Appendix

In this appendix, we briefly describe some properties of the three Jacobi elliptic functions and two of the complete elliptic integrals, having relevance to the works presented in this thesis.

\[
\begin{align*}
\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2)}} &= \text{cn}^{-1}(y, \kappa) \\
\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2)}} &= \text{dn}^{-1}(y, \kappa) \\
\int_0^\infty \frac{dt}{\sqrt{(1-t^2)(1-k^2)t^2}} &= \text{sn}^{-1}(y, \kappa),
\end{align*}
\]

where the parameter \(k^2\) is referred to as the elliptic modulus and the parameter \(k'^2\), defined by \(k'^2 = 1 - k^2\), is referred to as the complementarity elliptic modulus. Here, the elliptic modulus is restricted to the range, \(0 < \kappa < 1\). \(\text{cn}(x, \kappa)\) and \(\text{sn}(x, \kappa)\) functions have real period \(4K(\kappa)\), whereas \(\text{dn}(x, \kappa)\) has the real period \(2K(\kappa)\). Here, \(K(\kappa)\) is the complete elliptic integral of the first kind given by

\[
K(\kappa) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2 \sin^2(\theta)}} d\theta. \tag{6.4}
\]

The complete elliptic integral of the second kind, \(E(\kappa)\), involving squares of the Jacobi elliptic functions, is defined by

\[
E(\kappa) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2(\theta)} d\theta = \int_0^{K(\kappa)} \text{dn}^2 du. \tag{6.5}
\]

Limiting values of \(K(\kappa)\) and \(E(\kappa)\) are listed in the following table.
The limiting values of the three elementary elliptic functions are given below.

<table>
<thead>
<tr>
<th>function</th>
<th>$\kappa = 0$</th>
<th>$\kappa = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K(\kappa)$</td>
<td>$\pi/2$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$E(\kappa)$</td>
<td>$\pi/2$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

The Jacobi elliptic functions satisfy the algebraic relations

\[
\begin{align*}
sn^2(x, \kappa) + cn^2(x, \kappa) &= 1, \\
\kappa^2 sn^2(x, \kappa) + dn^2(x, \kappa) &= 1, \\
dn^2(x, \kappa) - \kappa^2 cn^2(x, \kappa) &= \kappa^2, \\
\kappa^2 sn^2(x, \kappa) + cn^2(x, \kappa) &= dn^2(x, \kappa).
\end{align*}
\]

The derivatives of the Jacobi elliptic functions are given by

\[
\begin{align*}
\frac{\partial}{\partial x} cn(x, \kappa) &= -sn(x, \kappa)dn(x, \kappa), \\
\frac{\partial}{\partial x} dn(x, \kappa) &= -\kappa^2 cn(x, \kappa)sn(x, \kappa), \\
\frac{\partial}{\partial x} sn(x, \kappa) &= cn(x, \kappa)dn(x, \kappa).
\end{align*}
\]