Chapter 5

CALCULATION OF WAVE-FUNCTIONS FOR QES MODELS

In the previous chapter we have seen that the condition for exact solvability arises from a simple requirement on the behavior of QMF at infinity. We continue our study of QES models and we take up an investigation of wave-functions. We find that the wave-functions can be computed by proceeding as in the case of exactly solvable models. We begin with our simplifying assumption mentioned in the previous chapter for the QES models and proceed in the same fashion as for the case of exactly solvable models in chapter 3. Thus the QMF is meromorphic and the corresponding residues at the poles are known, and also the behavior at infinity is known, with this information the bound state wave-functions can be obtained as in chapter 3. We give our results for two potential models viz. the sextic oscillator and the hyperbolic potential. This study reveals a new interesting feature of the zeros of the wave-functions, which will be discussed at the end of this chapter.
5.1 Sextic Oscillator

The potential for the sextic oscillator is:

\[ V(x) = ax^2 + \beta x^4 + \gamma x^6, \quad \gamma > 0 \]  

(5.1.1)

with the following values for \( \alpha, \beta, \gamma \) and the condition for \( \mu, n, p \) where \( p \) stands for parity

\[ \alpha = h^2 - a(3 + 2n), \quad \beta = 2ab, \quad \gamma = a^2, \quad 4\mu + 2p = 2n \text{ with } p = 0 \text{ or } 1 \]

The QHJ equation is \((\hbar = 1 = 2m)\)

\[ p(x, E) - ip'(x, E) - (E - \alpha x^2 - \beta x^4 - \gamma x^6) = 0 \]  

(5.1.2)

We assume that the point at infinity is a pole. Therefore \( p(x, E) \) behaves as \( x^n \) for some \( n \)

\[ p(x, E) \sim x^n \]

for large \( x \). Hence \( p(x, E) \) takes the form for large \( x \).

\[ p(x, E) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 + O(\frac{1}{x}) \]  

(5.1.3)

where \( a_0, a_1, \ldots, a_3 \) are constants, on the assumption that \( p(x, E) \) have no other singular points and substitute (5.1.3) in equation (5.1.2). Next we equate the coefficient of powers of \( x^6 \) to zero, gives

\[ a_z^2 + 7 = 0 \]  

(5.1.4)

Since \( 7 = a^2 \), we have

\[ a_3 = \pm ia \]  

(5.1.5)

As \( a_3 \) has two values, the correct value is fixed by the condition of square integrability on the wave function.
\[ \psi(x) = \exp \left( i \int p(x, E) \, dx \right) = \exp \left( i \int (a_3 x^3 + a_1 x) \, dx \right) \]

If the above integral have to bounded at infinity, the we require that

\[ a_3 = +ia \quad (5.1.6) \]

Next equating the coefficient of successive powers \( x^5, x^4, \ldots \) to zero we get

\[ a_2 = 0 \quad (5.1.7) \]
\[ a_1 = -\frac{ab}{a_3} \quad (5.1.8) \]
\[ a_0 = 0 \quad (5.1.9) \]

Hence

\[ a_1 = ib \quad (5.1.10) \]

Therefore \( p(x, E) \) becomes

\[ p(x, E) = \sum_{k=1}^{n} \frac{-i}{x - x_k} + iax^3 + ibx + c \quad (5.1.11) \]

To determine \( x_k \), or equivalently \( P(x) = \prod_{k=1}^{n} (x - x_k) \), we substitute (5.1.11) in (5.1.2) and get

\[ -[E - ax^2 - fix^4 - \gamma x^6] = 0 \quad (5.1.12) \]

Therefore, the above equation becomes

\[ c^2 + 2ax^3 \frac{P'}{P} + 2ibcx - 2i \frac{P'}{P} c + 2bx \frac{P'}{P} + 2iacx^3 - \frac{P''}{P} + b - E - 2anx^2 = 0 \quad (5.1.13) \]

Equating the coefficient of \( x^3 \) term we have

\[ 2iacx^3 = 0 \implies c = 0 \]
CHAPTER 5. CALCULATION OF WAVE-FUNCTIONS FOR QES MODELS

and hence we have

\[ 2ax^3 \frac{P'}{P} + 2bx \frac{P'}{P} - \frac{P''}{P} + b - E - 2anx^2 = 0 \]  \hspace{1cm} (5.1.14)

The above equation thus gives the following differential equation in \( P(x) \)

\[ P'' - P'(2ax^3 + 2bx) - P(b - E - 2anx^2) = 0 \]  \hspace{1cm} (5.1.15)

We get the expression for energies and wave functions for various values of \( n \) as follows:

We will derive explicit form of wave-functions for \( n = 0, 1 \) and 2. Later we will discuss the general form of the wave-function for arbitrary \( n \). The general strategy for obtaining the wave-functions is the same as discussed for exactly solvable models in chapter 3.

**Wave-function for \( n=0 \):** Only one energy level can be solved in this case. Since the number \( n \), representing the number of moving poles is zero (5.1.11), with \( c=0 \) as already found, becomes

\[ p(x,E) = iax^3 + ibx \]\hspace{1cm} (5.1.16)

and hence the wave-function is given by

\[ \psi(x) = \exp \left( i \int p(x)dx \right) = \exp \left( i \int [iax^3 + ibx]dx \right) = \exp \left( -a\frac{x^4}{4} - b\frac{x^2}{2} \right) \]

and the corresponding energy is obtained from (5.1.15) by equating the constant term and is given as

\[ E = b. \]  \hspace{1cm} (5.1.18)

**Wave-function for \( n=1 \):** In this case we take \( P(x,E) \) to be a first degree polynomial, \( (x - x_0) \). There (5.1.15) gives, \( x_0 = 0 \) and the energy is given as

\[ E = 3b. \]  \hspace{1cm} (5.1.19)
Therefore the wave-function comes out to be

\[ \psi(x) = N x \exp \left( -\frac{ax^4}{4} - bx^2 \right). \quad (5.1.20) \]

**Wave-function for \( n=2 \):** We seek a solution of (5.1.15) with \( P(x) \) as a second degree polynomial. Substituting \( P(x) \) as

\[ P(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2. \quad (5.1.21) \]

Using the above equation in (5.1.15) and comparing different powers of \( x \) gives

\[ \alpha_1 = 0 \quad (5.1.22) \]

\[ 4a\alpha_0 - \alpha_2(5b - E) = 0 \quad (5.1.23) \]

\[ \alpha_0(b - E) - 2\alpha_2 = 0. \quad (5.1.24) \]

The last two equation have non-trivial solution for \( \alpha_0 \) and \( \alpha_2 \) only if the

\[ \begin{vmatrix} 4a & 5b - E \\ b - E & -2 \end{vmatrix} = 0. \]

This gives two energy eigen-values

\[ E = 3b \pm 2\sqrt{b^2 + 2a}. \quad (5.1.25) \]

To get the wave function we compute \( \alpha_1 \) and \( \alpha_2 \) from equation (5.1.23) and (5.1.24) and use

\[ p(x) = -i \frac{2\alpha_2 x}{\alpha_0 + \alpha_2 x^2} + iax^3 + ibx. \quad (5.1.26) \]

Therefore the wave function is given by:

\[ \psi(x) = N \exp \left( i \int p(x) dx \right) = N \exp \left( i \left[ -i \frac{2\alpha_2 x}{\alpha_0 + \alpha_2 x^2} + iax^3 + ibx \right] dx \right) \quad (5.1.27) \]

\[ \psi(x) = N(\alpha_0 + \alpha_2 x^2) \exp \left( -\frac{ax^4}{4} - \frac{bx^2}{2} \right) \quad (5.1.28) \]
where \( N \) is the normalizing factor. The value of \( \alpha_0 \) is given by
\[
\alpha_0 = \frac{5b - E}{4a}.
\] (5.1.29)

Replacing the value of \( \alpha_0 \) and energy value \( E \) in the above equation one gets the expression for wave-function as
\[
\psi(x) = N \frac{E - 5b}{4a} \left[ b \pm \sqrt{b^2 + 2ax^2 - 1} \right] \exp \left( -a\frac{x^4}{4} - b\frac{x^2}{2} \right). \tag{5.1.30}
\]

The wave-functions and eigen-values explicitly obtained for the cases \( n = 0, 1 \) and 2 agree with the known results [6].

For an arbitrary value of \( n \) the polynomial \( P(x) \) will be obtained by solving (5.1.15). If we take \( P(x) \) to be of the form
\[
P(x) = \sum_{k=0}^{n} \alpha_k x^n, \tag{5.1.31}
\]
then the differential equation (5.1.15) leads to a set of homogenous equations for the corresponding coefficients \( \alpha_0, \alpha_1, \ldots, \alpha_n \). These equations will have a non-trivial solution only if determinant of the coefficients vanishes. This condition will determine the energy eigen-value, corresponding to each eigen-value we can find the coefficients \( \alpha_0, \alpha_1, \ldots, \alpha_n \). Thus we get \( n \) independent wave-function each having the form
\[
\psi(x) \sim P(x) \exp \left( -a\frac{x^4}{4} - b\frac{x^2}{2} \right). \tag{5.1.32}
\]

Notice that all these eigen-functions corresponding to a fixed value of \( n \) have a polynomial of the same degree \( n \) as a factor. Thus for a fixed value of \( n \), and hence for a given set of potential parameters, wave-functions for all the states which can be solved have the same number of zeros equal to \( n \). If these levels are arranged according to increasing energy, the number of zeros on the real axis (nodes) will increase. Hence the number of complex zero will decrease with increasing energy. This feature appears to be a general property of quasi-exactly solvable models.
CHAPTER 5. CALCULATION OF WAVE-FUNCTIONS FOR QES MODELS

5.2 Hyperbolic Potential

The hyperbolic potential is

\[ V(x) = -\frac{A}{\cosh^2 x} + \frac{B}{\sinh^2 x} - C \cosh^2 x + D \cosh^4 x, \quad (5.2.1) \]

where

\[ A = 4(s_1 - \frac{1}{4})(s_1 - \frac{3}{4}), \]
\[ B = 4(s_2 - \frac{1}{4})(s_2 - \frac{3}{4}), \]
\[ C = [q_1^2 + 4q_1(s_1 + s_2 + \mu)], \]
\[ D = q_1^2. \]

The QHJ equation is \((\hbar = 1 = 2m)\)

\[ p^2(x, E) - ip'(x, E) - [E + \frac{A}{\cosh^2 x} - \frac{B}{\sinh^2 x} + C \cosh^2 x - D \cosh^4 x] = 0. \quad (5.2.2) \]

We effect a transformation by

\[ y = \cosh x \quad (5.2.3) \]

The QHJ equation in the new variable is

\[ \tilde{p}(y, E) - i\hbar \sqrt{y^2 - 1}p'(y, E) - [E + \frac{A}{y^2} - \frac{B}{y^2 - 1} + Cy^2 - Dy^4] = 0 \quad (5.2.4) \]

Let

\[ \tilde{p}(y, E) = -i\sqrt{y^2 - 1}\phi(y, E) \quad (5.2.5) \]

Therefore the QHJ equation becomes

\[ \left[ \phi + \frac{1}{2} \frac{y}{y^2 - 1} \right]^2 - \frac{1}{4} \frac{y^2}{(y^2 - 1)^2} + \phi' + \frac{1}{y^2 - 1} [E + \frac{A}{y^2} - \frac{B}{y^2 - 1} + Cy^2 - Dy^4] = 0. \quad (5.2.6) \]

Let

\[ \chi = \phi + \frac{1}{2} \frac{y}{y^2 - 1}. \quad (5.2.7) \]
Therefore the above equation becomes
\[ \chi^2 + \chi' + \frac{3}{4} \frac{y^2}{(y^2 - 1)^2} \left[ 1 - \frac{1}{2} \frac{1}{y^2 - 1} \right] + \frac{1}{y^2 - 1} \left[ E + \frac{A}{y^2} - \frac{B}{y^2 - 1} + Cy^2 - Dy^4 \right] = 0 \quad (5.2.8) \]

\( X \) has poles at \( y = 0, \) and \( y = \pm 1, \) and there are moving poles between the turning points. We assume that there are no more poles in the complex plane other than a pole at infinity. We will first compute the residues at \( y = 0, \pm 1 \) and then in the general form of \( \chi \) (5.2.19) the constants \( b_1, b_1', b_1'' \) will be known and then we give the general form of the wave-function.

Computation of residues: For \( y = 0 \) let
\[ \chi = \frac{b_1}{y} + a_0 + a_1 y + \cdots \quad (5.2.9) \]

Therefore equation (5.2.8) becomes
\[ \left[ \frac{b_1}{y} + a_0 + a_1 y + \cdots \right]^2 + \left[ -\frac{b_1}{y^2} + a_1 + \cdots \right] + \frac{3}{4} \frac{y^2}{(y^2 - 1)^2} \left[ E + \frac{A}{y^2} - \frac{B}{y^2 - 1} + Cy^2 - Dy^4 \right] = 0 \quad (5.2.10) \]

Equating the coefficient of 4 on both sides gives
\[ b_1 = \frac{1}{2} [1 \pm (4s_1 - 2)] \]

By the condition of square integrability, the positive sign has to be taken. Hence the value of \( b_1 \) is
\[ b_1 = 2s_1 - \frac{1}{2} \quad (5.2.12) \]

For \( y = 1 \) let
\[ \chi = \frac{b_1'}{y - 1} + a_0' + a_1'(y - 1) + \cdots \quad (5.2.13) \]

Therefore equation (5.2.8) becomes
\[ \left[ -\frac{b_1'}{y - 1} + a_0' + a_1'(y - 1) + \cdots \right]^2 + \left[ -\frac{b_1'}{(y - 1)^2} + a_1' + \cdots \right] + \frac{3}{4} \frac{y^2}{(y^2 - 1)^2} \left[ E + \frac{A}{y^2 - 1} - \frac{B}{y^2 - 1} + Cy^2 - Dy^4 \right] = 0 \]
CHAPTER 5. CALCULATION OF WAVE-FUNCTIONS FOR QES MODELS

\[ + \frac{1}{y^2 - 1} \left[ E + \frac{A}{y^2} - \frac{B}{y^2 - 1} + Cy^2 - Dy^4 \right] = 0 \]

Equating the coefficient of \( \frac{1}{(y-1)^2} \) on both sides gives

\[ b'_1 = \frac{1}{2} \left[ 1 \pm (2s_2 - 1) \right] \tag{5.2.14} \]

By the condition of square integrability, the positive sign has to be taken. Hence the value of \( b'_1 \) is

\[ b'_1 = s_2 \tag{5.2.15} \]

For \( y = -1 \) let

\[ \chi = \frac{b''_1}{y + 1} + a''_0 + a''_1(y + 1) + \cdots \tag{5.2.16} \]

Therefore equation (5.2.8) becomes

\[ \left[ \frac{b''_1}{y + 1} + a''_0 + a''_1(y + 1) + \cdots \right]^2 + \left[ -\frac{b''_1}{y + 1} + a''_1 + \cdots \right] + \frac{3}{4} \left( \frac{y^2}{(y^2 - 1)^2} - 1 \right) - \frac{1}{2 y^2 - 1} \]

Equating the coefficient of \( \frac{1}{(y+1)^2} \) on both sides gives

\[ b''_1 = \frac{1}{2} \left[ 1 \pm (2s_2 - 1) \right] \tag{5.2.17} \]

By the condition of square integrability, the positive sign has to be taken. Hence the value of \( b''_1 \) is

\[ b''_1 = s_2. \tag{5.2.18} \]

For the fixed poles at \( y = 0, \pm 1 \) let \( \chi \) have the form

\[ \chi = b_1 + \frac{b'_1}{y - 1} + \frac{b''_1}{y + 1} + c_1 y + c_2, \tag{5.2.19} \]

where \( c_1 \) and \( c_2 \) are constants to be determined. This form of \( \chi \) is because the equation (5.2.8) has a \( y^2 \) term.
Form of wave-function: Using (5.2.19), equation (5.2.8) transforms to

\[
\left[ \frac{P''(y)}{P(y)} + \frac{b_1}{y} + \frac{b_2}{y-1} + \frac{b_3}{y+1} + c_1y + c_2 \right]^2 + \left[ \frac{P''(y)}{P(y)} - \left( \frac{P'(y)}{P(y)} \right)^2 \right]
\]

\[-\frac{b_1}{y^2} - \frac{b_2}{(y-1)^2} - \frac{b_3}{(y+1)^2} + c_1 + \frac{3}{4} \frac{y^2}{(y^2-1)^2} - \frac{1}{2y^2-1} + \frac{1}{y^2-1} \left[ E + \frac{A}{y^2} - \frac{B}{y^2-1} + Cy^2 - Dy^4 \right] = 0
\]

(5.2.20)

and for large \(y\) equating the coefficients of \(y^2\) to zero gives,

\[c_1^2 - D = 0,\]

hence

\[c_1 = \pm \sqrt{D} = \pm q_1.\]

(5.2.21)

For large \(y\) equating the coefficients of \(y\) gives

\[2c_1c_2 = 0,\]

hence

\[c_2 = 0.\]

(5.2.22)

The correct sign of \(c_1\) is fixed by square integrability and is found to be \(c_1 = -q_1\).

As the potential is symmetric, there are moving poles on either side and hence we take \(P(y)\) to have the form given below

\[P(y) = \prod_{k=1}^{n} (y^2 - y_k^2).\]

(5.2.23)

The wave-function for this model is computed as follows

\[
\psi(y) = \exp \int \left[ \frac{x}{y - \frac{1}{2y^2-1}} \right] dy.
\]

(5.2.24)

\[= \exp \int \left[ \frac{P'(y)}{P(y)} + \frac{b_1}{y} + \frac{b_1'}{y-1} + \frac{b_1''}{y+1} + c_1y \right] dy.
\]

(5.2.25)
On integrating and substituting the values of $b_1, b_2$ and $c_1$ we get the expression for the wave-function in the $x$ variable as

$$\psi(x) = (\cosh^2 x)^{s_1 - \frac{1}{2}} (\sinh^2 x)^{s_2 - \frac{1}{2}} \exp \left(-\frac{q_1}{2} \cosh^2 x\right) \prod_{k=1}^{n} (\cosh^2 x - y_k^2). \quad (5.2.26)$$

Computation of energy-eigenvalue: We shall now show how our analysis leads to the correct answer for energy spectrum. With $c_2 = 0$ and substituting the values of $b_1, b_2$ and $c_1$ (5.2.20) takes the form

$$\frac{P''(y)}{P(y)} + \frac{P'(y)}{P(y)} \left[ \frac{(4s_1 - 1)}{y} + \frac{2s_2}{y - 1} + \frac{2s_2}{y + 1} - 2q_1 y \right]$$

$$+ \left[ q_1^2 y^2 + \frac{4s_1 s_2}{y - 1} - \frac{4s_1 s_2}{y + 1} + \frac{s_2}{y - 1} - \frac{s_2}{y + 1} - \frac{s_2^2}{y - 1} - \frac{s_2^2}{y + 1} - 2s_2 q_1 \frac{y}{y - 1} - 2s_2 q_1 \frac{y}{y + 1} \right]$$

$$- 4s_1 q_1 - \frac{1}{8 y^2 - 1} + \frac{E + A}{y^2 - 1} + \frac{B}{2 y^2 - 1} + \frac{C}{y^2 - 1} - \frac{D}{y^2 - 1} = 0 \quad (5.2.27)$$

Using (5.2.23) in (5.2.27) we get the following equation.

$$\left[ \sum_{k=1}^{n} \frac{2y}{y^2 - y_k^2} \right]^2$$

$$- \sum_{k=1}^{n} \left[ \frac{1}{(y + y_k)^2} + \frac{1}{(y - y_k)^2} \right] + \sum_{k=1}^{n} \frac{2y}{y^2 - y_k^2} \left[ \frac{(4s_1 - 1)}{y} + \frac{2s_2}{y - 1} + \frac{2s_2}{y + 1} - 2q_1 y \right]$$

$$+ \left[ q_1^2 y^2 + \frac{4s_1 s_2}{y - 1} - \frac{4s_1 s_2}{y + 1} + \frac{s_2}{y - 1} - \frac{s_2}{y + 1} - \frac{s_2^2}{y - 1} - \frac{s_2^2}{y + 1} - 2s_2 q_1 \frac{y}{y - 1} - 2s_2 q_1 \frac{y}{y + 1} \right]$$

$$- 4s_1 q_1 - \frac{1}{8 y^2 - 1} + \frac{E + A}{y^2 - 1} + \frac{B}{2 y^2 - 1} + \frac{C}{y^2 - 1} - \frac{D}{y^2 - 1} = 0 \quad (5.2.28)$$

Multiplying the above throughout by $l^2$ and integrating along a closed contour enclosing $y = 0$ we get the expression for energy as

$$E = -4 \left[ s_1 + s - 2 - \frac{1}{2} \right]^2 - 8s_1 \left[ \frac{q_1}{2} + \sum_{k=1}^{n} \frac{1}{y_k^2} \right] \quad (5.2.29)$$

Using $y'i = \xi_k$ the above equations for energy become
CHAPTER 5. CALCULATION OF WAVE-FUNCTIONS FOR QES MODELS 73

The results (5.2.26), (5.2.30) and (5.2.32) and in agreement with those given in [6].

The general feature of the zeros of the wave-function for the sextic oscillator are also true for the QES hyperbolic potential. In particular, for a given potential it is correct that all the exactly solvable wave-functions have the same total number, (real and complex) of zeros. This feature is found to be correct for all QES model studied so far including the QES periodic potentials [12].

5.3 Concluding Remarks

Our study of bound state wave-functions in this chapter shows the following similarities and differences between the exactly solvable and QES models.

1. In both the models, the "QMF" turns out to be a rational function after a suitable change of variables.
2. In both the cases, the integer \( n \) in the right hand side of quantization condition coincides with the number of moving poles.

3. For every bound state in one dimension the \( k^{th} \) excited state wave-function have \( k \)-nodes on the real axis. This statement is a general one and is true for all models including exactly solvable and QES potentials. The study in chapter 3 shows that QMF for exactly-solvable models has moving poles which are in correspondence with the nodes of the wave-function. There are no poles off the real axis. However this property fails to be true for QES potentials where the QMF has poles off the real axis, in addition to the poles on the real axis corresponding to the nodes of the wave-function.

4. For the QES potentials only a part of the energy spectrum and the corresponding wave-functions can be computed exactly. An interesting property of the QMF for all these levels is that the total number of moving poles is the same and equal to the integer \( n \) of the quantization.

5. Different values of integer \( n \) correspond to different QES potentials within a family, and it does not refer to different excited state of a single potential, as was the case for exact-solvable model.