CHAPTER 2

OSCILLATION CRITERIA FOR A CLASS OF
THIRD ORDER NONLINEAR DIFFERENCE
EQUATIONS

2.1 INTRODUCTION

In this chapter, the oscillation of a class of third order nonlinear difference equations have been investigated of the form

\[ \Delta \left( a_n (\Delta b_n (\Delta x_n)^\alpha)^\beta \right) + c_n x_{n+\tau}^\mu = 0 \quad (2.1.1) \]

where \( \alpha, \beta, \mu \) are the ratios of odd positive integers, \( \tau \in \mathbb{Z} \) is a deviating argument, \( \{a_n\}, \{b_n\} \) & \( \{c_n\} \) are positive real sequences defined for \( n \in N_0 = \{n_0, n_0+1, \ldots\} \), \( n_0 \) is a positive integer. Here \( \Delta \) is the forward difference operator. It is defined by \( \Delta x_n = x_{n+1} - x_n \).

By a solution of (2.1.1), a real sequence \( \{x_n\} \) that satisfies (2.1.1) for all \( n \in N_0 \). A nontrivial solution \( \{x_n\}, n \in N_0 \) of (2.1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be non-oscillatory. Equation (2.1.1) is said to be oscillatory, if all its solutions are oscillatory.
A solution \( \{x_n\} \) of (2.1.1) is called quickly oscillatory if
\[
x_n = (-1)^n q_n, \quad q_n > 0 \quad \text{for} \quad n \in N_0.
\]

Equation (2.1.1) is called almost oscillatory if either \( \{x_n\} \) is oscillatory or \( \Delta x_n \) is oscillatory or \( \lim_{n \to \infty} x_n = 0 \).

Consider (2.1.1) as a three dimensional system, let
\[
y_n = b_n(\Delta x_n)^\alpha, \quad z_n = a_n(\Delta y_n)^\beta.
\]

Construct the nonlinear system,
\[
\begin{cases}
\Delta x_n = B_n y_n^{1/\alpha}, \\
\Delta y_n = A_n z_n^{1/\beta}, \\
\Delta z_n = -C_n x_n^\mu
\end{cases}
\]
(2.1.3)

where \( B_n = b_n^{-1/\alpha}, \quad A_n = a_n^{-1/\beta} \& C_n = c_n \). If \((x, y, z)\) is a solution of (2.1.3) and any one of its components is positive (or negative) sign, then all its components are positive (or negative) sign.

The canonical form of the difference operator in (2.1.1) is defined by,
\[
\sum_{n=n_0}^{\infty} a_n^{-1/\alpha} = \sum_{n=n_0}^{\infty} b_n^{-1/\beta} = \infty.
\]
(2.1.4)

In section 2.1, the sufficient conditions are obtained for quickly oscillatory solutions. In section 2.2, some sufficient conditions for oscillatory and non-oscillatory solutions have been given. Section 2.3, deals with the almost oscillatory solutions. In section 2.4, examples are given to illustrate the results.
2.2 QUICKLY OSCILLATORY SOLUTIONS

In this section, the sufficient conditions for quickly oscillatory solutions have been obtained.

**Theorem 2.2.1.** Assume that \( n > 0 \) and \( n \) is even. If \( \tau \) is odd, then (2.1.1) has no quickly oscillatory solutions.

**Proof.** Let \( x_n = (-1)^n o_n \) be a quickly oscillatory solution of (2.1.1) with positive even terms.

Then there exists \( n \in N_0, o_n > 0 \) such that \( \Delta x_n = (-1)^{n+1} (o_{n+1} + o_n) \).

From the first equation of (2.1.3), we have

\[
y_n = \left( \frac{\Delta x_n}{B_n} \right)^\alpha = (-1)^{n+1} \left( \frac{o_{n+1} + o_n}{B_n} \right)^\alpha = (-1)^{n+1} p_n
\]

where \( p_n = \left[ \frac{o_{n+1}}{B_n} + \frac{o_n}{B_n} \right]^\alpha > 0. \)

From second equation of (2.1.3), we get

\[
z_n = \left( \frac{\Delta y_n}{A_n} \right)^\beta = (-1)^n \left[ \frac{p_{n+1} + p_n}{A_n} \right]^\beta = (-1)^n q_n.
\]

where \( q_n = \left[ \frac{p_{n+1}}{A_n} + \frac{p_n}{A_n} \right]^\beta > 0. \) Repeating this process, we get

\[
\Delta z_n = (-1)^{n+1} (q_{n+1} + q_n) = -C_n (-1)^{(n+\tau)} \mu \phi_{n+\tau}^\mu
\]

\[
= C_n (-1)^{(n+1+\tau)} \mu \phi_{n+\tau}^\mu.
\]

Since \( \tau \) is odd, therefore (2.1.1) has quickly oscillatory solution with positive odd terms, which is a contradiction.

**Remark 2.2.2.** If \( \tau \) is even, \( n > 0 \), and \( n \) is odd, then (2.1.1) has no quickly oscillatory
Theorem 2.2.3. Let $\alpha$, $\beta$ and $\mu$ be the ratios of odd positive sequences. If $\tau = 0$, then (2.1.1) has quickly oscillatory solution.

Proof. Let $x_n$ be a not quickly oscillatory solution of (2.1.1),
That means $x_n = (-1)^n o_n$, $o_n < 0$
Since $\tau = 0$ and $\alpha$, $\beta$ and $\mu$ are the ratios of odd positive integers. Without loss of generality assume that $\alpha = \beta = \mu = 1$.
Since $x_n = (-1)^n o_n$, $o_n < 0$, then it follows
$\Delta x_n = (-1)^n [o_{n+1} + o_n]$.

$\therefore \Delta (b_n \Delta x_n) = (-1)^{n+2} [b_{n+1} o_{n+2} + (b_{n+1} + b_n) o_{n+1} + b_n o_n]$ 

$\Delta (c_n \Delta (b_n \Delta (x_n))) = (-1)^{n+1} [b_{n+2} o_{n+3} + (b_{n+2} c_{n+1} + b_{n+1} c_{n+2} + b_{n+1} c_{n+1} + b_{n+1} c_{n+1} + b_{n+1} c_{n+1}) o_{n+1} + b_{n+1} c_{n+1} + b_n c_{n+1} + b_n c o_n]$ 

(2.2.1)

Taking $\alpha = \beta = \mu = 1$ and $\tau = 0$ in (2.1.1), then it becomes
$\Delta (c_n \Delta (b_n \Delta (x_n))) = -c_n x_n$ 

(2.2.2)

Comparing (2.2.1) and (2.2.2), we get

$(-1)^{n+1} [b_{n+2} o_{n+3} + (b_{n+2} c_{n+1} + b_{n+1} c_{n+1} + b_{n+1} c_{n+1} + b_{n+1} c_{n+1} + b_{n+1} c_{n+1}) o_{n+1} + b_{n+1} c_{n+1} + b_n c_{n+1} + b_n c o_n]$ 

(2.2.3)

According to the assumption $o_n < 0$, then the left side terms of (2.2.3) are positive. But the right side terms of (2.2.3) are negative, which is a contradiction.
Hence the solutions $x_n$ of (2.1.1) is quickly oscillatory solution.

This completes the proof of the theorem.

### 2.3 OSCILLATORY SOLUTIONS

In this section, the sufficient conditions for oscillatory solutions of (2.1.1) have been studied.

**Lemma 2.3.1.** The following statements are equivalent

(i) $x$ is a solution of (2.1.1).

(ii) $y = y_n$, where $y_n = b_n (\Delta x_n)^{\alpha}$, is a solution of

$$\Delta \left( \frac{1}{c_n \mu} \left( \Delta (a_n (\Delta y_n)^{\beta}) \right)^{1/\mu} \right) + \frac{1}{b_n^{1/\alpha}} y_{n+\tau}^{1/\alpha} = 0 \quad (2.3.1)$$

(iii) $z = z_n$, where $z_n = a_n (\Delta y_n)^{\beta}$, is a solution of

$$\Delta \left( b_n^{1/\tau} \left( \Delta \frac{1}{c_n \mu} (\Delta z_n)^{1/\mu} \right) \right) + \frac{1}{a_n^{1/\beta}} z_{n+\tau}^{1/\beta} = 0. \quad (2.3.2)$$

**Proof.** First, we will prove (i) is equivalent to (ii).

Consider the third equation in (2.1.3) and (2.1.1), we get

$$x_{n+\tau} = -\frac{1}{c_n \mu} (\Delta z_n)^{1/\mu} = -\frac{1}{c_n \mu} \left( \Delta (a_n (\Delta y_n)^{\beta}) \right)^{1/\mu}. \quad (2.3.3)$$

Next, we consider the first equation in (2.1.3), we have

$$\Delta x_{n+\tau} = -\Delta \left( \frac{1}{c_n \mu} \left( \Delta (a_n (\Delta y_n)^{\beta}) \right)^{1/\mu} \right) + \frac{1}{b_n^{1/\alpha}} y_{n+\tau}^{1/\alpha}$$

$$\Delta \left( \frac{1}{c_n \mu} \left( \Delta (a_n (\Delta y_n)^{\beta}) \right)^{1/\mu} \right) + \frac{1}{b_n^{1/\alpha}} y_{n+\tau}^{1/\alpha} = 0$$

which gives (ii).
Next, to prove (i) is equivalent to (iii). From (2.3.3),

\[ \Delta x_n = -\Delta \left( \frac{1}{c_n^{1/\mu}} \left( \Delta \left( a_n - \tau \Delta (a_n - \tau \beta) \right) \right)^{1/\mu} \right). \]

Substitute this into \( \Delta y_n = \Delta \left( b_n (\Delta x_n)^\alpha \right) \), we get

\[ \Delta y_n = \Delta \left( b_n \left( -\Delta \frac{1}{c_n^{1/\mu}} \left( \Delta \left( a_n - \tau \Delta (a_n - \tau \beta) \right) \right)^{1/\mu} \right)^\alpha \right) \]

From second equation in (2.1.3), we get

\[ \Delta y_{n+\tau} = \Delta \left( b_{n+\tau} \left( -\Delta \frac{1}{c_n^{1/\mu}} \left( \Delta \left( a_n (\Delta y_n)^\beta \right) \right)^{1/\mu} \right)^\alpha \right) \]

\[ = -\Delta \left( b_{n+\tau} \left( \Delta \left( \frac{1}{c_n^{1/\mu}} (\Delta z_n)^{1/\mu} \right) \right) \right) = \frac{1}{a_{n+\tau}} z_{n+\tau}^{1/\beta} \]

which gives (iii).

\[ \blacksquare \]

**Theorem 2.3.2.** Equation (2.1.1) is oscillatory \( \iff \) Equations (2.3.1) & (2.3.2) are oscillatory.

**Proof.** Equation (2.1.1) is oscillatory.

\( \iff \) Every solution of (2.1.1) is oscillatory.

\( \iff \) \( x_n \) is an oscillatory solution of (2.1.1) for \( n \in \mathbb{N}_0 \)

\( \iff \) \( y_n \) is an oscillatory solution of (2.3.1) for \( n \in \mathbb{N}_0 \), by lemma 2.3.1

This completes the proof. \[ \blacksquare \]

**Lemma 2.3.3.** Assume (2.1.4), then there exists any solution of \((x, y, z)\) of (2.1.3) such that \( x_n > 0 \) for large \( n \) is of the following types:

- \((B_1)\) \( x_n > 0, y_n > 0, z_n > 0 \) for all large \( n \),
- \((B_2)\) \( x_n > 0, y_n < 0, z_n > 0 \) for all large \( n \),

**Proof.** Let \((x, y, z)\) be non-oscillatory solution of (2.1.3).
Therefore, there exists a solution such that \( y_n > 0, z_n < 0 \) for large \( n \).

Since \( \Delta z_n < 0 \), there exists \( k > 0 \) such that \( z_n \leq -k \), for large \( n \). Summing the second equation of (2.1.3),

\[
y_n - y_{n_0} = \sum_{i=n_0}^{n-1} A_i z_i^{1/\beta} \leq z_n^{1/\beta} \sum_{i=n_0}^{n-1} A_i \leq (-k)^{1/\beta} \sum_{i=n_0}^{n-1} A_i.
\]

Taking \( n \to \infty \), we get

\[
\lim_{n \to \infty} y_n = -\infty,
\]

which is a contradiction.

Similarly, we can prove that there exists a solution \( y_n < 0, z_n < 0 \), for large \( n \). Since \( y \) is negative decreasing, there exists \( k > 0 \) so that \( y_n \leq -k \) for large \( n \). Summing the first equation in (2.1.3), we get

\[
x_n - x_{n_0} = \sum_{i=n_0}^{n-1} B_i y_i^{1/\alpha} \leq y_n^{1/\alpha} \sum_{i=n_0}^{n-1} B_i \leq (-k)^{1/\alpha} \sum_{i=n_0}^{n-1} B_i.
\]

Taking \( n \to \infty \), then \( \lim_{n \to \infty} x_n = -\infty \). Which is a contradiction.

This completes the proof of the theorem. \( \blacksquare \)

**Lemma 2.3.4.** Equation (2.1.1) has no solution of type \( (B_1) \), if any one of the following conditions hold

\[
(i) \sum_{n=n_0}^{\infty} c_n \left( \sum_{i=n_0}^{n-1} \frac{1}{b_i^{1/\alpha}} \right)^{\mu} = \infty, \quad (2.3.4)
\]

\[
(ii) \sum_{n=n_0}^{\infty} c_n \left( \sum_{i=n_0}^{n-1} \frac{1}{b_i^{1/\alpha}} \left( \sum_{j=n_0}^{i-1} \frac{1}{a_j^{1/\beta}} \right)^{\alpha} \right)^{\mu} = \infty. \quad (2.3.5)
\]

**Proof.** Let \( (x, y, z) \) be a type \( (B_1) \) solution of (2.1.3), that is, all components of the solution are positive.

There exists \( k > 0 \) and \( z \) is positive and increasing such that \( z_n^{1/\beta} \geq k \) for large \( n, n \geq n_0 \).
From the first and second equations in (2.1.3), we get
\[ x_i = \sum_{i=n_0}^{j-1} B_i y_i^{1/\alpha} \]
\[ y_i = \sum_{i=n_0}^{j-1} A_i z_i^{1/\beta} \geq z_i^{1/\beta} \sum_{i=n_0}^{j-1} A_i \geq k \sum_{i=n_0}^{j-1} A_i. \]

\[ x_i \geq y_i^{1/\alpha} \sum_{n=n_0}^{j-1} B_n \geq k^{1/\alpha} \sum_{n=n_0}^{j-1} B_n \left( \sum_{k=n_0}^{n-1} A_i \right)^{1/\alpha}. \] (2.3.6)

Let us assume (2.3.4) and (2.3.5) are hold. By summing third equation of (2.1.3) and using (2.3.6), we get
\[ z_{n_0} - z_n = -\sum_{i=n_0}^{n-1} \Delta z_i \geq \sum C_i x^{\mu}_{n+\tau} \]
\[ z_{n_0} - z_n \geq k^{\mu/\alpha} \sum_{i=n_0}^{n-1} C_i \left( \sum_{j=n_0}^{i+\tau-1} B_j \left( \sum_{k=n_0}^{j-1} A_k \right)^{1/\alpha} \right)^{\mu}. \] 2.3.6(i)

Taking \( n \to \infty \), by the condition (ii) left hand side of (2.3.6 (i)) tends to \( \infty \), we get a contradiction to the boundedness of \( z_n \). Hence (2.1.1) has no solution of type \( (B_1) \).

This completes the proof.

\[ \square \]

**Lemma 2.3.5.**

Let \( \sum_{n=n_0}^{\infty} c_n < \infty \) be hold.

Then (2.1.1) has no solution of type \( (B_2) \) if any one of the following conditions hold,

(i) \( T := \sum_{n=n_0}^{\infty} \frac{1}{a_n} \left( \sum_{k=n}^{\infty} c_k \right)^{1/\beta} = \infty \) (2.3.7)

(ii) \( T < \infty \) and
\[ \sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\alpha}} \left( \sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\beta}} \left( \sum_{i=n_0}^{\infty} c_i \right)^{1/\beta} \right)^{1/\alpha} = \infty \] (2.3.8)
Proof. Let \((z, a, b)\) be a solution of (2.1.3) and satisfying \(z_n > 0, a_n < 0, b_n > 0\).

Since, the solutions \(z\) and \(-y\) are positive and decreasing, we get

\[
\lim_{n \to \infty} z_n = z_{\infty}, \quad z_n \geq 0
\]

\[
\lim_{n \to \infty} y_n = y_{\infty}, \quad y_n \leq 0
\]

By summing the third equation of (2.1.3), we get

\[
z_n - z_{n_0} = -\sum_{k=n_0}^{n-1} C_k x_{k+\tau}^{\mu}
\]

which implies,

\[
z_n = z_{\infty} + \sum_{k=n_0}^{\infty} C_k x_{k+\tau}^{\mu} \geq x_{k+\tau}^{\mu} \sum_{k=n_0}^{\infty} C_k.
\]

Let us assume (i) is hold, then summing second equation of (2.1.3), we get

\[
y_m - y_{n_0} = \sum_{n=n_0}^{m-1} A_n z_n^{1/\beta} \geq x_{k+\tau}^{\mu/\beta} \sum_{n=n_0}^{m-1} A_n \left( \sum_{k=n}^{\infty} C_k \right)^{1/\beta}
\]

\[
y_m \geq y_{n_0} + x_{k+\tau}^{\mu} \sum_{n=n_0}^{\infty} A_n \left( \sum_{k=n}^{\infty} C_k \right)^{1/\beta}
\]

which gives a contradiction to the boundedness of \(y\).

Let us assume \((ii)\) is hold, consider the second equation of (2.1.3), we have

\[
y_m = -y_{\infty} + \sum_{k=n}^{\infty} A_k z_k^{1/\beta}
\]

\[
y_m \geq x_{n+\tau}^{\mu/\beta} \sum_{k=n}^{\infty} A_k \left( \sum_{j=n}^{\infty} C_j \right)^{1/\beta}
\]

(2.3.9)

Since \(x\) is positive decreasing and using (2.1.3), we have
\[ x_{n_0} = x_n + \sum_{k=n_0}^{n-1} B_k (-y_k)^{1/\alpha} \]

\[ \geq \sum_{k=n}^{\infty} B_k \left( \sum_{i=n}^{\infty} A_i \left( \sum_{j=n}^{\infty} C_j \right)^{1/\beta} \right)^{1/\alpha} \]

which is a contradiction.

Therefore, (2.1.1) has no solution of type \((B_2)\).

This completes the proof.  

\textbf{Theorem 2.3.6.} Assume that (2.1.4), \( \sum_{n=n_0}^{\infty} c_n < \infty \) and \( \tau \in \mathbb{Z} \), if (2.3.5) & (2.3.8) hold then (2.1.1) is oscillatory.

\textit{Proof.} From the lemma 2.3.4 and lemma 2.3.5, if the conditions (2.3.5) and (2.3.7) hold, then (2.1.1) has no solutions of type \((B_1)\) and \((B_2)\).

By lemma 2.3.3, (2.1.1) has oscillatory solutions.  

\textbf{Theorem 2.3.7.} Assume that (2.1.4), \( \sum_{n=n_0}^{\infty} c_n < \infty \) and \( \tau \in \mathbb{Z} \), if (2.3.6) & (2.3.9) hold then (2.1.1) is oscillatory.

\textit{Proof.} From lemma 2.3.4 and lemma 2.3.5, (2.1.1) has no solution of type \((B_1)\) and \((B_2)\) if the conditions (2.3.6) and (2.3.8) hold.

Then by lemma 2.3.3, (2.1.1) is oscillatory.  

\textbf{2.4 ALMOST OSCILLATORY SOLUTIONS}

Throughout this section, the conditions of almost oscillatory solutions of equation (2.1.1) are obtained.

\textbf{Corollary 2.4.1.} If \((x, y, z)\) is a solution of system (2.1.3), with bounded first component and such that one of its components is of one sign, then there exists limit of sequence \((x_n)\) and exactly one of the following two cases are hold
(i) $\lim_{n \to \infty} x_n \neq 0$ and sequence $x, y$ and $z$ are monotonic for large $n$, or
(ii) sequence $(y_n)$ is of one sign and $\lim_{n \to \infty} x_n = 0$.

Proof. Let $x_n$ be positive. (Analogs to prove for $x_n$ is negative) Since $C_n$ is of one sign for $n > n_0$. From third equation in (2.1.3), we get $\Delta z_n$ is of one sign.
That means if $\Delta z_n > 0$, then $\{z_n\}$ is increasing. If $\Delta z_n < 0$, then $\{z_n\}$ is decreasing.
For that both cases $\{z_n\}$ is of one sign for large $n$.
This implies that, the second equation in (2.1.3), $\Delta y_n$ is of one sign for large $n$. Proceeding like this, we get $\{x_n\}$ is of one sign for large $n$.
Since $C_n$ is of one sign, from the third equation in (2.1.3), we get $z$ is monotonic.
Analogs to prove $x$ and $y$ are monotonic.
Consider the first equation in system (2.1.3),

$$\Delta x_n = B_n y_n^{1/\alpha}.$$  

By summing the equation, we get

$$x_n = x_{n_0} + \sum_{i=n_0}^{\infty} B_i y_i^{1/\alpha}.$$  

Let us take $\lim_{n \to \infty} y_n > 0$

$$\therefore \lim_{n \to \infty} y_n^{1/\alpha} > 0$$

Since $B_n$ is positive and taking $n$ tends to infinity, we have $\lim_{n \to \infty} x_n = 0$. ■

**Corollary 2.4.2.** Assume

$$\sum_{n=n_0}^{\infty} A_n = \sum_{n=n_0}^{\infty} B_n = \infty$$
and \((x, y, z)\) is a solution of system (2.1.3), such that \(\lim_{n \to \infty} x_n \in \mathbb{R}\), then
\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n = 0.
\]

**Proof.** Since \(\lim_{n \to \infty} x_n\) is finite. Consider the first equation in system (2.1.3),
\[
\Delta x_n = B_n y_1^{1/\alpha}.
\]

By summing the equation, we get
\[
x_n = x_{n_0} + \sum_{i=n_0}^{\infty} B_i y_i^{1/\alpha}.
\]

Now, assume the contrary that \(\lim_{n \to \infty} y_n > 0\)
\[
\therefore \lim_{n \to \infty} y_n^{1/\alpha} > 0
\]

Since \(B_n\) is positive and taking \(n\) tends to infinity, we have \(\lim_{n \to \infty} x_n = 0\). which is a contradiction to the fact that \(\lim_{n \to \infty} x_n\) is finite.

Hence \(\lim_{n \to \infty} y_n = 0\).

Similarly we can prove \(\lim_{n \to \infty} z_n = 0\).

\(\blacksquare\)

**Theorem 2.4.3.** If \(\lim_{n \to \infty} x_n \in \mathbb{R}\) and \(\sum_{i=1}^{\infty} c_i\) is divergent, then every solution of (2.1.1) is almost oscillatory.

**Proof.** Assume the contrary that (2.1.1) has a non-oscillatory solution does not approach zero. Now, we assume that \(x_n > 0\) for large \(n\).

From corollary 2.4.1, \(\{x_n\}\) exists. We have \(\lim_{n \to \infty} x_n = k \in (0, \infty)\).
Since $x_{n+\tau} > 0$, there exists a positive integer $n_1$ such that

$$x_{n+\tau} \geq \frac{k\mu}{2} \text{ for } n \geq n_1 \quad (2.4.1)$$

Since \( \{c_n\} \) is a real positive sequence. Summing the equation (2.1.3), we get

$$\sum_{i=n_1}^{\infty} c_i x_i^{\mu} \geq \frac{k\mu}{2} \sum_{i=n_1}^{\infty} c_i = \infty$$

Summing the third equation of system (2.1.3), we have

$$z_n - z_1 = -\sum_{i=1}^{n-1} c_i x_i^{\mu}$$

By corollary 2.4.2, we have \( \lim_{n \to \infty} z_n = 0 \).

Taking \( n \) tends to infinity in above equation, we get

$$z_1 = \sum_{i=1}^{\infty} c_i x_i^{\mu}$$

which gives a contradiction to the fact that $z_1$ is a constant term. Therefore, any bounded solution of (2.1.1) is almost oscillatory.

\[■\]

### 2.5 EXAMPLES

**Example 2.5.1.** Consider the third order difference equation

$$\Delta \left( 2^n (\Delta^4 (\Delta x_n)) \right) + 1400(2^{3n})x_n = 0. \quad (2.5.1)$$

Here \( a_n = 2^n \), \( b_n = 4^n \), \( c_n = 1300(2^{3n}) \) and \( \alpha = \beta = \mu = 1 \). \( x_n = (-1)^n 3^n \) is one of the quickly oscillatory solution of (2.5.1).
Solution 2.5.2. Let \( x_n = (-1)^n 3^n \)

\[
\Delta x_n = (-1)^n 3^{n+1} - (-1)^n 3^n \\
= (-1)^{n+1} 3^n [3 + 1] \\
= 4(-1)^{n+1} 3^n
\]

\[
\Delta 4^n (\Delta x_n) = \Delta(4^{n+1}(-1)^{n+1} 3^n) \\
= 4^{n+2}(-1)^{n+2} 3^{n+1} - (-1)^{n+1} 3^n 4^{n+1} \\
= 4^{n+1}(-1)^n 3^n (13)
\]

\[
\Delta(2^n \Delta 4^n (\Delta x_n)) = \Delta(2^n(13(4)^{n+1}(-1)^n 3^n)) \\
= 13(2^{n+1} 4^{n+2} 3^{n+1}(-1)^n - (-1)^n 4^{n+1} 3^n 2^n) \\
= 13(-1)^{n+1} 4^{n+1} 3^n 2^n [2(4)(3) + 1] \\
= -1400(-1)^n 3^n 2^{3n} \\
= -1400x_n 2^{3n}
\]

Example 2.5.3. Consider the third order difference equation

\[
\Delta((n - 1)(\Delta^2 x_n)) + \frac{1}{n-1} x_{n+3}^\mu = 0, \quad (\mu \geq 1). \tag{2.5.2}
\]

Here \( a_n = n - 1, b_n = 1 \) and \( c_n = \frac{1}{n-1} \) and \( \alpha = \beta = 1 \). We have

\[
\sum_{n=n_0}^{\infty} (n - 1)^{-1} = \sum_{n=n_0}^{\infty} 1 = \infty,
\]

\[
\sum_{n=n_0}^{\infty} \left( \frac{1}{n-1} \right) = \left( \sum_{n=n_0}^{\infty} \frac{1}{n-1} \right) = \infty
\]

and \( \left( \sum_{n=n_0}^{n-1} 1 \right) \left( \sum_{n=n_0}^{n-1} \frac{1}{n-1} \right) \left( \sum_{n=n_0}^{n-1} \frac{1}{n-1} \right) = \infty. \)
Therefore if $\mu > 1$, the conditions (2.3.5) and (2.3.8) are satisfied and by theorem 2.3.6, then (2.5.2) has no solution of type $(B_2)$, therefore (2.5.2) is oscillatory.

**Example 2.5.4.** Suppose that $a_n = \frac{1}{n}$, $b_n = \frac{1}{n-1}$ and $c_n = 2n$. Let $\alpha = \beta = 1$. Take the deviating argument $\tau$ is 2 then the equation (2.1.1) becomes

\[
\Delta \left( \frac{1}{n} (\Delta \frac{1}{n-1}\Delta x_n) \right) + 2nx_n^{\mu} + 2 = 0 \quad (\mu \geq 1)
\]  

(2.5.3)

Thus,

\[
\sum_{n=n_0}^{\infty} 2n \left( \sum_{n=n_0}^{n+1} (n-1) \left( \sum_{n=n_0}^{n-1} n \right) \right)^{1/\mu} = \infty
\]

and

\[
\sum_{n=n_0}^{\infty} (n-1) \left( \sum_{n=n_0}^{n-1} n \left( \sum_{n=n_0}^{2n} \right) \right) = \infty.
\]

Therefore if $\mu \geq 1$, the conditions (2.3.6) and (2.3.8) are satisfied and by theorem 2.3.7, then (2.5.3) has no solution of type $(B_2)$.

Hence (2.5.3) is oscillatory.

**Example 2.5.5.** By considering the third order difference equation

\[
(i) \Delta^2 (3^n (\Delta x_n)^3) + \frac{25}{4} 3^{n+3} x_{n+1}^3 = 0
\]

has the oscillatory solution $\frac{(-1)^n}{2^n}$, and deviating argument 1. Here $a_n = 1$, $b_n = 3^n$, $c_n = \frac{25}{4}(3^{n+3})$, $\alpha = 1$, and $\beta = \mu = 3$.

\[
(ii) \Delta (2n (\Delta^2 x_n)) + 8(2n - 1)x_{n+3} = 0
\]

(2.5.4)

has deviating argument 3, and $a_n = 2n$, $b_n = 1$, $c_n = 8(2n - 1)$, $\alpha = \beta = \mu = 1$. Hence $x_n = -\frac{1}{3^n}$ is negative solution of (2.5.4).
Solution 2.5.6. Let $x_n = -\frac{1}{3^n}$ be the solution of (2.5.4).

\[
\Delta^2 x_n = \Delta\left[ -\frac{1}{3^{n+1}} - \frac{1}{3^n} \right] \\
= \Delta\left[ \frac{1}{3^n} \left( \frac{2}{3} \right) \right] \\
= \frac{2}{3} \left[ \frac{1}{3^n+1} - \frac{1}{3^n} \right] \\
= \frac{2}{3} \left[ \frac{1}{3^n} \left( \frac{1}{3} - 1 \right) \right] \\
= -\frac{2}{3^{n+1}}
\]

\[
\Delta(2n\Delta^2(x_n)) = \Delta\left( 2n\left( -\frac{2}{3^{n+1}} \right) \right) \\
= -\frac{4}{3} \Delta\left( \frac{n}{3^n} \right) \\
= -\frac{4}{3} \left( \frac{n+1}{3^{n+1}} - \frac{n}{3^n} \right) \\
= -\frac{4}{3^{n+1}} \left[ \frac{n+1}{3} - n \right] \\
= -\frac{4}{3^{n+1}} \left[ 1 - 2n \right] \\
= -4(1 - 2n) \frac{1}{3^{n+1}} \\
= -4(1 - 2n)x_n.
\]

Example 2.5.7. Suppose that $a_n = 1, \quad b_n = n + 1, \quad c_n = \frac{2(4n^3 + 21n^2 + 27n + 1)}{(n+1)(n+2)}$.

Let $\alpha = \beta = 1$. Then (2.1.1) becomes,

\[
\Delta^2(n + 1(\Delta x_n)) + \frac{2(4n^3 + 21n^2 + 27n + 1)}{(n+1)(n+2)} x_n = 0. \quad (2.5.5)
\]

Take

\[
\lim_{n \to \infty} \frac{(-1)^n}{2n} \in R \\
\& \sum_{n=1}^{\infty} c_n = \infty.
\]

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Hence by theorem 2.4.3, \(\frac{(-1)^n}{2n}\) is almost oscillatory solution of (2.5.5).

2.6 CONCLUSION

In this chapter, the sufficient conditions for the oscillatory solutions, quickly oscillatory solutions and almost oscillatory solutions of third order nonlinear difference equations have been obtained.