and g have a unique common fixed point.
Putting $a_3=a_2=0$, we get the following corollary:

**Corollary 5.3:** Let $f$ and $g$ be weakly reciprocally continuous of weakly compatible self-mappings of a complete metric space $(X, d)$ such that

1. $fX \subseteq gX$
2. $d(fx, fy) \leq a_1 d(gx, gy)$, where $0 \leq a_1 < 1$. If $g$ is $f$-absorbing or $f$ is $g$-absorbing then $f$ and $g$ have a unique common fixed point.

**Remark 5.1:** Putting $g =$ identity map, we get some of the famous fixed point theorems as a particular case of the above theorem.

**Corollary 5.4** *(Hardy and Rogers fixed point theorem [30]):* Let $f$ be a self-mappings of a complete metric space $(X, d)$ such that

1. $d(fx, fy) \leq a_1 d(x, y) + a_2 [d(fx, x) + d(fy, y)] + a_3 [d(fx, y) + d(fy, x)]$, where $a_1, a_2, a_3 \geq 0$, $0 \leq a_1 + 2a_2 + 2a_3 < 1$. Then $f$ has a unique fixed point.

**Corollary 5.5:** *(Kannan fixed point theorem)*: Let $f$ be a self-mappings of a complete metric space $(X, d)$ such that

1. $d(fx, fy) \leq a_2 [d(fx, x) + d(fy, y)]$, where $0 \leq a_2 < 1/2$. Then $f$ has a unique fixed point.

**Corollary 5.6:** *(Chatterjee fixed point theorem [15]):* Let $f$ be a self-mappings of a complete metric space $(X, d)$ such that

1. $d(fx, fy) \leq a_3 [d(fx, y) + d(fy, x)]$, where $0 \leq a_3 < 1/2$. Then $f$ has a unique fixed point.

**Corollary 5.7:** *(Banach fixed point theorem [6]):* Let $f$ be a self-mappings of a complete metric space $(X, d)$ such that
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(i) \[ d(fx, fy) \leq a_1 d(x, y), \]

where \( 0 \leq a_1 < 1. \) Then \( f \) has a unique fixed point.

We now establish a common fixed point theorem for a pair of mappings satisfying an \((\varepsilon, \delta)\) type contractive condition. The next theorem demonstrates the usefulness of weak reciprocal continuity and shows that the new notion ensures the existence of a common fixed point under an \((\varepsilon, \delta)\) contractive condition.

**Theorem 5.2:** Let \( f \) and \( g \) be weakly reciprocally continuous of a weakly compatible self-mappings of a complete metric space \((X, d)\) such that

(i) \( fX \subseteq gX \)

(ii) \( d(fx, fy) < d(gx, gy) \) whenever \( gx \neq gy \)

(iii) given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[ \varepsilon < d(gx, gy) < \varepsilon + \delta \Rightarrow d(fx, fy) \leq \varepsilon \]

If \( g \) is \( f\)-absorbing or \( f \) is \( g\)-absorbing then \( f \) and \( g \) have a unique common fixed point.

**Proof:** Let \( x_0 \) be any point in \( X \). Define sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) by

\[ y_n = fx_n = gx_{n+1} \]

We claim that \( \{y_n\} \) is a Cauchy sequence. Using (ii) we obtain

\[ d(y_n, y_{n+1}) = d(fx_n, fx_{n+1}) < d(gx_n, gx_{n+1}) = d(y_{n-1}, y_n). \]

Thus \( \{d(y_n, y_{n+1})\} \) is a strictly decreasing sequence of positive real numbers and, therefore, tends to a limit \( r \geq 0 \), that is, \( \lim_{n \to \infty} d(y_n, y_{n+1}) = r \), \( r \geq 0 \). We assert that \( r = 0 \). For, if not, suppose that \( r > 0 \). Then given \( \delta > 0 \), however small \( \delta \) may be, there exists a positive integer \( N \) such that for each \( n \geq N \) we have
that is,

\[ r < d(y_n, y_{n+1}) = d(fx_n, fx_{n+1}) < r + \delta \]

Selecting \( \delta \) in (5.3) in accordance with (iii), for each \( n \geq N \) we get \( d(fx_{n+1}, fx_{n+2}) \leq r \), that is, \( d(y_{n+1}, y_{n+2}) \leq r \), a contradiction to (5.3). Therefore \( \lim_{n \to \infty} d(y_n, y_{n+1}) = 0 \).

We now show that \{\( y_n \)} is a Cauchy sequence. Suppose it is not. Then there exists an \( \epsilon > 0 \) and a subsequence \{\( y_{n_i} \)} of \{\( y_n \)} such that \( d(y_{n_i}, y_{n_i+1}) \geq 2 \epsilon \). Select \( \delta \) in (iii) so that \( 0 < \delta \leq \epsilon \). Since \( \lim_{n \to \infty} d(y_n, y_{n+1}) = 0 \), there exists an integer \( N \) such that \( d(y_n, y_{n+1}) \leq \frac{\epsilon}{6} \) whenever \( n \geq N \).

Let \( n \geq N \). Then, there exist integers \( m_i \) satisfying \( n_i < m_i < n_i+1 \) such that \( d(y_{n_i}, y_{m_i}) \geq \epsilon + \frac{\delta}{3} \). If not, then

\[ d(y_{n_i}, y_{n_i+1}) \leq d(y_{n_i}, y_{n_i+1-1}) + d(y_{n_i+1-1}, y_{n_i+1}) < \epsilon + \frac{\delta}{6} \leq 2\epsilon, \]

a contradiction. Let \( m_i \) be the smallest integer such that \( d(y_{n_i}, y_{m_i}) \geq \epsilon + \frac{\delta}{3} \). Then

\[ d(y_{n_i}, y_{m_i-2}) < \epsilon + \frac{\delta}{3} \text{ and } \epsilon + \frac{\delta}{3} \leq d(y_{n_i}, y_{m_i}) \leq d(y_{n_i}, y_{m_i-2}) + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i}) < \epsilon + \frac{\delta}{3} + \frac{\delta}{6} + \frac{\delta}{6} < \epsilon + \frac{2\delta}{3}, \]

(5.4)

That is, \( \epsilon < \epsilon + \frac{\delta}{3} \leq d(gx_{n_i+1}, gx_{m_i+1}) < \epsilon + (\frac{\delta}{3}) \delta \). In view of (iii), this yields \( d(y_{n_i+1}, y_{m_i+1}) \leq \epsilon \). But then

\[ d(y_{n_i}, y_{m_i}) \leq d(y_{n_i}, y_{n_i+1}) + d(y_{n_i+1}, y_{m_i+1}) + d(y_{m_i+1}, y_{m_i}) \]
which contradicts (5.4). Hence \( \{y_n\} \) is a Cauchy sequence. Since \( X \) is complete, there exists a point \( t \) in \( X \) such that \( y_n \to t \). Moreover, \( y_n = f x_n = g x_{n+1} \to t \).

Suppose that \( g \) is \( f \)-absorbing. Now, weak reciprocal continuity of \( f \) and \( g \) implies that \( g f x_n \to ft \) or \( g f x_n \to gt \). By virtue of (5.2) this also yields \( g g x_{n+1} = g f x_n \to gt \). Since \( g \) is \( f \)-absorbing, \( d(f x_n, g f x_n) \leq R d(f x_n, g x_n) \). On letting \( n \to \infty \), we get \( g f x_n \to t \). Using (ii) we get \( d(ft, g f x_n) < d(gt, g g x_n) \). On making \( n \to \infty \) we get \( g f x_n \to ft \). Hence \( t = ft \). Since \( f X \subseteq g X \), there exists \( u \) in \( X \) such that \( t = ft = gu \). Now using (ii), we obtain \( d(f x_n, fu) < d(g x_n, gu) \). On letting \( n \to \infty \), we get \( fu = t \). Thus \( fu = gu \). Since \( f \) and \( g \) are weakly compatible, that is, \( f gu = g fu \) and hence \( f gu = g fu = ggu = ffu \). If \( fu \neq ffu \) then using (ii) we get \( d(fu, ffu) < d(gu, gfu) = d(fu, ffu) \), a contradiction. Hence \( fu = ffu = gfu \) and \( fu \) is a common fixed point of \( f \) and \( g \).

Next suppose that \( f g x_n \to ft \). Since \( g \) is \( f \)-absorbing, \( d(f x_n, f g x_n) \leq R d(f x_n, g x_n) \). On letting \( n \to \infty \), we get \( t = ft \). Since \( f X \subseteq g X \), there exists \( u \) in \( X \) such that \( t = ft = gu \). Now using (ii), we obtain \( d(f x_n, fu) < d(g x_n, gu) \). On letting \( n \to \infty \), we get \( fu = t \). Thus \( fu = gu \). Since \( f \) and \( g \) are weakly compatible that is, \( f gu = g fu \) and hence \( f gu = g fu = ggu = ffu \). If \( fu \neq ffu \) then using (ii) we get \( d(fu, ffu) < d(gu, gfu) = d(fu, ffu) \), a contradiction. Hence \( fu = ffu = gfu \) and \( fu \) is a common fixed point of \( f \) and \( g \).

When \( f \) is assumed \( g \)-absorbing the proof follows on similar lines as in the corresponding part of Theorem 5.1.

**Remark 5.2:** Theorem 5.2 generalizes the well-known fixed point theorem of Meir and Keeler [59].

**Corollary 5.8:** Let \( f \) be a self mapping of a complete metric space \((X, d)\) such that

(i) \( d(fx, fy) < d(x, y) \) whenever \( x \neq y \)
(ii) given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon < d(x, y) < \varepsilon + \delta \Rightarrow d(fx, fy) \leq \varepsilon.$$ 

Then $f$ has a unique fixed point.

**Corollary 5.9:** Let $f$ be a self mapping of a complete metric space $(X, d)$ such that

(i) given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon < d(x, y) < \varepsilon + \delta \Rightarrow d(fx, fy) \leq \varepsilon.$$ 

Then $f$ has a unique fixed point.

**Theorem 5.3:** Let $f$ and $g$ be weakly reciprocally continuous non compatible self-mappings of a metric space $(X, d)$ satisfying

(i) $fX \subseteq gX$

(ii) $d(fx, fy) \leq k d(gx, gy)$, $k \geq 0$.

If $f$ and $g$ are weakly compatible and $g$ is $f$-absorbing or $f$ is $g$-absorbing then $f$ and $g$ have a common fixed point.

**Proof:** Since $f$ and $g$ are non compatible maps, there exists a sequence $\{x_n\}$ in $X$ such that $fx_n \to t$ and $gx_n \to t$ for some $t$ in $X$ but either $\lim_{n \to \infty} d(fgx_n, gfx_n) \neq 0$ or the limit does not exist. Since $fX \subseteq gX$, for each $x_n$ there exists $y_n$ in $X$ such that $fx_n = gy_n$. Thus $fx_n \to t$, $gx_n \to t$ and $gy_n \to t$ as $n \to \infty$. By virtue of this and using (ii) we obtain $fy_n \to t$. Therefore, we have

$$fx_n = gy_n \to t, gx_n \to t, fy_n \to t.$$ 

Suppose that $g$ is $f$-absorbing. Then $d(fx_n, fgx_n) \leq Rd(fx_n, gx_n)$ and $d(fy_n, fgy_n) \leq Rd(fy_n, gy_n)$. On letting $n \to \infty$ these inequalities yield

$$fgx_n \to t \text{ and } fgy_n (= ffx_n) \to t.$$ 

...(5.5)
Weak reciprocal continuity of $f$ and $g$ implies that $fgx_n \to ft$ or $gf x_n \to gt$. By virtue of (ii) we get $d(ffx_n, ft) \leq kd(gfx_n, gt)$. On letting $n \to \infty$, we get $ffx_n \to ft$.

In view of (5.5) this yields $t = ft$. Since $fX \subseteq gX$, there exists $u$ in $X$ such that $t = ft = gu$. Now using (ii), we obtain $d(fx_n, fu) \leq kd(gx_n, gu)$. On letting $n \to \infty$, we get $fu = t$. Thus $fu = gu$. Weak compatibility of $f$ and $g$ implies that $f$ and $g$ commute at $u$, $fgu = gf u$. Also $ff u = fg u = gf u = gg u$. Since $g$ is $f$-absorbing $d(fu, fg u) \leq Rd(fu, gu)$. This yields $fu = fg u$. Hence $fu = ff u = gf u$ and $fu$ is a common fixed point of $f$ and $g$.

Next suppose that $fgx_n \to ft$. In view of (5.5) we get $t = ft$. Since $fX \subseteq gX$, there exists $u$ in $X$ such that $t = ft = gu$. Now using (ii), we obtain $d(fx_n, fu) \leq k d(gx_n, gu)$. On letting $n \to \infty$, we get $fu = t$. Thus $fu = gu$. This, in view of weakly compatible and $f$-absorbing property of $g$, implies that $f$ and $g$ have a common fixed point.

Now suppose that $f$ is $g$-absorbing. Then $d(gx_n, gfx_n) \leq Rd(fx_n, gx_n)$ and $d(gy_n, gfy_n) \leq Rd(fy_n, gy_n)$. On letting $n \to \infty$, these inequalities yield

$$gfx_n = ggy_n \to t \text{ and } gfy_n \to t.$$  …(5.6)

Weak reciprocal continuity of $f$ and $g$ implies that $fg y_n \to ft$ or $gfy_n \to gt$. Let us first assume that $gfy_n \to gt$. In view of (5.6) this yields $t = gt$. Using (ii) we get $d(fx_n, ft) \leq kd(gx_n, gt)$. On letting $n \to \infty$, we obtain $t = ft$. Hence $t = ft = gt$ and $t$ is a common fixed point of $f$ and $g$.

Next suppose that $fg y_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$. Therefore, $fg y_n \to ft = gu$. Using (ii) and in view of (5.6), we get $d(fy_n, fgy_n) \leq kd(gy_n, ggy_n)$. On letting $n \to \infty$, we get $t = gu$. Again, by virtue of (ii), we obtain $d(fy_n, fu) \leq kd(gy_n, gu)$. Making $n \to \infty$ we get $t = fu$. Hence $fu = gu$. Weak compatibility of $f$ and $g$ implies that $f$ and $g$ commute at $u$, $fg u = gf u$. Also $ff u = fg u = gu$. Since $f$ is $g$-absorbing $d(gu, gf u) \leq Rd(fu, gu)$. This yields $gu =
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gfu. Hence fu = ffu = gfu and fu is a common fixed point of f and g. This completes the proof of the theorem.

We now give examples to illustrate Theorem 5.3.

**Example 5.2 [8]:** Let $X = [0, 1]$ and $d$ be the usual metric on $X$. Define $f, g : X \to X$ as follows:

$$fx = \frac{1}{2} - |x - \frac{1}{2}|,$$

$$gx = \left(\frac{2}{3}\right) \text{ fractional part of } (1-x)$$

Then $f$ and $g$ satisfy all the conditions of the above theorem and have three coincidence points $x = 0, \frac{2}{5}, 1$ and two common fixed point $x = 0, \frac{2}{5}$. It may be verified in this example that $f(X) = [0, \frac{1}{2}]$, $g(X) = [0, \frac{2}{3}]$ and $fX \subseteq gX$. Also, $f$ and $g$ are weakly compatible maps since they commute at each of their coincidence points viz. $x = 0, \frac{2}{5}, 1$. To see that $f$ and $g$ are noncompatible, let us consider the sequence $\{x_n\}$ given by $x_n = 1 - \frac{1}{n}$. Then $fx_n \to 0$, $gx_n \to 0$, $fgx_n \to 0$, and $gfx_n \to \frac{2}{3}$. Hence $f$ and $g$ are noncompatible. It is also easy to verify that $f$ and $g$ satisfy the Lipschitz type condition $d(fx, fy) \leq \left(\frac{3}{2}\right) d(gx, gy)$. The mapping $g$ is $f$-absorbing since $d(fx, fgx) \leq d(fx, gx)$ for all $x$. It can also be noted that $f$ and $g$ are weakly reciprocally continuous. To see this, let $\{x_n\}$ be a sequence in $X$ such that $fx_n \to t$, $gx_n \to t$, for some $t$. Then $t = 0$ and either $x_n = 0$ for each $n$ or $x_n \to 1$. If $x_n = 0$ for each $n$, then $fx_n \to 0, gx_n \to 0, fgx_n \to 0 = f(0)$ and $gfx_n \to 0 = g(0)$. If $x_n \to 1$ then $fx_n \to 0, gx_n \to 0, fgx_n \to 0 = f(0)$ and $gfx_n \to \frac{2}{3} \neq g(0)$. Thus $\lim_{n \to \infty} fx_n = f(0)$ but $\lim_{n \to \infty} gfx_n \neq g(0)$. Hence $f$ and $g$ are weakly reciprocally continuous.

Putting $k = 1$ in Theorem 5.3, we get a common fixed point theorem for a non-expansive type mapping pair:
Corollary 5.10: Let \( f \) and \( g \) be weak reciprocally continuous noncompatible self-mappings of a metric space \((X, d)\) satisfying

(i) \( fX \subseteq gX \)

(ii) \( d(fx, fy) \leq d(gx, gy) \).

If \( f \) and \( g \) are weakly compatible and \( g \) is \( f \)-absorbing or \( f \) is \( g \)-absorbing then \( f \) and \( g \) have a common fixed point.

In the next theorem we prove a common fixed point theorem under strict contractive condition satisfying property (E. A.).

Theorem 5.4: Let \( f \) and \( g \) be weakly reciprocally continuous self-mappings of a metric space \((X, d)\) satisfying (E. A.) property and

(i) \( fX \subseteq gX \)

(ii) \( d(fx, fy) < d(gx, gy) \), whenever \( gx \neq gy \).

If \( f \) and \( g \) are weakly compatible and \( g \) is \( f \)-absorbing or \( f \) is \( g \)-absorbing then \( f \) and \( g \) have a common fixed point.

Proof: Since \( f \) and \( g \) satisfy property (E.A.), there exists a sequence \( \{x_n\} \) in \( X \) such that \( fx_n \rightarrow t \) and \( gx_n \rightarrow t \) for some \( t \) in \( X \). Since \( fX \subseteq gX \), for each \( x_n \) there exists \( y_n \) in \( X \) such that \( fx_n = gy_n \). Thus \( fx_n \rightarrow t \), \( gx_n \rightarrow t \) and \( gy_n \rightarrow t \) as \( n \rightarrow \infty \). By virtue of this and using (ii) we obtain \( fy_n \rightarrow t \). Therefore, we have

\[
fx_n = gy_n \rightarrow t, \quad gx_n \rightarrow t, \quad fy_n \rightarrow t.
\]

Suppose that \( g \) is \( f \)-absorbing. Then \( d(fx_n, fgx_n) \leq Rd(fx_n, gx_n) \) and \( d(fy_n, fgy_n) \leq Rd(fy_n, gy_n) \). On letting \( n \rightarrow \infty \) these inequalities yield

\[
fgx_n \rightarrow t \quad \text{and} \quad fgy_n \rightarrow t \quad \text{as} \quad n \rightarrow \infty.
\]
Weak reciprocal continuity of $f$ and $g$ implies that $fgx_n \to ft$ or $gfx_n \to gt$. Let $gfx_n \to gt$. By virtue of (ii) we get $d(ffx_n, ft) < d(gfx_n, gt)$. On letting $n \to \infty$, we get $ffx_n \to ft$. In view of (5.7) this yields $t = ft$. Since $fX \subseteq gX$, there exists $u$ in $X$ such that $t = ft = gu$. Now using (ii), we obtain $d(fx_n, fu) < d(gx_n, gu)$. On letting $n \to \infty$, we get $fu = t$. Thus $fu = gu$. Weak compatibility of $f$ and $g$ implies that $f$ and $g$ commute at $u$, $fgu = gfu$. Also $ffu = fg = gf = ggu$. Since $g$ is $f$-absorbing $d(fu, gfu) \leq Rd(fu, gu)$. This yields $gu = fgu$. Hence $fu = ffu = gfu$ and $fu$ is a common fixed point of $f$ and $g$.

Next suppose that $fgx_n \to ft$. In view of (5.7) we get $t = ft$. Since $fX \subseteq gX$, there exists $u$ in $X$ such that $t = ft = gu$. Now using (ii), we obtain $d(fx_n, fu) < d(gx_n, gu)$. On letting $n \to \infty$, we get $fu = t$. Thus $fu = gu$. This, in view of weakly compatible and $f$-absorbing property of $g$, implies that $f$ and $g$ have a common fixed point.

Now suppose that $f$ is $g$-absorbing. Then $d(gx_n, gfx_n) \leq Rd(fx_n, gx_n)$ and $d(gy_n, gfy_n) \leq Rd(fy_n, gy_n)$. On letting $n \to \infty$, these inequalities yield

$$gfx_n (= ggy_n) \to t \text{ and } gfy_n \to t.$$  \hspace{1cm} (5.8)

Weak reciprocal continuity of $f$ and $g$ implies that $fgy_n \to ft$ or $gfy_n \to gt$. Let us first assume that $gfy_n \to gt$. In view of (5.8) this yields $t = gt$. Using (ii) we get $d(fx_n, ft) < d(gx_n, gt)$. On letting $n \to \infty$, we obtain $t = ft$. Hence $t = ft = gt$ and $t$ is a common fixed point of $f$ and $g$.

Next suppose that $fgy_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$. Therefore, $fgy_n \to ft = gu$. Using (ii) and in view of (5.8), we get $d(fy_n, fgy_n) < d(gy_n, ggy_n)$. On letting $n \to \infty$, we get $t = gu$. Again, by virtue of (ii), we obtain $d(fy_n, fu) < d(gy_n, gu)$. Making $n \to \infty$ we get $t = fu$. Hence $fu = gu$. Weak compatibility of $f$ and $g$ implies that $f$ and $g$ commute at $u$, $fgu = gf$. Also $ffu = fgu = gf = ggu$. Since $f$ is $g$-absorbing $d(gu, gfu) \leq Rd(fu, gu)$. This yields $gu =
gfu. Hence fu = ffu = gfu and fu is a common fixed point of f and g. This completes the proof of the theorem.

**Corollary 5.11:** Let f and g be weakly reciprocally continuous noncompatible self-mappings of a metric space (X, d) satisfying

(i) \( fX \subseteq gX \)

(ii) \( d(fx, fy) < d(gx, gy) \), whenever \( gx \neq gy \).

If f and g are weakly compatible and g is f-absorbing or f is g-absorbing then f and g have a common fixed point.

**Corollary 5.12:** Let f and g be reciprocally continuous noncompatible self-mappings of a metric space (X, d) satisfying

(i) \( fX \subseteq gX \)

(ii) \( d(fx, fy) < d(gx, gy) \), whenever \( gx \neq gy \).

If f and g are weakly compatible and g is f-absorbing or f is g-absorbing then f and g have a common fixed point.

**Theorem 5.5:** Let f and g be weakly reciprocally continuous self-mappings of a metric space (X, d) satisfying (E. A.) property and

(i) \( fX \subseteq gX \)

(ii) \( d(fx, fy) \leq kd(gx, gy) \), \( k \geq 0 \).

If f and g are weakly compatible and g is f-absorbing or f is g-absorbing then f and g have a common fixed point.

**Proof:** Since f and g satisfy property (E.A.), there exists a sequence \( \{x_n\} \) in X such that \( fx_n \rightarrow t \) and \( gx_n \rightarrow t \) for some \( t \) in X. Rest of the proof follows from the Theorem 5.3.
Corollary 5.13: Let $f$ and $g$ be reciprocally continuous self-mappings of a metric space $(X, d)$ satisfying (E. A.) property and

(i) $fX \subseteq gX$

(ii) $d(fx, fy) \leq kd(gx, gy)$, $k \geq 0$.

If $f$ and $g$ are weakly compatible and $g$ is $f$-absorbing or $f$ is $g$-absorbing then $f$ and $g$ have a common fixed point.