CHAPTER 2

A COMPARATIVE STUDY OF COMMUTATIVITY AND ITS WEAKER FORMS

Recently many researchers established some interesting common fixed point theorems by imposing the condition like compatibility, weak commutativity, weak compatibility, R-weak commutativity etc. Jungck [39] in 1976 gave the concept of commutativity condition on two maps and generalized Banach contraction principle for existence of common fixed points for two mappings. Followed by this, a lot of work has been done to weaker the commutativity conditions. As a result many new concepts came into existence. The aim of this chapter is to make a systematic comparison and interrelations amongst these conditions.

Preliminaries:

Throughout this study $(X, d)$ is a metric space $f$ and $g$ are self maps of $X$. A point $x$ is called a coincidence point of $f$ and $g$ iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of $f$ and $g$. Let $C(f, g)$ and $PC(f, g)$ denote the sets of coincidence point’s and points of coincidence, respectively, of the pair $(f, g)$, [43], [46], [31].

In the fixed point theory, common fixed point for two or more than two mappings plays an important role because of its applicability in nonlinear analysis.
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as well as other branches of study. Jungck[39] started the study of existence of common fixed points for pair of mappings by generalizing Banach Contraction principle by taking the extra condition of commutativity and extending contraction condition for pair of maps.

**Definition 2.1 [39]:** Two self maps of a metric space \((X, d)\) are said to be commuting if \(fgx = gfx\) for all \(x \in X\).

**Example 2.1:** Let \(X = [0, \infty)\) be endowed with the usual metric.

\[ fx = x, \text{ and } gx = x^2 \]

then \(fgx = f(x^2) = x^2\) and \(gfx = g(x) = x^2\).

Therefore \(fgx = gfx = x^2\) for all \(x \in X\).

Jungck contraction principle has been extended and generalized by several researchers for the study of existence of common fixed points by relaxing commutativity condition on the pair of maps. As a result several new definitions like weak commutativity, compatibility, R-weakly commuting mapping, weak compatibility, occasionally weakly compatible mapping, conditionally commuting etc. came into existence. Murthy [61] and Pathak and Khan [79] compared some of these conditions. Recently Singh and Tomar [103] also gave a detailed study of commutativity and its weaker forms and also established some coincidences and fixed point theorems. Before making a comparative flow chart of these conditions we gave following definitions.

**Definition 2.2 [92]:** Two self mappings \(f\) and \(g\) of a metric space \((X, d)\) are said to be weakly commuting at a point \(x \in X\) whenever \(d(fgx, gfx) \leq d(fx, gx)\).

**Definition 2.3 [41]:** Two self mappings \(f\) and \(g\) of a metric space \((X, d)\) are said to be compatible if \(\lim_n d(fgx_n, gfx_n) = 0\), whenever, \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_n fx_n = \lim_n gx_n = t \in X\).
Definition 2.4 [48]: Two self mappings $f$ and $g$ of a metric space $(X, d)$ are said to be compatible of type A if $\lim_{n}d(fgx_n, ggx_n) = 0$ and $\lim_{n}d(gfx_n, ffx_n) = 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n}fx_n = \lim_{n}gx_n = t \in X$.

Definition 2.5 [80]: Two self mappings $f$ and $g$ of a metric space $(X, d)$ are said to be compatible of type B if $\lim_{n}d(ffx_n, ggx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n}fx_n = \lim_{n}gx_n = t \in X$.

Definition 2.6 [63]: Two self mappings $f$ and $g$ of a metric space $(X, d)$ are said to be $R$-weakly commuting if there exist $R > 0$ such that $d(fgx, gfx) \leq R d(fx, gx)$ for each $x \in X$. It is clear that $R$-weakly commuting maps commute at their coincidence point.

Definition 2.7 [65]: Two self mappings $f$ and $g$ of a metric space $(X, d)$ are said to be point wise $R$-weakly commuting on $X$ if given $x \in X$, there exist $R > 0$ such that $d(fgx, gfx) \leq R d(fx, gx)$.

Pant [65] also showed that $f$ and $g$ can fail to be point wise $R$-weakly commuting if and only if there is some $x \in X$ such that $fx = gx$ but $fgx \neq gfx$, that is, only if they possess a coincidence point at which they do not commute. Therefore the concept of point wise $R$-weak commutativity is equivalent to commutativity at coincidence points.

Definition 2.8 [81]: Two self mappings $f$ and $g$ of a metric space $(X, d)$ are said to be $R$-weakly commuting of type $A_f$ if there exist $R > 0$ such that $d(fgx, ggx) \leq Rd(fx, gx)$ for every $x \in X$.

Definition 2.9 [81]: Two self mappings $f$ and $g$ of a metric space $(X, d)$ are said to be $R$-weakly commuting of type $A_g$ if there exist $R > 0$ such that $d(gfx, ffx) \leq Rd(fx, gx)$ for every $x \in X$.

Definition 2.10 [44]: Two self mappings $f$ and $g$ of a metric space $(X, d)$ are said to be $f$-biased iff $\{x_n\}$ is a sequence in $X$ and $\lim_{n}fx_n = \lim_{n}gx_n = t \in X$ then,
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\[ a \, d(fgx_n, fx_n) \leq a \, d(gfx_n, gx_n) \text{ if } \alpha = \lim_n \sup \text{ or } \alpha = \lim_n \inf. \]

Jungck and Pathak [44] have shown that the pair \( \{ f, g \} \), is compatible, then it is \( f \)-biased and \( g \)-biased, but the converse is not true.

**Definition 2.11** [42], [45]: Two self mappings \( f \) and \( g \) of a metric space \( (X, d) \) are said to be weakly compatible if they commute at their coincidence point, that is, if \( fgx = gfx \) whenever, \( fx = gx \) for \( x \in X \).

**Definition 2.12** [79]: Two self mappings \( f \) and \( g \) of a metric space \( (X, d) \) are said to be \( f \)-compatible if
\[
\lim_n d(fgx_n, gfx_n) = 0,
\]
whenever, \( \{ x_n \} \) is a sequence in \( X \) such that \( \lim_n fx_n = \lim_n gx_n = t \in X \).

**Definition 2.13** [79]: Two self mappings \( f \) and \( g \) of a metric space \( (X, d) \) are said to be \( g \)-compatible if
\[
\lim_n d(gfx_n, ffx_n) = 0,
\]
whenever, \( \{ x_n \} \) is a sequence in \( X \) such that \( \lim_n fx_n = \lim_n gx_n = t \in X \).

**Definition 2.14** [89]: Two self mappings \( f \) and \( g \) of a metric space \( (X, d) \) are said to be \( f \)-intimate if and only if,
\[
a \, d(fgx_n, fx_n) \leq a \, d(ggx_n, gx_n), \text{ where } a = \lim_n \sup \text{ or } a = \lim_n \inf \text{ and } \{ x_n \} \text{ is a sequence in } X \text{ such that } \lim_n fx_n = \lim_n gx_n = t \in X.\]

**Definition 2.15** [4]: Two self mappings \( f \) and \( g \) of a metric space \( (X, d) \) are said to be occasionally weakly compatible iff there is point \( x \) in \( X \) which is a coincidence point of \( f \) and \( g \) at which \( f \) and \( g \) commute.

**Definition 2.16** [76]: Two self mappings \( f \) and \( g \) of metric space \( (X, d) \) are called conditionally commuting if they commute on a non empty subset of the set of coincidence points whenever the set of their coincidence is nonempty.

**Definition 2.17** [83]: Two self mappings \( f \) and \( g \) of metric space \( (X, d) \) are called \( P \)-operators iff there is a point \( u \) in \( X \) such that \( u \in C(f, g) \) and \( d(u, fu) \leq \delta (C(f, g)) \).
Clearly, occasionally weakly compatible and nontrivial weakly compatible maps \( f \) and \( g \) which do have a coincidence point are \( P \)-operators.

**Definition 2.18 [31]:** Two self mappings \( f \) and \( g \) of metric space \((X, d)\) are called \( JH \)-operators iff there is a point \( w = fx = gx \) in \( PC \) \((f, g)\) such that, \( d(w, x) \leq \delta \ (PC \ (f, g)) \).

**Definition 2.19 [44]:** The ordered pair \((f, g)\) of two self maps of a metric space \((X, d)\) is called weakly \( g \)-biased, if and only if \( d(gfx, gx) \leq d(fgx, fx) \) whenever \( fx = gx \).

**Definition 2.20 [31]:** The ordered pair \((f, g)\) of two self maps of a metric space \((X, d)\) is called occasionally weakly \( g \)-biased, if and only if there exists some \( x \in X \) such that \( fx = gx \) and \( d(gfx, gx) \leq d(fgx, fx) \).

Clearly, an occasionally weakly compatible and a nontrivial weakly \( g \)-biased pair \((f, g)\) are occasionally weakly \( g \)-biased pairs, but the converse does not hold, in general.

**Definition 2.21 [54]:** A pair of self-mappings \((f, g)\) of a metric space \((X, d)\) is said to be \( R \)-weakly commuting of type (P) if there exists some \( R > 0 \) such that \( d(ffx, ggx) \leq Rd(fx, gx) \) for all \( x \in X \).

Now we present following relationship amongst the above definitions and the proofs are also illustrated by counter examples.
Main Results:

Theorem 2.1

(i). Commutativity $\Rightarrow$ weak commutativity (2.1 $\Rightarrow$ 2.2), but weak commutativity $\nRightarrow$ commutativity (2.2 $\nRightarrow$ 2.1).

(ii). Weak commutativity $\Rightarrow$ compatibility (2.2 $\Rightarrow$ 2.3) but compatibility $\nRightarrow$ weak commutativity (2.3 $\nRightarrow$ 2.2).

(iii). Compatibility $\Rightarrow$ $f$-biased (2.3 $\Rightarrow$ 2.10), but, $f$-biased $\nRightarrow$ compatibility (2.10 $\nRightarrow$ 2.3).

(iv). $R$-weakly commuting $\Rightarrow$ Point wise $R$-weakly commuting (2.6 $\Rightarrow$ 2.7), but, Point wise $R$-weakly commuting $\nRightarrow$ $R$-weakly commuting (2.7 $\nRightarrow$ 2.6).

(v) Occasionally weakly compatible $\Rightarrow$ conditionally commuting (2.15 $\Rightarrow$ 2.16), but, conditionally commuting $\nRightarrow$ occasionally weakly compatible (2.16 $\nRightarrow$ 2.15).

(vi) Weakly compatible $\Rightarrow$ occasionally weakly compatible (2.11 $\Rightarrow$ 2.15), but, occasionally weakly compatible $\nRightarrow$ weakly compatible (2.15 $\nRightarrow$ 2.11).

(vii) Compatible of type A $\Rightarrow$ $f$-intimate (2.4 $\Rightarrow$ 2.14), but, $f$-intimate $\nRightarrow$ compatible of type A (2.14 $\nRightarrow$ 2.4).

(viii) $R$-Weakly commuting $\Rightarrow$ occasionally weakly compatible (2.6 $\Rightarrow$ 2.15), but, occasionally weakly compatible $\nRightarrow$ $R$-weakly commuting (2.15 $\nRightarrow$ 2.6).

(ix) Compatible $\Rightarrow$ compatible of type B (2.3 $\Rightarrow$ 2.5), but, compatible of type B $\nRightarrow$ compatible (2.5 $\nRightarrow$ 2.3).

(x) Compatible $\Rightarrow$ weakly compatible (2.3 $\Rightarrow$ 2.11), but, weakly compatible $\nRightarrow$ compatible (2.11 $\nRightarrow$ 2.3).
(xi) R- weakly commuting ⇒ compatible (2.6 ⇒ 2.3), but, compatible ⇄ R-weakly commuting (2.3 ⇄ 2.6).

(xii) Occasionally weakly compatible ⇒ occasionally weakly g-biased (2.15 ⇒ 2.20), but, occasionally weakly g-biased ⇄ occasionally weakly compatible (2.20 ⇄ 2.15).

(xiii) Weakly compatible ⇒ occasionally weakly g-biased (2.11 ⇒ 2.20), but, occasionally weakly g-biased ⇄ weakly compatible (2.20 ⇄ 2.11).

(xiv) Occasionally weakly compatible ⇒ P-operator (2.15 ⇒ 2.17), but, P-operator ⇄ occasionally weakly compatible (2.17 ⇄ 2.15).

(xv) Weakly compatible ⇒ P-operator (2.11 ⇒ 2.17), but, P-operator ⇄ weakly compatible (2.17 ⇄ 2.11).

(xvi) Occasionally weakly compatible ⇒ JH-operator (2.15 ⇒ 2.18), but, JH-operator ⇄ occasionally weakly compatible (2.18 ⇄ 2.15).

(xvii) Definition 2.8, definition 2.9 and definition 2.21 are independent.

**Proof:** (i). It is trivial that 2.1 ⇒ 2.2. Following example [103] - shows that 2.2 ⇄ 2.1.

Let X = [0, 1] be endowed with the usual metric. Let f(x) = x/6, g(x) = x/2 (1+x) .Then fgx ≠ gfx However, |fx - gx| ≤ |fx - gx| for all and x ∈ X. f and g are weakly commuting.

**Proof:** (ii). It is trivial that 2.2 ⇒ 2.3. Following example [41] shows that 2.3 ⇄ 2.2.
Let $X = [0, \infty)$ be endowed with the usual metric. Let $f(x) = x^3$ and $g(x) = 2x^3$. So $f$ and $g$ are not commuting on $X$ and $|f(x) - g(x)| > |f(x) - g(x)|$. Therefore, $f$ and $g$ are not weakly commuting on $X$. However, $\lim_{x \to 0} |f(x) - g(x)| = 0 \in X$ and it implies $\lim_{x \to 0} |f(x) - g(x)| = 0$. Therefore $f$ and $g$ are compatible.

**Proof:** (iii). It is trivial that $2.3 \Rightarrow 2.10$. Following example [44] shows that $2.10 \not\Rightarrow 2.3$.

Let $X = [0,1], f(x) = 1 - 2x, g(x) = 2x$ for $x \in [0,1/2]$ and $f(x) = 0, g(x) = 1$ for $x \in (1/2,1]$. Then $\{g, f\}$ are both $g$-biased and $f$-biased but not compatible.

**Proof:** (iv). It is trivial that $2.6 \Rightarrow 2.7$. Following example shows that $2.7 \not\Rightarrow 2.6$.

Let $X = [2, 20]$ with the usual metric. Define

$f(x) = 2$, if $x = 2$, $f(x) = 13 - x$, if $2 < x \leq 5$, $f(x) = x - 3$, if $x > 5$;

$g(x) = 2$, if $x \in 2 \cup (5, 20]$, $g(x) = 8$, if $2 < x \leq 5$.

In this example $f(2) = 2 = g(2)$ and $f(5) = 8 = g(5)$. Hence 2 and 5 are the coincidence points of $f$ and $g$. Further, $f(g(2)) = 2 = g(f(2))$ and $f(g(5)) = g(f(5))$. Hence the maps $f$ and $g$ are point wise R-weak commutativity.

**Proof:** (v). It is trivial that $2.15 \Rightarrow 2.16$. Following example shows that $2.16 \not\Rightarrow 2.15$.

Let $X = [0, \infty)$ be endowed with the usual metric. Let $f(x) = x$ and $g(x) = x + 1$. The maps $f$ and $g$ have no coincidence point’s. Then the maps $f$ and $g$ are not occasionally weakly compatible.
Proof (vi). It is trivial that $2.11 \Rightarrow 2.15$. Following example [76] shows that $2.15 \not\Rightarrow 2.11$.

Let $X = [2, 20]$. Define $f, g : X \to X$ as follows;

$f(x) = 2$, if $x = 2$ or $x > 5$, $f(x) = 6$ if $2 < x \leq 5$

$g(x) = 2$, if $2 < x \leq 5$, $g(x) = (x/3 + 1/3)$ if $x > 5$.

In this example $f(2) = 2 = g(2)$ and $f(3) = 6 = g(3)$. Hence 2 and 3 are the coincidence points of $f$ and $g$. Further $f(g(2)) = 2 = g(f(2))$. However, $f(g(3)) = f(6) = 2$, $g(f(3)) = g(6) = 7/3$ and therefore $f(g(3)) \neq g(f(3))$, $f$ and $g$ are occasionally weakly compatible.

Proof (vii). It is trivial that $2.4 \Rightarrow 2.14$. Following example [89] shows that $2.14 \not\Rightarrow 2.4$.

Let $X = [0, 1]$ be endowed with the usual metric. Let $f(x) = 2/(2+x)$ and $g(x) = 1/(x+1)$ for $x \in [0,1]$. Let $\{x_n\} = \{1/n\}$, $n \in \mathbb{N}$ be a sequence in $X$. Then,

$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = 1, \lim_{n \to \infty} |f(x_n) - f(x_0)| = 1/3, \lim_{n \to \infty} |g(x_n) - g(x_0)| = 1/2$.

$\lim_{n \to \infty} |f(x_n) - x_0| < \lim_{n \to \infty} |g(x_n) - x_0|$, that is, $\{f, g\}$ is $f$-intimate. Also, $\lim_{n \to \infty} |f(x_n) - g(x_n)| = 1/6$. Thus $\{f, g\}$ is not compatible of type A.

Proof (viii). It is trivial that $2.6 \Rightarrow 2.15$. Following example shows that $2.15 \not\Rightarrow 2.6$.

Let $X = [2, 20]$. Define $f, g : X \to X$ as follows;
fx = 2, if x = 2 or x > 5, fx = 6 if 2 < x ≤ 5

g2 = 2, gx = 6, if 2 < x ≤ 5, gx = (x + 1) if x > 5.

In this example the closed interval [2, 5] is the set of coincidence points of f and g. fg(2) = 2 = gf(2) but the mappings f and g do not commute at coincidence points in (2, 5]. The map f and g are occasionally weakly compatible but not R-weakly commuting.

(ix). It is trivial that 2.3 ⇒ 2.5. Following example shows that 2.5 ≠ 2.3.

Let X = [2, 20]. Define f, g: X → X as follows;

fx = 3, if x = 2, fx = 2, if 2 < x ≤ 5, fx = x - 3, if x > 5.

gx = 2, if x = 2 or x > 5, gx = 8, if 2 < x ≤ 5.

We consider a sequence \{x_n\} = \{5 + (1/n): n ≥ 1\} in X.

fx_n = f(5 + 1/n) = 5 + (1/n) - 3 = 2 + (1/n),

gx_n = g(5 + 1/n) = 2,

lim_n fx_n = lim_n \{2 + (1/n)\} = 2, lim_n gx_n = 2,

lim_n fx_n = lim_n gx_n = 2 ∈ X,

ffx_n = f(2 + 1/n) = 2, ggx_n = g(2) = 2, fgx_n = f(2) = 3, gffx_n = g(2 + 1/n) = 8,

Therefore lim_n d (ffx_n, ggx_n) = 0.

The map f and g are compatible of type B but not compatible.
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(x) It is trivial that 2.3 $\Rightarrow$ 2.11. Following example [98] shows that 2.11 $\not\Rightarrow$ 2.3. Let $X = [0, \infty)$ be the metric space with the,

\[ f(x) = x \text{ if } x \in [0, 1), \quad f(x) = 1 \text{ if } x \in [1, \infty), \quad \text{and } g(x) = x / (1 + x) \text{ if } x \in X. \]

Then $f$ and $g$ are not compatible on $X$ but $f$ and $g$ are commuting at their coincidence point at $x = 0$. Indeed $f$ and $g$ are weakly compatible.

(xi) It is trivial that 2.6 $\Rightarrow$ 2.3. Following example [41] shows that 2.3 $\not\Rightarrow$ 2.6. Let $X = [0, \infty)$ be endowed with the usual metric. Let $f(x) = x^3$ and $g(x) = 2x^3$.

\[ f(0) = 0, \quad g(0) = 0, \]

\[ f(0) = g(0) = 0, \quad fg(0) = gf(0) = 0, \]

\[ fg(x) = f(2x^3) = (2x^3)^3 = 8x^9, \]

\[ gf(x) = g(x^3) = 2(x^3)^3 = 2x^9, \]

\[ d(fgx, gfx) = 6x^9, \quad d(fx, gx) = x^3, \]

\[ d(fgx, gfx) \not\leq Rd(fx, gx). \]

The map $f$ and $g$ are not $R$-weakly commuting but $f$ and $g$ commuting at their coincidence point at $x = 0$.

(xii) It is trivial that 2.15 $\Rightarrow$ 2.20. Following example [31] shows that 2.20 $\not\Rightarrow$ 2.15. Let $X = [0, \infty)$ be endowed with the usual metric. Define $f, g: X \rightarrow X$ as follows:

\[ f(x) = 2x \text{ if } x \in [0, \frac{1}{2}), \quad f(x) = 1 \text{ if } x \in [\frac{1}{2}, 1], \quad \text{and } g(x) = 1 - 2x \text{ if } x \in [0, \frac{1}{2}), \quad g(x) = 0 \text{ if } x \in [\frac{1}{2}, 1]. \]
Here $C(f, g) = \{1/4\}$,

\[ |gf(1/4) - g(1/4)| = |0 - 1/2| = \frac{1}{2} \leq |fg(1/4) - f(1/4)| = |1 - \frac{1}{2}| = \frac{1}{2} \]

implies that $(f, g)$ is an occasionally weakly g-biased pair. Further, $fg(1/4) \neq gf(1/4)$. Hence $(f, g)$ is not an occasionally weakly compatible pair.

(xiii). It is trivial that $2.11 \Rightarrow 2.20$. Following example [31] shows that $2.20 \not\Rightarrow 2.11$.

Let $X = [0, \infty)$ be endowed with the usual metric. Define $f, g : X \to X$ as follows;

\[ fx = 2x^2, \quad gx = 2x, \text{ for all } x \neq 0 \text{ and } f(0) = g(0) = \frac{1}{2}. \]

Here $C(f, g) = \{0, 1\}$,

\[ |gf(0) - g(0)| = |1 - \frac{1}{2}| = \frac{1}{2} \neq |fg(0) - f(0)| = |\frac{1}{2} - \frac{1}{2}| = 0, \text{ but}, \]

\[ |gf(1) - g(1)| = |4 - 2| = 2 \leq |fg(1) - f(1)| = |8 - 2| = 6 \]

implies that $(f, g)$ is an occasionally weakly g-biased pair. Further, $fg(0) \neq gf(0)$ and $fg(1) \neq gf(1)$. Hence $(f, g)$ is not weakly compatible pair.

(xiv). It is trivial that $2.15 \Rightarrow 2.17$. Following example shows that $2.17 \not\Rightarrow 2.15$.

Let $X = [0, \infty)$ be endowed with the usual metric. Define $f, g : X \to X$ as follows;

\[ fx = x/2, \quad gx = x^2/2 \text{ for all } x \neq 0 \text{ and } f(0) = g(0) = \frac{1}{4}, \]

$C(f, g) = \{0, 1\}$, $d(u, fu) \leq \delta(C(f, g))$;

\[ |0 - \frac{1}{4}| = \frac{1}{4} \leq \text{sup}\{\text{max}\{d(0,1), d(1,0)\}: 0, 1 \in C(f, g)\} = \text{sup}\{\text{max}\{1,1\}\} = 1, \text{ and} \]
| $1 - \frac{1}{2}$ | = $\frac{1}{2}$ $\leq$ sup{max{d(0,1), d(1,0)}: $0,1 \in C(f, g)$} = sup{max{1,1}} = 1, implies that $(f, g)$ is a $p$-operator. Further, $fg(0) \neq gf(0)$ and $fg(1) \neq gf(1)$. Hence $(f, g)$ is not an occasionally weakly compatible.

(xv). It is trivial that 2.11 $\Rightarrow$ 2.17. Following example shows that 2.17 $\not\Rightarrow$ 2.11.

Let $X = [0, \infty)$ be endowed with the usual metric. Define $f, g: X \rightarrow X$ as follows;

$f(x) = x, g(x) = x^2$, for all $x \neq 0$ and $f(0) = g(0) = \frac{1}{2}$,

$C(f, g) = \{0, 1\},$ $d(u, fu) \leq \delta(C(f, g))$;

| $0 - \frac{1}{2}$ | = $\frac{1}{2}$ $\leq$ sup{max{d(0,1), d(1,0)}: $0,1 \in C(f, g)$} = sup{max{1,1}} = 1, and

| $1 - 1$ | = 0 $\leq$ sup{max{d(0,1), d(1,0)}: $0,1 \in C(f, g)$} = sup{max{1,1}} = 1,

implies that $(f, g)$ is a $\rho$ operator. Further $fg(0) \neq gf(0)$ and $fg(1) = gf(1)$. Hence $(f, g)$ is not a weakly compatible.

(xvi). It is trivial that 2.15 $\Rightarrow$ 2.18. Following example [31] shows that 2.18 $\not\Rightarrow$ 2.15.

Let $X = [0, \infty)$ be endowed with the usual metric. Define $f, g: X \rightarrow X$ as follows;

$f(x) = x^2, g(x) = 2x$ for all $x \neq 0$ and $f(0) = g(0) = 1$. Then $C(f, g) = \{0,2\}$ and $PC(f, g) = \{1,4\}$

$d(w, x) \leq \delta(PC(f, g))$;

| $1 - 0$ | = 1 $\leq$ sup{max{d(1,4), d(4,1)}: $1,4 \in PC(f, g)$} = sup{max{3,3}} = 3 and,

| $4 - 2$ | = 2 $\leq$ sup{max{d(1,4), d(4,1)}: $1,4 \in PC(f, g)$} = sup{max{3,3}} = 3.
implies that \((f, g)\) is a JH-operator. Further \(fg(0) \neq gf(0)\) and \(fg(2) \neq gf(2)\). Hence \((f, g)\) is not an occasionally weakly compatible.

(xvii) Example 2, 3 and 4 of Kumar S [54].