Chapter 3
Calculation of Wave-Function for ES models

In this chapter we apply the QHJ formalism outlined in the previous chapter, to find bound state wave-functions for several exactly solvable potential problems in one dimension. We will show that, by making use of elementary theorems in complex variables, the form of QMF can be determined completely and hence the bound state wave-functions are easily obtained. To determine the form of QMF we begin with the QHJ equation \( (\hbar = 1 = 2m) \)

\[
p^2(x,E) - ip'(x,E) - [E - V(x)] = 0
\]

where \( p(x,E) \) is the QMF continued in the complex \( x \)-plane, and is related to the wave-function by

\[
p(x,E) = -i \frac{\psi'(x)}{\psi(x)}
\]

The zeros of the wave-function will appear as poles in the QMF. According to the well known theorems about the nodes of wave-function, the \( n^{th} \) excited state corresponds to \( n \) zeros on the real line, and there will be corresponding \( n \) (moving) poles in the QMF and the residue at each pole will be \(-i\) as has been discussed in chapter 2. In addition to these moving poles, there are fixed poles corresponding to the
singularities of the potential. **We will make an assumption that QMF has no other singularities in the finite complex plane.** The QMF turns out to be meromorphic, and to fix its form one needs to know the behavior of QMF for large \( x \) in the complex \( x \)-plane. This information can be easily read from the QHJ equation and hence the form of QMF can then be fixed completely. In the next section we show how this strategy works for the harmonic oscillator. In the remaining sections of this chapter we give the details of the calculation of the bound state wave-functions for harmonic oscillator, Morse oscillator, Poschl Teller and Eckart potentials and hydrogen atom. For these potentials a change of variable becomes necessary and we always try to bring the QHJ equation in the new variable to a same form as the above equation. We also mention that several other potentials have been studied [5] and the bound state wave-functions in each case agree with the known results [8].

### 3.1 Harmonic Oscillator

The potential energy of the harmonic oscillator is

\[
V(x) = \frac{1}{2}m\omega^2x^2
\]  

(3.1.1)

The quantum Hamilton-Jacobi equation is given by \((\hbar = 1 = 2m)\)

\[
p^2(x, E) - ip'(x, E) - \left(E - \frac{1}{4}\omega^2x^2\right) = 0
\]  

(3.1.2)

The QMF \( p(x, E) \) has \( n \) poles corresponding to the zeros of the wave-function, and residue at each of these poles is \(-i\). It can be proved that \( p(x, E) \) has no other poles except at infinity [1].

For large \( x \)

\[
p(x, E) \approx \pm \frac{1}{2}i\omega x
\]  

(3.1.3)
and we write
\[ p(x, E) \approx \pm \frac{1}{2} i \omega x + \phi(x) \] (3.1.4)
where \( \phi(x) \) is to be determined.

The sign of \( p(x, E) \) is determined by the condition of square integrability of the wave-function.

The wave-function is expressed as
\[ \psi(x) = \exp \left( i \int p(x, E) dx \right) \] (3.1.5)
When the above value of \( p(x, E) \) is substituted in the equation of wave-function, the wave-function is bounded at at large \( x \) if we choose the positive sign of \( \frac{1}{2} i \omega x \).

Hence we write the quantum momentum function \( p(x, E) \) as
\[ p(x, E) = \sum_{k=1}^{n} \frac{-i}{x - x_k} + \frac{1}{2} i \omega x + \phi(x) \] (3.1.6)
where \( x_1, x_2, \cdots, x_n \) are the location of \( n \) poles on the \( x \)-axis and \( \phi(x) \) is analytic everywhere and bounded at infinity. Therefore Liouville's theorem tells us, it has to be a constant. Hence let \( \phi(x) = c \) a constant. Hence the above equation becomes
\[ p(x, E) = \sum_{k=1}^{n} \frac{-i}{x - x_k} + \frac{1}{2} i \omega x + c \] (3.1.7)
Substituting (3.1.7) in (3.1.2) we have
\[ \left[ \sum_{k=1}^{n} \frac{-i}{x - x_k} + \frac{1}{2} i \omega x + c \right]^2 + \sum_{k=1}^{n} \frac{1}{(x - x_k)^2} - [E - \frac{1}{4} \omega^2 x^2] = 0 \] (3.1.8)
Equating the power of \( x \), we have \( c = 0 \). Equating the constant term to zero on both sides in equation (3.1.8) gives
\[ E = (n + \frac{1}{2}) \omega \]
which is the well known expression for energy of the harmonic oscillator in our notation \( (2m = 1) \).
CHAPTER 3. CALCULATION OF WAVE-FUNCTION FOR ES MODELS

The sum moving pole terms
\[ \sum_{k=1}^{n} \frac{-i}{x - x_k} \]
can be expressed as \( \frac{P'(x)}{P(x)} \) where \( P(x) \) is the polynomial
\[ P(x) = \prod_{k=1}^{n} (x - x_k). \] (3.1.10)

The QMF (3.1.7) can be expressed as
\[ p(x, E) = -i \frac{P'(x)}{P(x)} + \frac{1}{2} i \omega x, \] (3.1.11)

Using (3.1.11) in (3.1.2) and on simplification yields
\[ P''(x) - \omega x P'(x) + n \omega P(x) = 0 \] (3.1.12)

We effect a transformation \( \xi = \alpha x \) where \( \alpha^2 = \% \). Hence equation (3.1.12) changes to
\[ P''(\xi) - 2\xi P'(\xi) + 2n P(\xi) = 0 \] (3.1.13)

This above equation resembles the well known Hermite differential equation. Hence on comparison, we get
\[ P(\xi) \equiv H_n(\alpha x) \] (3.1.14)

and is the Hermite polynomial.

The wave-function is expressed as
\[ \psi(x) = \exp \left[ i \int p(x, E) dx \right] = \exp \left[ i \int \left( -i \frac{P'(x)}{P(x)} + \frac{1}{2} i \omega x \right) dx \right]. \]

and hence we have
\[ \psi(x) = H_n(\alpha x) \exp \left( -\frac{1}{4} \omega x^2 \right). \] (3.1.15)

This is the desired wave-function for the harmonic oscillator.
3.2 Morse Oscillator

The potential energy of the Morse oscillator is

\[ V(x) = A^2 + B^2 e^{-2\alpha x} - 2B(A + \frac{\alpha}{2})e^{-\alpha x} \]  \hspace{1cm} (3.2.1)

with the super potential

\[ W(x) = A - Be^{-\alpha x} \]  \hspace{1cm} (3.2.2)

and

\[ s = \frac{\alpha}{\alpha} \]  \hspace{1cm} (3.2.3)

The quantum Hamilton-Jacobi equation is given by \((\hbar = 1 = 2m)\)

\[ p^2(x, E) - ip'(x, E) - \left[ E - A^2 - B^2 e^{-2\alpha x} + 2B(A + \frac{\alpha}{2})e^{-\alpha x} \right] = 0 \]  \hspace{1cm} (3.2.4)

We effect a transformation to the variable

\[ y = \frac{2B}{\alpha} e^{-\alpha x} \]  \hspace{1cm} (3.2.5)

The quantum Hamilton-Jacobi equation in the new variable is

\[ \tilde{p}^2(y, E) + i\alpha y \tilde{p}'(y, E) - \left[ E - A^2 - \frac{\alpha^2}{4} y^2 + (A + \frac{\alpha}{2})\alpha y \right] = 0 \]  \hspace{1cm} (3.2.6)

We define

\[ \tilde{p}(y, E) = i\alpha y \phi(y, E) \]  \hspace{1cm} (3.2.7)

Then (3.2.6) transforms to

\[ \left( \phi + \frac{1}{2y} \right)^2 + \phi' - \frac{1}{4y^2} + \frac{1}{\alpha^2 y^2} [E - A^2 - \frac{y^2 \alpha^2}{4} + (A + \frac{\alpha}{2})\alpha y] = 0 \]  \hspace{1cm} (3.2.8)

We define

\[ \chi(y, E) = \phi(y, E) + \frac{1}{2y} \]  \hspace{1cm} (3.2.9)
Therefore (3.2.8) transforms to

\[ \chi^2 + \frac{1}{4y^2} + \chi' + \frac{1}{\alpha^2y^2}[E - A^2 - \frac{y^2\alpha^2}{4} + (A + \frac{\alpha}{2})y\alpha] = 0 \] (3.2.10)

This equation suggests that \( \chi \) has a pole at \( y = 0 \). It will also have \( n \) moving poles corresponding to the nodes of the wave-function. We assume that there are no other poles in the finite complex plane. For large \( y \) the behavior of \( \chi \) has already been obtained in section (3.1) and is seen to be bounded for large \( y \). Hence we get using Liouville's theorem

\[ \chi = \frac{b_1}{y} + \sum_{k=1}^{n} \left( \frac{1}{y - y_k} \right) + c \] (3.2.11)

where \( b_1 \) and \( c \) are constants to be fixed. The residue of \( \chi \) at \( y = 0 \) is \( b_1 \) and has been obtained in section (3.1) and

\[ b_1 = \frac{A}{\alpha} + \frac{1}{2} \] (3.2.12)

We write, once again,

\[ \sum_{k=1}^{n} \frac{1}{y - y_k} = \frac{P'(y)}{P(y)} \]

where

\[ P(y) = \prod_{k=1}^{n} (y - y_k) \]

Substituting (3.2.11) in (3.2.10) gives

\[ \frac{P''}{P} + \frac{b_1}{y} \frac{P'}{P} + 2 \frac{P'}{P} c + 2 \frac{b_1}{y} c + c^2 - \frac{1}{4} + \frac{1}{\alpha y}(A + \frac{\alpha}{2}) = 0 \] (3.2.13)

In order to proceed further we look at the behavior of each term for large \( y \). Using the leading terms

\[ \frac{P''(y)}{P(y)} \sim \frac{n(n-1)}{y^2}, \quad \frac{P'(y)}{P(y)} \sim \frac{n}{y} \]
in equation (3.2.13) and equating the constant term on both sides gives, $c = \pm \frac{1}{2}$.
The correct sign for $c$ is chosen by the condition of square integrability on the wave function which fixes $c = -\frac{1}{2}$.

Comparing the coefficient of $-\frac{1}{2}$ for large $y$ on both sides we get

$$2b_1 c + 2nc + \left(A + \frac{\alpha}{2}\right) \frac{1}{\alpha} = 0$$

(3.2.14)

which on using the values of $b_1$ and $c$ and on simplification gives the energy eigenvalue

$$E = A^2 - (A - n\alpha)^2$$

(3.2.15)

Substituting the value of $b_1$ and $c$ in equation (3.2.13) we have

$$yP''(y) + \{1 - y + 2(s - n)\} P'(y) + nP(y) = 0$$

(3.2.16)

Compare this with the standard Laguerre differential equation

$$xy'' + (\beta + 1 - x)y' + ny = 0$$

we have $P(y) = L^\beta_n(y)$.

The wave function for the Morse oscillator is given by

$$\psi(x) = \exp \left( i \int p(x, E) dx \right)$$

(3.2.17)

In terms of $y$ variable we have

$$\psi(y) = \exp \left( i \int \left[ \frac{b_1}{y} + \frac{P'(x)}{P(x)} - \frac{1}{2} - \frac{1}{2y} \right] dy \right)$$

(3.2.18)

On integrating and simplifying we get

$$\psi_n(y) = y^{s-n} \exp(-\frac{1}{2}y)P(x)$$

(3.2.19)

Replacing the value of $P(y)$ we have

$$\psi_n(y) = y^{s-n} \exp(-\frac{1}{2}y)L^\beta_n(y)$$

(3.2.20)
CHAPTER 3. CALCULATION OF WAVE-FUNCTION FOR ES MODELS

3.3 Poschl-Teller Potential

The Poschl-Teller potential is

\[ V(x) = A^2 + (B^2 + A^2 + A\alpha) \csc^2 \alpha x - B(2A + a) \coth \alpha x \csc \alpha x \]  

(3.3.1)

with the super potential given by

\[ W(x) = A \coth \alpha x - B \csc \alpha x \quad (A < B) \]  

(3.3.2)

and

\[ s = \frac{A}{\alpha}, \quad \lambda = \frac{B}{\alpha} \]  

(3.3.3)

The quantum Hamilton-Jacobi equation is given by \((h = 1 = 2m)\)

\[ p^2(x, E) - i\dot{p}(x, E) \]

\[- [E - A^2 - (B^2 + A^2 + Aa) \csc^2 \alpha x + B(2A + a) \coth \alpha x \csc \alpha x] = 0 \]  

(3.3.4)

We effect a transformation to a new variable

\[ y = \cosh \alpha x \]  

(3.3.5)

The quantum Hamilton-Jacobi equation in the new variable becomes

\[ \ddot{p}(y, E) - i\alpha \sqrt{y^2 - 1} \dddot{p}(y, E) \]

\[- \left[ E - A^2 - (B^2 + A^2 + A\alpha) \frac{1}{y^2 - 1} + B(2A + \alpha) \frac{y}{y^2 - 1} \right] = 0 \]  

(3.3.6)

We define

\[ \ddot{p}(y, E) = -i\alpha \sqrt{y^2 - 1} \phi(y). \]  

(3.3.7)

Therefore (3.3.6) transforms to

\[ \left( \phi + \frac{1}{2} \frac{y}{y^2 - 1} \right)^2 + \phi' - \frac{1}{4} \frac{y^2}{(y^2 - 1)^2} \]  

\[ = 0 \]
For \( y = -1 \), we define

\[
\chi = \phi + \frac{1}{2y^2 - 1}. \quad (3.3.9)
\]

Therefore (3.3.8) transforms to

\[
\chi^2 + \chi' + \frac{3}{4} \frac{y^2}{(y^2 - 1)^2} - \frac{1}{2} \frac{1}{y^2 - 1} + \frac{1}{\alpha^2(y^2 - 1)} \left[ E - A^2 - (B^2 + A^2 + A\alpha) \frac{1}{y^2 - 1} + B(2A + \alpha) \frac{y}{y^2 - 1} \right] = 0 \quad (3.3.10)
\]

\( \chi \) has poles at \( y = \pm 1 \) and there are moving poles between the classical turning points. We assume that there are no more poles in the complex line. We now determine the residue at each of these poles.

For \( y = +1 \), we define

\[
\chi = \frac{b_1}{y - 1} + a_0 + a_1(y - 1) + \cdots \quad (3.3.11)
\]

Using (3.3.11) in (3.3.10) and equating the coefficient of \( \frac{1}{y - 1} \) yields

\[
b_1 = \frac{1}{2} \left[ 1 \pm \frac{1}{2\alpha} [2(B - A) - \alpha] \right] \quad (3.3.12)
\]

The correct value of \( b_1 \) is selected by imposing the condition

\[
\lim_{E \to 0} \text{Res } \tilde{p}(y, E) = i \text{Res } \tilde{W}(y, E)
\]

on super potential as explicitly shown for Morse oscillator gives

\[
b_1 = \frac{1}{4} - \frac{1}{2\alpha} (A - B) \quad (3.3.13)
\]

For \( y = -1 \), we define

\[
\chi = \frac{b_1'}{y + 1} + a_0' + a_1'(y + 1) + \cdots \quad (3.3.14)
\]
Using (3.3.14) in (3.3.10) and equating the coefficient of \( \frac{1}{y+1} \) and following the above procedure yields

\[ b'_1 = \frac{1}{4} - \frac{1}{2\alpha}(A + B) \]  

(3.3.15)

The residue at a moving pole is seen from (3.3.10) to be 1. Hence we arrive at the form

\[ \chi = \frac{b_1}{y-1} + \frac{b'_1}{y+1} + \frac{P''(y)}{P(y)} + c \]  

(3.3.16)

for \( \chi \) where \( c \) is a constant to be determined.

Substituting (3.3.16) in (3.3.10) gives the following equation.

\[
c^2 + \frac{2b_1b'_1}{y^2-1} + \frac{2b'_1}{y+1} \frac{P'}{P} + 2 \frac{P'}{P} + \frac{2b_1c}{y-1} + \frac{2b_1}{y-1} \frac{P'}{P} + \frac{2b'_1}{y+1} \frac{c}{y-1} - \frac{1}{2} \frac{1}{y^2-1} + \frac{P''}{P} \\
\frac{3}{8} \frac{1}{y^2-1} + \frac{1}{\alpha^2} (E - A^2) \frac{1}{y^2-1} + \frac{1}{2\alpha^2} (B^2 + A^2 + A\alpha) \frac{1}{y^2-1} = 0 \]  

(3.3.17)

Now we look at different terms in equation (3.3.17) for large \( y \) and equate their coefficient to zero. Equating the constant term to zero gives \( c = 0 \). With \( c = 0 \) the above equation becomes

\[
\frac{P''}{P} + \frac{P'}{P} \left[ \frac{2b_1}{y-1} + \frac{2b'_1}{y+1} \right] + \frac{2b_1b'_1}{y^2-1} - \frac{1}{2} \frac{1}{y^2-1} \\
\frac{3}{8} \frac{1}{y^2-1} + \frac{1}{\alpha^2} (E - A^2) \frac{1}{y^2-1} + \frac{1}{2\alpha^2} (B^2 + A^2 + A\alpha) \frac{1}{y^2-1} = 0 \]  

(3.3.18)

For large \( y \), \( P(y) \) behaves as \( P(y) \sim y^n + \cdots \) and \( \frac{P''(y)}{P(y)} \sim \frac{n(n-1)}{y^2} \), \( \frac{P'(y)}{P(y)} \sim \frac{n}{y} \).

Using these in (3.3.18) and equating the coefficient of \( \frac{1}{y^2} \) gives

\[
2b_1b'_1 + 2nb'_1 + 2nb_1 + n(n-1) + \frac{3}{8} + \frac{1}{\alpha^2} (E - A^2) + \frac{1}{2\alpha^2} (B^2 + A^2 + A\alpha) - \frac{1}{2} = 0 \\
\]  

and substituting the values of \( b_1 \) and \( b'_1 \) gives the expression for energy as
\[ E = A^2 - (A - no)^2 \]  

(3.3.19)

Substituting the value of \( E, b_1 \) and \( b'_1 \) in (3.3.18) we get

\[
(1 - y^2)P''(y) + [(2s - 1)y - 2\lambda]P'(y) + [n(n - 2s)]P(y) = 0
\]

(3.3.20)

The above equation resembles the standard Jacobi polynomial, such that

\[
P(y) aP_n^{(a,b)}(y) = P_n^{(\lambda-s-\frac{1}{2},-\lambda-s-\frac{1}{2})}(y)
\]

(3.3.21)

The wave function for the Poschl Teller potential is obtained on the same line as that for the Morse oscillator and is given by

\[
\psi(y) = (y - 1)^{(\lambda-s)}y + 1)^{-(\lambda+s)}P_n^{(\lambda-s-\frac{1}{2},-\lambda-s-\frac{1}{2})}(y)
\]

(3.3.22)

which agrees well with the values given in literature. [5]

### 3.4 Eckart Potential

The potential energy of the Eckart potential is

\[
V(x) = A^2 + \frac{B^2}{A^2} - 2B \coth \alpha x + A(\alpha - \alpha) \text{csch}^2 \alpha x
\]

(3.4.1)

with the super potential given by

\[
W(x) = -A \coth \alpha x + \frac{B}{A}, \quad (B > A^2)
\]

(3.4.2)

and

\[
s = \frac{A}{\alpha}, \quad \lambda = \frac{B}{\alpha^2}, \quad \alpha = \frac{\lambda}{n + s}
\]

(3.4.3)

The quantum Hamilton-Jacobi equation is given by \((\hbar = 1 = 2m)\)

\[
\frac{p^2(x, E)}{2} - i p'(x, E) - \left[ E - A^2 - \frac{B^2}{A^2} + 2B \ coth \alpha x - A(\alpha - \alpha) \text{csch}^2 \alpha x \right] = 0
\]

(3.4.4)
We effect a transformation by the variable

\[ y = \coth ax \]  

(3.4.5)

The quantum Hamilton-Jacobi equation in the new variable is

\[ \tilde{p}^2(y, E) - i\alpha(1 - y^2)\tilde{p}(y, E) - \left[ E - A^2 - \frac{B^2}{A^2} + 2By - A(A - \alpha)(y^2 - 1) \right] = 0 \]

(3.4.6)

We define

\[ \tilde{p}(y, E) = -i\alpha(1 - y^2)\phi(y) \]  

(3.4.7)

Hence equation (3.4.6) simplifies to

\[ \frac{1}{\alpha^2(1 - y^2)^2} \left[ E - A^2 - \frac{B^2}{A^2} + 2By - A(A - \alpha)(y^2 - 1) \right] = 0 \]

(3.4.8)

Let

\[ \chi = \phi - \frac{y}{1 - y^2} \]  

(3.4.9)

Therefore the above equation changes to

\[ \chi^2 + \chi' + \frac{y^2}{(1 - y^2)^2} + \frac{1}{1 - y^2} \]

\[ + \frac{1}{\alpha^2(1 - y^2)^2} \left[ E - A^2 - \frac{B^2}{A^2} + 2By - A(A - \alpha)(y^2 - 1) \right] = 0 \]

(3.4.10)

\( \chi \) has poles at \( y = \pm 1 \) and there are moving poles between the classical turning points. We assume that there are no more poles in the complex plane. We determine the residue at each of these poles.

For \( y = +1 \), we define

\[ \chi = \frac{b_1}{y - 1} + a_0 + a_1(y - 1) + \cdots \]  

(3.4.11)
Using (3.4.11) in (3.4.10) and equating the coefficient of $\frac{1}{(\alpha)^2}$ yields

$$b_1 = \frac{1}{2} \left[ 1 \pm \frac{1}{\alpha} \sqrt{\left( A - \frac{B}{A} \right)^2 - E} \right] \quad (3.4.12)$$

As the residue has two values, the correct value is selected by imposing the condition on the super potential as done in Morse oscillator and the correct value is

$$b_1 = \frac{1}{2} \left[ 1 - \frac{1}{\alpha} \sqrt{\left( A - \frac{B}{A} \right)^2 - E} \right] \quad (3.4.13)$$

Similarly the residue for $y = -1$ is determined and is given as

$$b'_1 = \frac{1}{2} \left[ 1 - \frac{1}{\alpha} \sqrt{\left( A + \frac{B}{A} \right)^2 - E} \right] \quad (3.4.14)$$

We assume $\chi$ to have the form

$$\chi = \frac{b_1}{y-1} + \frac{b'_1}{y+1} + \frac{P'(y)}{P(y)} + c \quad (3.4.15)$$

where $c$ is a constant to be determined.

Using (3.4.15) in (3.4.10) and following the similar lines as that of Morse and Poschel Teller potential one gets the value of $c = 0$ and the resulting equation becomes

$$\frac{P^n(y)}{P(y)} + \frac{P'(y)}{P(y)} \left[ \frac{2b_1}{y-1} + \frac{b'_1}{y+1} \right] + \frac{2b_1b'_1}{y^2-1} = \frac{1}{2} \frac{1}{y^2-1} - \frac{1}{2\alpha^2} \frac{1}{y^2-1} \left[ E - A^2 - \frac{B^2}{A^2} \right] - \frac{1}{\alpha^2} \frac{1}{y^2-1} A(A - \alpha) = 0 \quad (3.4.16)$$

For large $y$ assuming $P(y) \sim y^n + \cdots$ and equating the coefficient of $\frac{1}{y^2}$ gives

$$2b_1b'_1 + 2b'_1n + 2b_1n + n(n-1) - \frac{1}{2} - \frac{1}{2\alpha^2} (E - A^2 - \frac{B^2}{A^2}) - \frac{1}{\alpha^2} A(A - \alpha) = 0$$

Substituting the values of $b_1$ and $b'_1$ gives the energy expression as

$$E = A^2 - (A + n\alpha)^2 - \frac{B^2}{(A + n\alpha)^2} + \frac{B^2}{A^2} \quad (3.4.17)$$
CHAPTER 3. CALCULATION OF WAVE-FUNCTION FOR ES MODELS

Using the values of \( b_1, b'_1 \) and \( E \) in (3.4.16) one gets the differential equation for Eckart potential as

\[
(1 - y^2)P''(y) + [-2a - 2(-n - s + 1)y]P'(y) + (-2ns)P(y) = 0 \tag{3.4.18}
\]

Equation (3.4.18) resembles the standard Jacobi polynomial and

\[
P(y) = P_n^{(a,b)}(y) = P_n^{(s_1,s_2)}(y) \tag{3.4.19}
\]

The wave function for the Eckart Potential is obtained from

\[
\psi(x) = e^{i \int p(x,E)dx}
\]

and is given by

\[
\psi(y) = (y - 1)^{2s_1/2} (y + 1)^{2s_2/2} P_n^{(s_1,s_2)}(y) \tag{3.4.20}
\]

The values for energy and wave function agree with those found in the literature.[5]

3.5 Hydrogen Atom

In this section we obtain the bound state wave functions of the radial part of the Schrödinger equation \((\hbar = 1 m = 11)\)

\[
\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \left( E + \frac{Ze^2}{r} - \frac{\lambda}{r^2} \right) R = 0 \tag{3.5.1}
\]

where \( A = l(l+1) \). Using the transformation \( R(r) = \phi(r)/r \), the Schrödinger equation becomes

\[
\frac{d^2 \phi}{dr^2} + \left( E + \frac{Ze^2}{r} - \frac{\lambda}{r^2} \right) \phi = 0. \tag{3.5.2}
\]

The QHJ equation in terms of

\[
q = \frac{d}{dr} \ln(\phi(r)) \tag{3.5.3}
\]
is given by
\[ q^2 + \frac{dq}{dr} + \left( E + \frac{Ze^2}{r} - \frac{\lambda}{r^2} \right) = 0. \] (3.5.4)
The range of \( r \) is from 0 to \( \infty \), and the wave function \( \phi(r) \) should vanish at \( r = 0 \). Thus \( q \) has a fixed pole at \( r = 0 \), along with the \( n \) moving poles with residue equal to one on the real line. Like harmonic oscillator there are no other singularities in the finite complex plane. Thus we can write \( q \), in a similar fashion as for harmonic oscillator, as
\[ q(r) = \frac{P'}{P} + \frac{b_1}{r} + C. \] (3.5.5)
where \( b_1 \) is the residue at \( r = 0 \) which can be obtained by doing a Laurent expansion of \( q \) around the pole at the origin. The two values of residues obtained are
\[ b_1 = -l, \quad b_1 = l + 1. \] (3.5.6)
One chooses the right residue by using the square integrability property of the wave function \( \phi \) and obtain
\[ b_1 = l + 1 \] (3.5.7)
as the right choice. Substituting (3.5.5) for \( q \) in (3.5.4) and expanding different terms of the resulting equation for large \( r \) and comparing the leading terms we get
\[ C^2 = -E, \quad E = -\frac{Z^2e^4}{(2n')^2} \] (3.5.8)
where \( n' = n + l + 1 \) and one is left with the differential equation
\[ rP'' + 2P' \left( l + 1 - \frac{Ze^2r}{2n'} \right) + \frac{(n' - l - 1)Ze^2}{n'} P = 0. \] (3.5.9)
Now defining
\[ \frac{Ze^2}{n'} = \rho \] (3.5.10)
(3.5.8) becomes

\[ pP''' + ((2l + 1) + 1 - \rho)P' + (\tau' - l - 1)P = 0 \]  \hspace{1cm} (3.5.11)

which is the associated Laguerre differential equation where \( P \) is the Laguerre polynomial denoted by \( L \). The bound state wave function obtained from (3.5.5) and (3.5.3) is

\[ \psi_n(\rho) = \rho^{l+1} \exp(-\rho/2)L_{n+1}^{2l+1}(\rho) \]  \hspace{1cm} (3.5.12)

which is seen to be identical with known correct answer.