CONSTRUCTION OF ROTATABLE DESIGNS FROM FACTORIAL DESIGNS

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Received: June, 1960

1. INTRODUCTION

Rotatable designs were introduced by Box and Hunter (1954). For the construction of these designs they used geometrical configurations. They obtained several second order rotatable designs by using the properties of regular configurations. Afterwards, Gardiner and others (1959) obtained through the same technique some third order designs for two and three factors and one design in four factors. They observed that through their technique designs with larger number of factors may be possible, but they will require very large number of points. Bose and Draper (1959) obtained some second order designs in three factors by using a different technique.

In the present paper second and third order rotatable designs with up to 8 factors have been obtained from factorial designs. The number of points in these designs including the third order designs are reasonably small.

2. ROTATABLE DESIGNS

Let there be k factors or variates each at s levels. If a design be formed with N of the s^k treatment combinations, it can be written in the following (N x k) matrix which we shall hereafter call the design matrix, D:
A variate $x_t$ has been associated with the $t$-th variate such that the entries in the $t$-th column of $D$ below the variate are its values. The treatment combinations will sometimes also be referred to as the points of the design.

A design of the above form will be a rotatable design of order $d$ if a polynomial response surface of degree $d$ of the response, $y$ as obtained from the treatments on the variates $x_t (i = 1, 2, \ldots k)$ can be so fitted that the variance of the estimated response from any treatment is a function of the sum of squares of the levels of the factors in that treatment combination. In other words, the variance of the estimated response at any point is a function of the square of the distance of the point from some suitable origin so that the variance of all responses at points equidistant from the origin is the same. When the response surface is of the second degree, that is, $d = 2$, such constancy of variance will be possible if the design points are so selected that the following relations hold:

**Relation A:**

$$
\begin{align*}
\Sigma x_t &= 0, \\
\Sigma x_t x_j &= 0, \\
\Sigma x_t x_i x_j &= 0, \\
\Sigma x_t x_i x_j^2 &= 0, \\
\Sigma x_t x_i x_j x_k &= 0, \\
\Sigma x_t x_i x_j x_k x_l &= 0, \\
\Sigma x_t x_i x_j x_k x_l x_m &= 0.
\end{align*}
$$

**Relation B:**

(i) $\Sigma x_t^4 = \text{constant} = N\lambda_2$

(ii) $\Sigma x_t^6 = \text{constant} = 3N\lambda_4$

**Relation C:**

$\Sigma x_t^4 x_j^4 = \text{constant for all pairs of } i \text{ and } j (i \neq j)$. 

\[D = \begin{bmatrix}
X_1 & X_2 & \cdots & X_k \\
X_{11} & X_{21} & \cdots & X_{k1} \\
X_{12} & X_{22} & \cdots & X_{k2} \\
\vdots & \vdots & \ddots & \vdots \\
X_{1n} & X_{2n} & \cdots & X_{kn}
\end{bmatrix}\]
Relation D:
\[ \sum x_i = 3 \sum x_i x_j x_k \] for all \( i \) and \( j \).

Relation E:
\[ \frac{\lambda_i}{\lambda_2} > \frac{k}{k + 2} \]

In the case of the third order rotatable designs the following further relations need also be satisfied:

Relation A:
Each of the sum of powers or each of the products of powers of \( x_i \)'s, in which at least one power is odd, is zero.

Relation B:
\[ \sum x_i = \text{constant} = 15N\lambda_6. \]

Relation C:
(i) \( \sum x_i x_j x_k = \text{constant} \).
(ii) \( \sum x_i x_j x_k x_l = \text{constant} \) \( (i \neq j \neq k) \).

Relation D:
(i) \( \sum x_i = 5 \sum x_i x_j x_k \}
(ii) \( \sum x_i x_j x_k x_l = 3 \sum x_i x_j x_k x_l \) for all \( i, j \) and \( k \) \( (i \neq j \neq k) \).

Relation E:
\[ \frac{\lambda_i}{\lambda_2} > \frac{k}{k + 4} \]

The problem of construction of rotatable designs consists not only in selecting a set of \( N \) treatment combinations out of \( 2^p \) combinations, but also in spacing properly the levels of the factors with a suitable origin on a suitably chosen scale such that the relations specified are satisfied. In ordinary factorial experiments the problem of spacing the levels does not arise. Hence for such designs the relative magnitudes of the different levels remain unspecified.

3. A Modified Form of Factorial Treatments

A factorial experiment involving three factors each at two levels is usually denoted as \( 2^3 \) and the levels as 0 and 1. Instead we may denote the levels as \( x (p) \) and \( x (q) \) where \( x (p) \) like \( x (q) \) means that \( p \)
is to be associated with $x$ according to certain rule of association. We may call $x$ as magnitude and $p$ and $q$ the associates of the magnitude. If $p = 1$ and $q = -1$ and the rule of association is multiplication, we get the levels as $x$ and $-x$ out of the magnitude $x$. From this angle we may denote this factorial design as $(1 \times 2)^2$ where 1 stands for the magnitude $x$ and 2, the two associates and the product sign, the rule of association. In general we may say that the design $(m_1 \times m_2)^s$ have $m_1 \times m_2$ levels formed of $m_1$ magnitudes and $m_2$ associates. The $(m_1 \times m_2)^s$ treatments can be obtained by 'multiplying' the $m_1^s$ magnitude combinations with $m_2^s$ associate combinations. For example, when $m_1 = 2$ and $m_2 = 2$, if the magnitudes are $a$ and $\beta$ and the associates 1 and $-1$, the magnitude combinations are $aa$, $a\beta$, $\beta a$ and $\beta \beta$ and the associate combinations, $11$, $1-1$, $-11$, $-1 -1$.

The 16 treatments can now be obtained by multiplying each of the magnitude combinations with the associate combinations. The way of multiplication is illustrated below: When a magnitude combination say $a\beta$ is multiplied with the 4 associate combinations we get $a\beta$, $a-\beta$, $-a\beta$ and $-a -\beta$. If we call the contents of any combination of $k$ factors, whether of magnitudes or associates, as its elements, the multiplication of a magnitude combination with an associate combination generates a treatment which consists of $k$ levels being the products of corresponding elements in the two combinations.

Designs with $m_2 = 1$, reduce to the ordinary factorial designs with $m_1$ levels. If some of the magnitudes as also the associates be zero, all the treatments obtained in this way will not be distinct. But for our purpose we shall use only distinct treatments unless otherwise mentioned.

4. **Rotatable Designs as Fractional Replicates of the Modified Factorial Designs**

For the purpose of construction of rotatable designs from the modified factorial designs the associates should always be numbers which are deviates from their mean so that their sum is zero. As will be evident later, for reducing the size of the design, the number of associates should be as small as possible, and hence 1 and $-1$ are the most suitable associates for this purpose.

The modified factorial design corresponding to any factorial design, $s^s$ can be written as $(m_1 \times 2)^s$ where $m_1 = s/2$ or $(s + 1) \times 2$ according as $s$ is even or odd. Such a design is always available for any $s$, provided 0 be taken as one of the magnitudes when $s$ is odd.
Having given one magnitude combination and \( n \) associate combinations of the design \((m_1 \times m_2)^n\), we have seen that \( n \) treatments can be obtained from their multiplication. This method of obtaining treatments from a magnitude combination will be referred to as multiplication.

If any magnitude combination of the design \((m_1 \times 2)^p\) contains \( p \) non-zero magnitudes, the number of distinct treatments obtainable from it by multiplication with the \( 2^p \) associate combinations will be only \( 2^p \) and not \( 2^s \).

If \( n \) combinations of the \( 2^p \) associate combination, with the associates as 1 and \(-1\) be taken to form the design matrix \( D \), and the factors be denoted as \( A_1, A_2, \ldots, A_k \) such that the variate \( x_i \) corresponds to \( A_i \), then \( \sum x_i \) will be zero if the main effect \( A_i \) is not confounded in \( D \), as the values of \( x_i \) in it are \(+1\) and \(-1\) in equal numbers. Similarly \( \sum x_i x_j \) will be zero if the interaction \( A_i A_j \) is not confounded in the \( n \) combinations. If each of the values of the variate \( x_i \) in the design be raised to \( r \)-th power, \( \sum x_i^r \) will remain equal to \( \sum x_i \), if \( r \) is odd and every entry in the column will be unity if \( r \) is even. Thus, if in the design, \( \sum x_i = 0 \), then \( \sum x_i^r \) is also zero when \( r \) is odd. Again as \( \sum x_i^r x_j \) is equal to \( \sum x_j \), the former will vanish if \( \sum x_i = 0 \). In terms of interaction we may express this fact by making the convention that \( A_i^r = 1 \) or \( A_i \), according as \( r \) is even or odd and then equating any sum of products, say, \( \sum x_i x_j x_k x_m \) with the interaction \( A_i A_j A_k A_m \) which is equivalent to \( A_i/A_m \). Thus, if the interaction \( A_i A_m \) be unconfounded in the treatments, \( \sum x_i x_j x_k x_m \) and all other sums of products corresponding to the interaction \( A_i A_m \) will be zero. If in these \( n \) associate combinations none of the main effects and interactions with less than 5 factors be confounded, they will satisfy all relations \( A \) which are necessary for second order rotatable designs. A little thought shows that all these relations will be satisfied by the treatments which are obtained by the multiplication of any one or more magnitude combinations with the \( n \) associate combinations in which no main effect or interaction with less than 5 factors are confounded. If in these \( n \) associate combinations no interaction with less than 7 factors be confounded, relations \( A \) also will be satisfied. A set of \( n \) associate combinations in which no interaction with less than 5 factors in the case of second order sign and 7 factors for third under designs is confounded, will be called an unaffected set of combinations. Any magnitude combination containing only one magnitude that is of the form \( aa \cdots a \) will be called a homogeneous set. Any homogeneous set multiplied with \( n \) unaffected
associate combinations will produce treatments which will satisfy all relations $A$, $A_1$, $B$, $B_1$ and $C$, $C_1$.

Having given the number of magnitudes for a design, if a magnitude combination contains more than one magnitude, we may obtain other magnitude combinations out of it by cyclically changing over the magnitudes. Thus if there be 4 magnitudes $a$, $b$, $c$, $d$ and a combination $a'b'c'd'$ be taken, we shall obtain out of $a'b'c'd'$ by cyclically changing over magnitudes three other combinations, viz., $b'c'd'a'$, $c'd'a'b'$ and $d'a'b'c'$. The magnitude combination, with which we start, will be called the initial set and the process by means of which other sets are generated out of it, as detailed above, will be called rotation over the magnitudes.

Given any initial set we shall obtain out of it a group of treatments by rotation and multiplication which will satisfy all relations $A$, $A_1$; $B$, $B_1$ and partly $C$, $C_1$.

If there be only two factors, relations $C$ and $C_1$ do not appear. In the case of three factors each with three magnitudes the initial set containing all of them generate treatments by rotation and multiplication which will satisfy all relations $C$ but will produce one equation from relations in $C_1$.

Just like the homogeneous set $a \cdots a$, the set $a'b'c' \cdots$ subjected to rotation and multiplication will generate treatments which will satisfy all relations $A$, $A_1$; $B$, $B_1$ and $C$, $C_1$. The set $a00 \cdots 0$ is a particular case of $a'b'c' \cdots$ and hence this set also satisfies all these relations. The treatment combination $0, 0, \cdots 0$ called the central point satisfies all relations excepting $E$ and $E_1$.

Relation $D$ can be satisfied in some cases by proper choice of several sets as will be illustrated afterwards. In general, relations $D$, $D_1$, $C$, $C_1$ are utilised to obtain a set of equations involving the magnitudes as unknowns, a solution of which gives the proper spacing of the magnitudes so as to satisfy relations $D$, $D_1$, $C$ and $C_1$.

If there be $k$ factors and the treatments are generated as described earlier the number of equations obtainable from the treatment for satisfying relations $C$ is $(k-2)/2$ when $k$ is even or the number just less, if $k$ is odd. The number of equations from relations $C_1$ is $k-2$. Thus, when the number of factors is more, larger number of magnitudes are necessary for getting designs excepting when sets like $a \cdots a$ and $a'b'c' \cdots$ only are used. Though second order designs are obtainable by increasing the number of magnitudes and by satisfying relations $C$ with their proper choice, in the case of third order designs
increase in magnitude is no easy remedy, particularly when there are
four or more factors. In such cases unless all relations \( C \) and \( C_1 \) are
satisfied by proper choice of sets, it may not always be possible to get
designs. One method of generating sets out of any given set so that
the treatments obtained from all these sets by multiplication, satisfy
all relations \( A, A_1, B, B_1, C \) and \( C_1 \), has been described below:

Let there be \( k \) factors and let us take a set in which there are
\( s \) (\(< k\)) distinct magnitudes. If \( s = k \), the total number of ways
in which these magnitudes can be allotted to the \( k \) factors is evidently
\( \underline{k} \). If we form a set out of each allotment, the treatments obtained
from these sets through multiplication will satisfy all these relations.
Again if \( s < k \), that is, when there are fewer magnitudes than factors,
the total number of ways of allotment of them to the factors is evidently

\[
\frac{\underline{k}}{\underline{r_1} \underline{r_2} \cdots \underline{r_s}}
\]

where \( r_1 + r_2 + \cdots + r_s = k \) and \( r_i \) denotes the number of times the
\( i \)-th magnitude occurs in the starting set. For example, if there be 5
factors and two magnitudes and the starting set is taken as \( aaBBB \),
the total number of allotments, and hence sets, is \( \underline{5}((\underline{2} \underline{3})) \), as
here \( r_1 = 2 \) and \( r_2 = 3 \). It will be found that this procedure
requires too many sets and hence so many points in the designs
obtained through them. One remedy is to take 0 as one of the
magnitudes and repeat it \((k - 2)\) times together with one more
magnitude repeated twice. In this case there will be \( k(k - 1) \) sets
giving \( 2k(k - 1) \) points in all. This procedure of obtaining sets will
be called permutation. As an example, when \( k = 5 \), permutation
of the set \( aa000 \) gives the following 10 sets.

\[
\begin{array}{cccc}
aa000 & 0aa00 & 00aa0 & 000aa \\
an000 & a00a0 & 0a00a & 00a0a \\
a00a0 & 0a00a & 000a0 & \\
a000a & & & \\
\end{array}
\]

It is known that relations \( E \) and \( E_1 \) cannot be satisfied when a
design is obtained out of one set only through the processes; as in
this case all points will have the same distance from the origin. In
such situations at least one central point \((0, 0, \cdots, 0)\) has to be taken
together with the others to satisfy \( E \), but not \( E_1 \).
It will be evident from the previous considerations of generating the design points that for keeping the size of a design low, the starting sets should be homogeneous and those non-homogeneous should contain smaller number of non-zero magnitudes together with zero. In the next section we shall discuss some of the second order designs in detail while for other designs only the starting sets and the solutions for the magnitudes will be indicated. The second order designs have been presented according to number of levels as in ordinary factorial designs.

5. Designs with Three Levels

The factorial design used for the construction of rotatable designs with 3 levels is \((2 \times 2)^3\) where the magnitudes are 0 and \(a\) and the associates 1 and \(-1\). The magnitude combinations for two factors are therefore \(aa, a0, 0a\) and \(00\) and the associate combinations, 1 1, 1 —1, —1 1 and —1 —1. By taking the starting set \(aa\), the treatments generated by the multiplication of it with all the associate combinations cannot satisfy relations \(D\). As there is only one magnitude \(a\) which has to be fixed to make \(\sum x^4 = N\), relation \(D\) has to be satisfied by suitable choice of the magnitude combinations. It can be seen easily that the treatments obtained from the sets (i) \(aa\) and (ii) \(a0\), when the second set is repeated 4 times, by rotation and multiplication, satisfy all the relations \(A, B, C, D\) and \(E\). If necessary the design can be augmented by adding some central points, \(00\). The value of \(a\) has to be obtained from \(\sum x^4 = N\) where \(N\) stands for the total number of design points including the central points. This design will then have at least 20 points, as shown below:

\[
\begin{array}{cccccc}
\text{aa} & \text{a0} & \text{a0} & \text{a0} & \text{a0} \\
\text{a—a} & \text{—a0} & \text{—a0} & \text{—a0} & \text{—a0} \\
\text{—aa} & \text{0a} & \text{0a} & \text{0a} & \text{0a} \\
\text{—a—a} & 0—a & 0—a & 0—a & 0—a \\
\end{array}
\]

In the general case of \(k\) factors a second order rotatable design can always be obtained by generating treatments through the two processes from the two sets (i) \(aa \cdots a\) and (ii) \(a0 \cdots 0\) when the second set is repeated \(n\) times where \(n\) denotes, as stated earlier, the minimum number of associate combinations in which no interaction with less than 5 factors is confounded. The value of \(a\) is given by \(a = \sqrt{N/3n}\).

The number of points in the design will be \((2k + 1) n\) excluding the central points which may or may not be added. It will be seen
that the number of points required for such designs is very large. For example, when \( k = 3 \), 56 points are required. Thus, when there is no compelling reasons for adopting designs with three levels, designs with larger number of levels may be adopted to bring down the size of the design.

When \( k = 3 \), there is another design obtainable from the sets (i) \( a = 0 \) and (ii) \( a = 0 \), the second set being repeated twice. This design is small in size as it requires only 24 points. The value of \( a \) in this design is given by \( \sqrt{N/12} \).

6. DESIGNS WITH 4 LEVELS

The factorial design for obtaining such rotatable designs is again \((2 \times 2)^b\) with the magnitude \( a \) and \( \beta \) and associates 1 and \(-1\). The only possible type of starting set in this case is \( a a \beta \cdots \beta \) where \( a \) is repeated \( r_1 \) times and \( \beta \), \( r_2 \) times. As the magnitudes \( a \) and \( \beta \) are to be chosen so as to satisfy the relations \( \Sigma x_i^2 = N \) and \( \Sigma x_i \beta = 3 \Sigma x_i x_i \beta \), all relations \( C \) must be satisfied by proper choice of sets. We know that the set \( a \beta \cdots \beta \) gives treatments through the two processes of rotation and multiplication which satisfy all relations, \( C \). Hence for constructing the design with \( k \) factors we may take the set \( a \beta \cdots \beta \) and generate the treatments through the processes of rotation and multiplication. Relation \( D \) gives the equation:

\[
N \{a^4 + (k - 1) \beta^4\} = 3n \{2a^2\beta^2 + (k - 2) \beta^4\}.
\]

Putting

\[
s = \frac{a^n}{\beta^r},
\]

the equation reduces to

\[
s^2 + k - 1 = 6s + 3(k - 2)
\]

whence

\[
s = 3 \pm \sqrt{4 + 2k}.
\]

When \( k = 2 \), there will thus be two designs for two values of \( s \), in other cases there will be only one design, as negative value of \( s \) cannot provide any real solution for \( a \) and \( \beta \). There will be \( kn \) points in the design with at least one central point for satisfying relation \( E \).

From relation \( B \) (1) the value of \( \beta \) comes as

\[
\beta^2 = \frac{N}{(s + k - 1) n}
\]
where \( n \) is the number of associate combinations taken and \( N \), the total numbers of design points including the central points.

From consideration of geometrical configuration this series of designs was obtained by Box and Hunter (1954) also.

7. Designs with 5 Levels

The factorial design \((3 \times 2)^n\) where the magnitudes are 0, \( \alpha \), \( \beta \) and the associates, 1 and -1, will be used to get the design with 5 levels.

The sets (i) \((\alpha \alpha \cdots \alpha)\) and (ii) \((\beta 0 \cdots 0)\) produce designs for all values of \( k \) through the two operations of rotation and multiplication. The number of points in the design is \( n + 2k \).

From relation \( D \) we get the equation:

\[
na^4 \times 2\beta^4 = 3na^4
\]

which gives \( s = 1/\sqrt{n} \) where \( s = a^2/\beta^2 \) and \( n \) is the number of associate combinations used for multiplication. In this case we need not add any central point to satisfy relation \( E \) excepting when \( k = 2 \) and 4 as in these cases the distance of all the design points from the centre becomes the same. The value of \( \beta \) can be obtained from

\[
\beta^2 = \frac{N}{ns + 2}
\]

These designs are the central composite rotatable designs. One more series of designs like the central composite designs is available when there are 5 magnitudes. This series is obtainable from the sets (i) \((\alpha \alpha \cdots \alpha)\) and (ii) \((\beta 0 \cdots 0)\) when the second set is permuted to give \( k(k-1)/2 \) sets. The number of treatments in the design will be \( n + 2k(k-1) \) where \( n \) is as before the number of associate combinations used for multiplication. Relation \( D \) gives the equation,

\[
na^4 + 4(k-1)\beta^4 = 3na^4 + 12\beta^4
\]

i.e.,

\[
\beta^2 = \frac{a^2}{\beta^2} = \frac{2(k-4)}{n}
\]

When \( k = 4 \), \( a \) is 0 and hence the design is available from the second set having 24 points together with one central point. The value of \( \beta \) can be obtained from

\[
\beta^2 = \frac{N}{ns + 4(k-1)}
\]
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In addition some other designs with 5 levels are also available. For example, a design can be obtained from the set \(0a\beta\). Relation \(D\) gives the equation, \(4(a^4 + \beta^4) = 12a^2\beta^2\).

Taking
\[
\frac{s}{\beta^2} = a^2
\]
it becomes
\[
s^2 + 1 = 3s
\]
and hence
\[
s = \frac{3 \pm \sqrt{5}}{2}
\]

As there are two values of \(s\), two designs are possible. The value of \(\beta\) is obtainable from \(\beta^2 = N/4(s + 1)\), where \(N\) includes at least one central point.

When \(k = 4\), designs with one set in which all the magnitudes are present but one of them is repeated, cannot be obtained by rotation and multiplication as relations \(C\) cannot be satisfied with such sets. It appears the same is also true when \(k > 4\).

8. DESIGNS WITH 6 LEVELS

Let the magnitudes be \(a\), \(\beta\) and \(\gamma\) and the associates, \(1\) and \(-1\). When there are two factors, a design can be obtained from the sets \(a\beta\), \(\beta\gamma\) and \(\gamma a\) by multiplying each one of them with the four associate combinations. This design will have 12 points. From relations \(D\) we get the equation
\[
a^4 + \beta^4 + \gamma^4 = 3(a^2\beta^2 + \beta^2\gamma^2 + \gamma^2a^2).
\]

Putting
\[
s = \frac{a^2}{\beta^2} \text{ and } t = \frac{\beta^2}{\gamma^2},
\]
it reduces to
\[
s^2 + t^2 + 1 = 3(st + s + t).
\]
As there are two unknowns in one equation, one of them can be taken arbitrarily so that the solution for the others is real and positive. The value of \(\gamma\) can be obtained from
\[
4\gamma^2(s + t + 1) = N.
\]
In case of three factors a design with the initial set \( ab\gamma \) can be obtained by rotation and multiplication with all the 8 associate combinations. There will be 24 points in the design with at least one central point. The equation from relation \( D \) is the same as in the case of the two-factor design. Hence the same solution of \( s \) and \( t \) will give the design but the value of \( \gamma \) will be different, \( \text{viz.}, \quad 8\gamma^4(s + t + 1) = N. \) This design was also obtained by Bose and Draper (1959). When \( k = 4 \), a design obtained from the set \( ab\gamma \) will have one equation from relation \( C \) and one from \( D \). As there are three unknowns and three equations in all it may be possible to get such a design. Equation from relation \( C \) comes out as

\[
\gamma^4 + a^2\beta^2 + \beta^2\gamma^2 + \gamma^2a^2 = 2(a^2\beta^2 + \beta^2\gamma^2)
\]

i.e.,

\[
(\gamma^2 - a^2) (\gamma^2 - \beta^2) = 0.
\]

This shows that relation \( C \) cannot be satisfied when \( a, \beta \) and \( \gamma \) are different. Hence no design with \( k = 4 \) and 6 magnitudes is possible through rotation and multiplication, though it may be possible by permutation from one set. It appears that in the general case of \( k \) factors no design with one set is possible likewise as relations \( C \) cannot be satisfied unless two of the magnitudes are equal.

9. DESIGNS WITH 7 LEVELS

Let the magnitude be \( a, \beta, \gamma \) and \( \delta \) and the associates 1 and \(-1\). When the number of factors is less than 4, designs with 7 or more levels have no special advantage over those obtainable from smaller number of levels, rather they suffer from the defect of requiring too many points.

(i) When \( k = 4 \):

We could not get any design with 6 levels when \( k \) is greater than three. With 7 levels if we take the set \( 0\alpha\beta\gamma \) a design can be obtained by developing the set through rotation and multiplication. There will be 32 points with at least one central point to satisfy relation \( E \).

Relations \( C \) and \( D \) give the equations

(i) \( a^2\gamma^2 + \beta^2\gamma^2 = 2a^2\beta^2 \).

(ii) \( a^4 + \beta^4 + \gamma^4 = 6a^2\beta^2 \).

Putting

\[
s = \frac{a^2}{\gamma^2} \quad \text{and} \quad t = \frac{\beta^2}{\gamma^2},
\]
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When there are six factors or more relations $C$ will give 2 or more equations. Unless it is possible to satisfy all the relations $C$ excepting one so as to get only one equation out of it, there will be more equations than unknowns. As such no design in general with 7 level seems...
10. DESIGNS WITH 6, 7 AND 8 FACTORS

(i) \( k = 6 \).—A design for 6 factors can be obtained by taking the set \( 000000 \) and developing it through rotation and multiplication. Relations \( C \) and \( D \) give the following equations:

\[
(i) \ a^2\beta^2 = \beta^2\beta^2 = 2a^2\beta^2, \\
(ii) \ a^4 + \beta^4 + \beta^4 = 6a^2\beta^2.
\]

Putting
\[
s = \frac{a^2}{\beta^2}, \quad t = \frac{\beta^2}{\beta^2}
\]
they become
\[
st = t = 2s
\]
and
\[
s^2 + t^2 + 1 = 6s.
\]
The solutions are \( s = 1, t = 2 \).

This design has 48 points and at least one central point is necessary.

(ii) \( k = 7 \).—A design for 7 factors can be obtained by taking the set \( 0000000 \) and developing it through rotation and multiplication. Relations \( C \) and \( D \) give the following equations:

\[
(i) \ a^2\beta^2 = \beta^2\beta^2 = a^2\beta^2, \\
(ii) \ a^4 + \beta^4 + \beta^4 = 3a^2\beta^2.
\]

Putting
\[
s = \frac{a^2}{\beta^2}, \quad t = \frac{\beta^2}{\beta^2}
\]
they become
\[
st = t = s
\]
and
\[
s^2 + t^2 + 1 = 3s.
\]
The solutions are \( s = 1, t = 1 \).

This design has 56 points and at least one central point is necessary.
(iii) $k = 8$—For a design in 8 factors take the set $000\beta\gamma0\omega$ and develop it through rotation and multiplication. Relations $C$ and $D$ give the following equations:

(i) $a^2\beta^2 + \beta^2\gamma^2 = a^2\gamma^2 + \gamma^2\omega^2 = \beta^2\omega^2 = 2a^2\omega^2$.

(ii) $a^4 + \beta^4 + \gamma^4 + \omega^4 = 6a^2\omega^2$.

Putting

$$\frac{a^2}{\omega^2} = s, \quad \frac{\beta^2}{\omega^2} = t, \quad \frac{\gamma^2}{\omega^2} = u$$

they become

$$st + tu = su + u = t = 2s$$

and

$$s^2 + t^2 + u^2 + 1 = 6s.$$  

As there are more independent equations than unknowns, we take one more set $(2\sqrt{p\omega})000000000$ involving another unknown $p$, together with the previous set. The relations $C$ and $D$ now give the equations:

$$st + tu = su + u = t = 2s$$

and

$$s^2 + t^2 + u^2 + 1 + 2p^2 = 6s.$$  

The solutions are

$$s = (\sqrt{2} - 1)$$
$$t = 2(\sqrt{2} - 1)$$
$$u = \sqrt{2}.(\sqrt{2} - 1)$$
$$p = 0.332.$$  

This design has 144 points and no central point is necessary.

11. Third Order Rotatable Designs in Two Factors

Gardiner et al. (1959) have constructed a series of 2 factor designs by taking equidistant points on two concentric circles. We could get a design with three magnitudes which do not come out as any particular case of their series. This design has 16 points and are obtainable from the three sets (1) $a\beta$, (2) $a\omega$ and (3) $\gamma0$ by developing them through rotation and multiplication. Relations $D$ and $D_1$ give the equations:

$$8a^4 + 4\beta^4 + 2\gamma^4 = 24a^2\beta^2 + 12a^2.$$
Putting $s = \alpha^2 / \beta^2$ and $t = \gamma^2 / \beta^2$, they become

\[ 2t^2 + 4 = 24s + 4s^2 \]

and

\[ 2t^2 + 4 = 20(s^2 + s) + 12s^2, \]

i.e.,

\[ t^2 = 2s^2 + 12s - 2 \]

and

\[ t = 6s^3 + 10(s^2 + s) - 2. \]

Solving these equations:

\[ s = 2.5538 \]

and

\[ t = 1.09320. \]

The value of $\beta$ can be obtained from

\[ \beta^2(8s + 2t + 4) = N. \]

As the points are not all equidistant from the centre, conditions $E$ and $E_1$ are satisfied.

(ii) Designs with four magnitudes can be obtained from the sets

(i) $a\beta$ and (ii) $\gamma\delta$. The number of points in the design will be again 16.

Relations $C$ and $C_1$ do not appear when $k = 2$.

Relations $D$ and $D_1$ give the equations

\[ 4(a^4 + \beta^4 + \gamma^4 + \delta^4) = 24(a^2\beta^2 + \gamma^2\delta^2), \]

\[ 4(a^4 + \beta^4 + \gamma^4 + \delta^4) = 20(a^2\beta^2 + \gamma^2\delta^2 + \gamma^2\delta^2 + \gamma^2\delta^2). \]

Putting

\[ s = \alpha^2 / \delta^2, \quad t = \beta^2 / \delta^2 \quad \text{and} \quad u = \gamma^2 / \delta^2. \]
we get

\[ s^2 + t^2 + u^2 + 1 = 6(st + u) \]

\[ s^3 + t^3 + u^3 + 1 = 5(st^2 + s^2t + u^2 + u) \]

As there are two equations and three unknowns, one of them can be chosen arbitrarily.

By putting \( t/s = u \), we find the equations become:

\[ (s^3 + 1)(u^2 - 6u + 1) = 0 \]

and

\[ (s^3 + 1)(u + 1)(u^2 - 6u + 1) = 0. \]

Thus, if \( u \) be so chosen that

\[ u^2 - 6u + 1 = 0, \]

i.e.,

\[ u = 3 \pm \sqrt{8} \]

the relations are satisfied whatever \( s \) may be. Thus a series of design is available from the sets:

(i) \( (\sqrt{s\delta}), (\sqrt{s\delta}) \)

(ii) \( (\sqrt{u\delta}), (\delta) \)

where \( u = 3 \pm \sqrt{8} \) and \( s \) is arbitrary. \( \delta \) can be fixed from the relation \( 4\delta^2(s + 1)(u + 1) = N \). If \( s = 1 \), relations \( E \) and \( E_1 \) will not be satisfied while for all other values of \( s \), they will be satisfied.

12. Third Order Rotatable Designs for Three and More Factors

It has been seen that with the help of the sets (i) \( aa \cdots a \) and (ii) \( 000 \cdots 0 \) second order central composite rotatable designs can be obtained. But no third order designs can be obtained with them as relations \( D_1 \) cannot be satisfied. But if one more set, \( viz., \gamma 00 \cdots 0 \) permuted to give \( k(k - 1)/2 \), sets of four points each be taken together with the above, designs are available when one more set out of the above three, which may have different magnitudes, is also taken.

The equations in the general case of \( k \) factors obtainable from the above three sets are:

\[ ma^4 + 2\beta^4 + 4(k - 1)\gamma^4 = 3ma^4 + 12\gamma^4 \]

\[ 4\gamma^6 + ma^6 = 3ma^6 \]

\[ ma^6 + 2\beta^6 + 4(k - 1)\gamma^6 = 5ma^6 + 20\gamma^6. \]
where \( m \) denotes the number of associate combinations in which no interactions with less than 7 factors are confounded.

Putting \( \frac{a^2}{\gamma_S} = s \), \( \frac{\beta^2}{\gamma_S} = t \),
these become

\[
\begin{align*}
t^2 &= ms^2 + 2 (4 - k) \\
s^2 &= \frac{2}{m} \\
t^3 &= 2ms^3 + 2 (6 - k) = 4 + 2 (6 - k).
\end{align*}
\]

As there are three equations and two unknowns, one more magnitude is to be taken to obtain their solution. There will thus be the following three cases according to the nature of the added set:

**Case 1.**—If a set \( \theta \theta \cdots \theta \), i.e., of the form \( \theta \theta \cdots \theta \) be taken and \( \delta^2/\gamma^2 = u \), the equation for this design based on the four sets will be

\[
\begin{align*}
t^2 + u^2 &= ms^2 + 2 (4 - k) \\
s^3 &= \frac{2}{m} \\
t^3 &= 4 + 2 (6 - k).
\end{align*}
\]

**Case 2.**—If again, the added set be \( \omega \omega \cdots \omega \) being of the form \( \omega \omega \cdots \omega \) and \( \omega^2/\gamma^2 = v \), the equations will be:

\[
\begin{align*}
m (s^2 + u^2) &= i^2 - 2 (4 - k) \\
s^3 + v^3 &= \frac{2}{m} \\
t^3 &= 4 + 2 (6 - k).
\end{align*}
\]

**Case 3.**—Lastly, if the added set be \( xx \cdots 0 \), which is of the form \( xx \cdots 0 \), the equations become when \( x^2/\gamma^2 = p \),

\[
\begin{align*}
t^2 &= ms^2 + 2 (4 - k) (1 + p^g) \\
s^3 &= \frac{2 (1 + p^g)}{m} \\
t^3 &= (4 + 2 (6 - k)) (1 + p^g).
\end{align*}
\]

It will be seen that in each set of the above equations one of the unknowns, viz., either \( s \) or \( t \) is automatically known. The equations in the first two sets are of the form:

\[
\begin{align*}
t^2 &= ms^2 + 2 (4 - k) \\
s^3 &= \frac{2}{m} \\
t^3 &= 4 + 2 (6 - k).
\end{align*}
\]
CONSTRUCTION OF ROTATABLE DESIGNS FROM FACTORIAL DESIGNS

\[ s^2 + v^2 = A \]
and
\[ s^3 + v^3 = B. \]

If the relation \( A^{3/2} \leq B^2 \leq A^3 \) holds and \( A \) and \( B \) are positive, there will be a positive real solution for \( s \) and \( v \) and one of the solutions will lie between \( \sqrt{A} \) and \( \sqrt[3]{A/2} \). Some of the designs obtainable for such sets for different values of \( k \) up to 8 have been presented below:

(i) When \( k = 3 \):

The following sets (i) \( \text{aaa} \), (ii) \( \text{b00} \) (iii) \( \text{gg0} \) and (iv) \( \text{g00} \) give a design which belongs to case 1. This design has 32 points and no central points are necessary.

The equations for the design are:
\[ t^2 + u^2 = 8s^2 + 2 = 5.17497 \]
\[ t^3 + u^3 = 10 \]
\[ s^3 = \frac{1}{4}, \text{ i.e., } s = 0.62996. \]

The solutions for \( t \) and \( u \) are:
\[ t = 2.1090 \]
\[ u = 0.8526. \]

(ii) When \( k = 4 \):

The following sets (i) \( \text{aaaa} \), (ii) \( \text{b000} \) (iii) \( \text{gg00} \) and (iv) \( \text{gg00} \) give a design which evidently belong to case 3. The equations for these sets are:
\[ t^2 = 16s^2, \text{ i.e., } t = 4s. \]
\[ s^3 = \frac{1 + p^3}{8} \]
\[ t^3 = 8 (1 + p^3). \]

As the third equation follows from the first two, whatever \( p \) may be, there will be a design with these sets for each value of \( p \). The values of \( t \) and \( s \) can be obtained as soon as \( p \) is fixed. When \( p \neq 0 \), there will be 72 points. If \( p = 0 \), there will be 48 points, but in this design all the points will be equidistant from the centre and hence relation \( E_1 \) cannot be satisfied even by adding central point.
When \( k = 5 \):

The following sets (i) aaaaa, (ii) \( 00000 \), (iii) \( yy000 \) and (iv) \( owoowo \) will give a design in 114 points. This design belongs to case 2 and the equations are:

\[
\begin{align*}
  s^2 + v^2 &= \frac{t^2 + 2}{32}, \\
  s^3 + v^3 &= \frac{1}{16}, \\
  t^3 &= 6, \quad \text{i.e.,} \quad t = 1.81712.
\end{align*}
\]

The solutions are \( s = -39.48 \) and \( v = 0.0991 \).

When \( k = 6 \):

From the sets (i) aaaaaa, (ii) \( 000000 \), (iii) \( yy0000 \), but without any fourth set, a design can be obtained for which the equations are:

\[
\begin{align*}
  t^2 &= 64s^2 - 4, \\
  s^2 &= \frac{1}{32}, \\
  t^3 &= 4.
\end{align*}
\]

It is found that the values of \( s \) and \( t \) as obtained from the last two equations almost satisfies the first equation. Hence this design in 136 points is very near by a third order rotatable design.

If, however, one more set \( owoowo \) be added, the design will have 200 points. It belongs to case 2 and the equations are:

\[
\begin{align*}
  s^2 + v^2 &= \frac{t^2 + 4}{64} = 10.187, \\
  s^3 + v^3 &= \frac{1}{32}, \\
  t^3 &= 4.
\end{align*}
\]

Solving these equations:

\[
\begin{align*}
  t &= 1.88740, \\
  s &= -3.1446, \\
  v &= 0.05466.
\end{align*}
\]

When \( k = 7 \):

The sets for a design are

(i) aaaaaaa
CONSTRUCTION OF ROTATABLE DESIGNS - FROM FACTORIAL DESIGNS

(ii) \$000000
(iii) yy000000

and

(iv) 00000000.

In this design \( m = 64 \) and not 128. Hence there will be 226 points. This design belongs to case 2 and the equations are:

\[
\begin{align*}
  s^2 + v^2 &= \frac{t^2 + 6}{64} = 11855 \\
  s^2 + v^2 &= \frac{1}{32} \\
  t^2 &= 2.
\end{align*}
\]

Solutions to these equations are

\[
\begin{align*}
  t &= 1.2599 \\
  s &= 0.2949 \\
  v &= 0.7777
\end{align*}
\]

(vi) When \( k = 8 \):

In this case if we take the sets:

(i) aaaaaaaaaa

and

(ii) yy000000

the equations come out by taking \( m = 128, \)

\[
\begin{align*}
  s^2 &= \frac{8}{m} = \frac{1}{16} \\
  s^2 &= \frac{2}{m} = \frac{1}{64}.
\end{align*}
\]

It is found that \( s = \frac{1}{2} \) is a solution and hence a rotatable arrangement is possible with these sets. With this solution for \( s \), we find \( 8a^2 = 2y^2 \) and hence all the points are equidistant from the centre. This indicates that relation \( E_1 \) cannot be satisfied with these points even by adding central points. It appears that the present technique it is not possible to get a design, by adding any further sets belonging to any of the four types tried.

13. THIRD ORDER SEQUENTIAL ROTATABLE DESIGNS

The third order designs presented earlier in the paper cannot be fitted into a sequential programming of experimentation as they are.
Gardiner et al. (1959) have shown that a third order rotatable design will be sequential if the treatments constituting the design can be divided into two groups to form the contents of two blocks such that the treatments in each group form a second order rotatable design with some central points, if necessary. After the treatments have been divided to form the two blocks, let \( \Sigma_1 \) and \( \Sigma_2 \) stand for the sum over the treatments in the first and second blocks respectively. Now, in addition to the relation which are to be satisfied by the third order rotatable designs one more relation, viz., \( \Sigma_1 x_i^2 - 3 \Sigma_2 x_i x_j \) must be satisfied for the design to be sequential, as this will ensure that each block is a second order rotatable design. We shall call this relation \( F \).

A further condition for sequential designs is that
\[
\frac{\Sigma_1 x_i^4}{\Sigma_2 x_i^4} = \frac{n_1 + n_{10}}{n_2 + n_{20}}
\]
where \( n_1 \) and \( n_2 \) are the numbers of treatments excluding the central points in the two blocks respectively and \( n_{10} \) and \( n_{20} \) denote the central points which may be added to the two blocks.

As \( \Sigma_1 x_i^4 \) and \( \Sigma_2 x_i^2 \) are functions of the magnitudes which are known while obtaining the design, the above relation can always be satisfied by suitably choosing \( n_{10} \) and \( n_{20} \).

A number of such designs for values of \( k = 3, 4, 5 \) and \( 6 \) have been presented below.

(i) When \( k = 3 \):

<table>
<thead>
<tr>
<th>Block No.</th>
<th>Block contents</th>
<th>No of treatments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block I</td>
<td>Set (i) ( \gamma'0 )</td>
<td>( n_1 = 18 )</td>
</tr>
<tr>
<td></td>
<td>Set (ii) ( 800 )</td>
<td></td>
</tr>
<tr>
<td>Block II</td>
<td>Set (i) ( \alpha'\alpha )</td>
<td>( n_2 = 22 )</td>
</tr>
<tr>
<td></td>
<td>Set (ii) ( 800 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Set (iii) ( o\omega \omega )</td>
<td></td>
</tr>
</tbody>
</table>

The equations from the relations \( D, D_1 \) and \( F \) are
\[
8 (\gamma^4 + \alpha^4 + \omega^4) + 2 (\beta^4 + \delta^4) = 12 \gamma^4 + 24 (\alpha^4 + \omega^4),
\]
\[
8 (\gamma^6 + \alpha^6 + \omega^6) + 2 (\beta^6 + \gamma^6) = 20 \gamma^6 + 40 (\alpha^6 + \omega^6),
\]
\[
4 \gamma^4 + 8 (\alpha^4 + \omega^4) = 24 (\gamma^4 + \alpha^4),
\]
\[
8 \gamma^4 + 2 \delta^4 = 12 \gamma^4.
\]
Putting

\[ s = \frac{a^2}{\gamma^2}, \quad t = \frac{b^2}{\gamma^2}, \quad u = \frac{w^2}{\gamma^2}, \quad \text{and} \quad v = \frac{\delta^2}{\gamma^2} \]

the equations become:

\[ r^2 + v^2 = 2 + 8(s^2 + u^2) \]
\[ r^2 + v^2 = 6 + 16(s^2 + u^2) \]
\[ 4(s^2 + v^2) = 1 \]
\[ v^2 = 2. \]

Solving these equations

\[ s = 0.6 \]
\[ t = 1.92849 \]
\[ u = 1.41421 \]
\[ v = 0.32390. \]

As

\[ \sum x_i^2 = \gamma^2(8 + 2v) \]
\[ \sum x_i^2 = \gamma^2(8s + 8u + 2t) \]
\[ n_1 = 18; \quad n_2 = 22 \]

the relation

\[ \frac{\sum x_i^2}{\sum x_i^2} = \frac{n_1 + n_{10}}{n_2 + n_{20}} \]

can be satisfied by suitably choosing \( n_{10} \) and \( n_{20} \) so that

\[ \frac{8 + 2v}{8s + 8u + 2t} = \frac{18 + n_{10}}{22 + n_{20}} \]

where \( s, t, u \) and \( v \) are as obtained above. The value of \( \gamma \) is to be obtained from:

\[ \gamma^2(8 + 2v + 8s + 8u + 2t) = N \]

where

\[ N = n_1 + n_2 + n_{10} + n_{20}. \]

(ii) \textit{When } \( k = 4; \)

The series of designs presented in Section 12 under \( k = 4 \) is also sequential when the block contents are taken as below:
The values of the magnitudes are the same as given for the design. Here \( n_1 = 24, n_2 = 48 \) and \( n_10 \) has to be taken greater than one.

(iii) When \( k = 5 \):

<table>
<thead>
<tr>
<th>Block No.</th>
<th>Block contents</th>
<th>No. of treatments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block I</td>
<td>Sets (i) aaaa</td>
<td>( n_1 = 52 )</td>
</tr>
<tr>
<td></td>
<td>(ii) ( \beta 000 )</td>
<td></td>
</tr>
<tr>
<td>Block II</td>
<td>Sets (i) ( \alpha 000 )</td>
<td>( n_2 = 72 )</td>
</tr>
<tr>
<td></td>
<td>(ii) ( \gamma 000 )</td>
<td></td>
</tr>
</tbody>
</table>

The equations for the design are:

\[
32 (\alpha^4 + \omega^4) + 16 \gamma^4 + 2 (\beta^4 + \delta^4) = 3 \times 32 (\alpha^4 + \omega^4) + 12 \gamma^4
\]

\[
32 (\alpha^5 + \omega^5) + 16 \gamma^5 + 2 (\beta^5 + \delta^5) = 5 \times 32 (\alpha^5 + \omega^5) + 20 \gamma^5
\]

\[
32 (\alpha^6 + \omega^6) + 4 \gamma^6 = 3 \times 32 (\alpha^6 + \omega^6)
\]

\[
32 \alpha^4 + 2 (\beta^4 + \delta^4) = 3 \times 32 \alpha^4.
\]

Putting

\[
s = \frac{\alpha^2}{\gamma^2}, \quad t = \frac{\beta^3}{\gamma^3}, \quad u = \frac{\omega^3}{\gamma^3}, \quad v = \frac{\delta^3}{\gamma^3}
\]

the equations become

\[
t^2 + u^2 = 32 (s^2 + u^2) - 2
\]

\[
t^2 + v^2 = 64 (s^2 + u^2) + 2
\]

\[
16 (s^2 + u^2) = 1
\]

\[
t^2 + v^2 = 32 s^2.
\]

Solution to these equations are:

\[
u = 6.25
\]

\[
s = 0.36056
\]

\[
t = v = 1.44225
\]
When \( k = 6 \):

<table>
<thead>
<tr>
<th>Block No.</th>
<th>Block contents</th>
<th>No. of treatments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block I</td>
<td>(i) 1 1 1 1 1 1</td>
<td>( n_1 = 124 )</td>
</tr>
<tr>
<td></td>
<td>(ii) ( \gamma \gamma \gamma \gamma \gamma \gamma )</td>
<td></td>
</tr>
<tr>
<td>Block II</td>
<td>(i) a a a a a a</td>
<td>( n_2 = 136 )</td>
</tr>
<tr>
<td></td>
<td>(ii) ( \beta \beta \beta \beta \beta \beta )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(iii) ( \delta \delta \delta \delta \delta \delta )</td>
<td></td>
</tr>
</tbody>
</table>

The equations for the design are:

\[
\begin{align*}
64 (a^4 + b^4) + 4 (\delta^4 + \gamma^4) &= 3 \times 64 (a^4 + b^4) + 12 (\delta^4 + \gamma^4) \\
64 (a^8 + b^8) + 20 (\delta^8 + \gamma^8) &= 5 \times 64 (a^8 + b^8) + 20 (\delta^8 + \gamma^8) \\
64 (a^4 + b^4) + 4 (\delta^4 + \gamma^4) &= 3 \times 64 (a^4 + b^4) \\
64 \alpha^4 + 20 \gamma^4 &= 3 \times 64 \alpha^4 + 12 \gamma^4.
\end{align*}
\]

Putting

\[
\begin{align*}
s &= \frac{a^2}{\gamma^2}, & t &= \frac{\beta^2}{\gamma^2}, & u &= \frac{\alpha^2}{\gamma^2}, & v &= \frac{\beta^2}{\gamma^2}
\end{align*}
\]

the equations become

\[
\begin{align*}
t^2 &= 64 (s^2 + u^2) - 4 (1 + v^2) \\
t^3 &= 128 (s^3 + u^3) \\
32 (s^3 + u^3) &= 1 + v^3 \\
16u^3 &= 1.
\end{align*}
\]

Solving these equations

\[
(\gamma = 0.4660, \quad t = 2.4636, \quad v = 1.3990 \text{ and } u = 0.25)
\]

**Summary**

Usually the rotatable designs are obtained from regular geometrical configuration. This makes it difficult to construct such designs with larger number of factors. In the present paper a modified form of factorial designs has been defined and they have been utilised to obtain rotatable designs in a simple manner. Second and third order...
rotatable designs for each of the number of factors from 2 to 7, have been constructed. Some third order designs which fit into sequential programming of experimentation have also been evolved. The number of points required for the designs are reasonably small. The method has the speciality that it makes easier and systematic the investigation for obtaining such designs. Through this technique a large number of new designs could be obtained for up to 8 factors.

ACKNOWLEDGMENT

I am grateful to Dr. V. G. Panse, Statistical Adviser to I.C.A.R., for encouragement to do further work as also for general guidance and providing opportunities for carrying out research.

REFERENCES

CONSTRUCTION OF ROTATABLE DESIGNS THROUGH BALANCED INCOMPLETE BLOCK DESIGNS

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Reprinted from The Annals of Mathematical Statistics
Vol. 33, No. 4, December, 1982
Printed in U.S.A.
CONSTRUCTION OF ROTATABLE DESIGNS THROUGH BALANCED INCOMPLETE BLOCK DESIGNS

BY M. N. DAS AND V. L. NARASIMHAM

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1. Introduction and summary. Rotatable designs were introduced by Box and Hunter (1954, 1957) for the exploration of response surfaces. They constructed these designs through geometrical configurations and obtained several second order designs. Afterwards, Gardiner and others (1959) obtained some third order designs through the same technique for two and three factors and a third order design for four factors. Bose and Draper (1959) obtained some second order designs by using a different method. Draper (1960 a) gave a method of construction of an infinite series of second order designs in three and more factors. Recently, Box and Behnken (1960 a) have obtained a class of second order rotatable designs from those of first order. Draper (1960 b) has obtained some third order rotatable designs in three dimensions and a third order rotatable design in four dimensions. Das (1961) has obtained such designs, both second and third orders up to 8 factors as fractional replicates of factorial designs. The method of construction of the designs presented in this paper is essentially based on that presented by Das (1961). After the manuscript of this paper was submitted for publication the authors’ attention was drawn to the work of Box and Behnken (1960 b). They have obtained some second order designs by following a procedure which uses balanced incomplete block designs in the same manner as described below. They did not, however, extend the method to include other complementary sets of points which, as will be shown, allow one to obtain rotatable second and third order designs based on any balanced incomplete block design.

In the present paper a method has been given by using the properties of balanced incomplete block designs through which second order rotatable designs with any number of factors, with a reasonably small number of points, can be obtained. By extending the method, third order rotatable designs, both sequential and nonsequential, up to 15 factors have been obtained with the help of doubly balanced incomplete block designs and complementary B.I.B. designs.

2. Rotatable designs. Let there be \( v \) variates, each at \( s \) levels. If a design be formed with \( N \) of the \( s^v \) treatment combinations, it can be written as the following \( N \times v \) matrix, which we shall call the design matrix:

\[
\begin{bmatrix}
  x_{11} & x_{21} & x_{31} & \cdots & x_{v1} \\
  x_{12} & x_{22} & x_{32} & \cdots & x_{v2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  x_{1N} & x_{2N} & x_{3N} & \cdots & x_{vN}
\end{bmatrix}
\]
For convenience, a variate $X_i$ has been associated with the $i$th factor to denote its levels. The treatment combinations will hereafter be called points of the design. According to Box and Hunter (1957) a design of the above form will be a rotatable design of order $d$ if a response polynomial surface

$$y = \beta_0 + \sum_i \beta_i x_i + \sum_{i<j} \beta_{ij} x_i x_j + \sum_{i<j<k} \beta_{ijk} x_i x_j x_k + \cdots$$

of order $d$ of the response $y$ as obtained from the treatments, on the variates $x_i$, $i = 1, 2, \ldots, v$, with some suitable origin and scale, can be so fitted that the variance of the estimated response from any treatment is a function of the sum of squares of the levels of the factors in that treatment combination. In other words, the variance of the estimated response at a point is a function of the square of the distance of the point from a suitable origin, so that the variances of all estimated responses at points equidistant from the origin are the same. When the response surface is of the second degree (i.e., $d = 2$), such constancy of variance is possible if the design points are so selected as to satisfy the following relations:

(A) $\sum x_i = 0$, $\sum x_i x_j = 0$, $\sum x_i^2 = 0$, $\sum x_i x_j = 0$, $\sum x_i x_j x_k = 0$, $\sum x_i x_j x_k x_l = 0$ for all $i \neq j \neq k \neq l$.
(B) (i) $\sum x_i^2 = \text{constant} = N\lambda_2$
    (ii) $\sum x_i^4 = \text{constant} = 3N\lambda_4$
    for all $i$.
(C) $\sum x_i^2 x_j = \text{constant}$
    for all $i \neq j$.
(D) $\sum x_i^4 = 3 \sum x_i^2 x_j^2$
    for all $i \neq j$.
(E) $\lambda_2/\lambda_4 > v/(v + 2)$.

In the above relations, the summation is over the design points.

In the case of third order designs the following further relations also should be satisfied:

(A) Each of the sums of powers or products of powers of $x_i$'s, in which at least one power is odd, is zero.
(B) $\sum x_i^2 = \text{constant} = 15 N\lambda_4$
    for all $i$.
(C) (i) $\sum x_i x_j x_k = \text{constant}$
    (ii) $\sum x_i^2 x_j^2 x_k^2 = \text{constant}$
    for all $i \neq j \neq k$.
(D) (i) $\sum x_i^4 = 5 \sum x_i^2 x_j^2$
    (ii) $\sum x_i^2 x_j x_k^2 = 3 \sum x_i^2 x_j^2 x_k^2$
    for all $i \neq j \neq k$.
(E) $(\lambda_4/\lambda_6)^2 > (v + 2)/(v + 4)$, the summations being over the design points.

3. Method of construction of rotatable designs. Each point in a design is essentially a combination of the levels of different factors. We thus propose first to take some unknown levels, to be denoted by $a$, $b$, $c$ etc., excepting that some of these may be zero also, and to get a factorial design in $v$ factors, say, out of these unknown levels. Thus, if there are four factors, each at two levels denoted by $a$ and $b$, the 16 combinations will be of the form
Next we shall have another design in $v$ factors of the form $2^v$ where the two levels of each factor are $+1$ and $-1$. We can now get one more set of combinations when any combination of the first design is associated with any combination of the second design, $2^v$ by multiplying the corresponding entries, that is, the levels of the same factor in the two combinations and writing the products in the same order. This method of association of any two combinations of the two designs will hereafter be called multiplication.

For example, in the design with four factors if the combination $a \ b \ b \ b$ be "multiplied" by each of the $2^4$ combinations of the levels $+1$ and $-1$, we shall get the following 16 combinations:

- $a \ b \ b \ b$ when multiplied by $1 \ 1 \ 1 \ 1$
- $a \ b \ b \ -b$ when multiplied by $1 \ 1 \ 1 \ -1$
- $a \ b \ -b \ b$ when multiplied by $1 \ 1 \ -1 \ 1$
- $a \ b \ -b \ -b$ when multiplied by $1 \ 1 \ -1 \ -1$
  
  etc.

If one of the unknown levels, say, $a$ be zero, all the sixteen combinations will not be distinct, but only eight of them will be distinct, as by associating $+1$ and $-1$ with zero we get the same thing. We shall consider, in future, only those combinations which are distinct unless otherwise mentioned. Thus, by "multiplying" any combination of the first design with all the combinations of the second design, $2^v$, we shall get $2^v$ distinct combinations where $p$ denotes the number of non-zero unknown levels in the combination considered of the first design. As a matter of fact, if there be only $p$ unknowns in a combination together with some zeros, we have to multiply only the non-zero levels in the combination with each of the $2^v$ combinations of $+1$ and $-1$.

We have by now come across three types of combinations, namely, (i) Factorial combinations of the unknown levels $a, b$, etc. together with '0.' (ii) Factorial combinations of levels $+1$ and $-1$. (iii) Combinations when each $0, a, b$ etc. is associated with $+1$ and $-1$ through "multiplication".

The first type of factorial combinations will be called combinations of unknown levels and the second will be called associate combinations. The third combinations will actually constitute the design points and hence they will be referred to as the design points.
It will be seen easily that if a design be formed by including all the distinct points which are got by “multiplying” any combination of the unknown levels with all the associate combinations, these points will always satisfy relations $A$ and $A_1$. When $v > 4$, or $p > 4$, the relations $A$ and $A_1$ will also be satisfied when a suitable fraction of the $2^v$ or $2^p$ associate combinations, as the case may be, are so chosen for “multiplication” to obtain a second order rotatable design that no interaction with less than five factors is confounded in these associate combinations. In case of third order designs the fraction should be so chosen that no interaction with less than seven factors is confounded. For satisfying the other relations, $B$, $C$, $D$, $E$ one or more combinations of the unknown levels will have to be chosen suitably. A method for the choice of such combinations through which second order rotatable designs can be obtained has been described below.

Let there be a balanced incomplete block design with the parameters $(v, b^*, r, k, \lambda)$. Let us write the design in the form of a $b^* \times v$ matrix, the elements of which are zero and $a$. If in any block a particular treatment occurs the element in that block corresponding to that treatment will be $a$, otherwise, zero. Each row of the matrix corresponding to a block of the B.I.B. design can be considered to give a combination of zero and the unknown level $a$. By “multiplying” each of these “$b^*$” combinations thus obtained through the B.I.B. design with $2^k$, since $p = k$ here, or a suitable fraction of the associate combinations, we shall get a number of design points less than or equal to $b^* \times 2^k$. These points which we will denote as $a-(v, b^*, r, k, \lambda) \times 2^k$ (or a suitable fraction of $2^k$) will satisfy all relations except $D$ and $E$, as constancy of replication will satisfy relation $B$ and that of replication of pair of treatments will satisfy relation $C$. These points can also be obtained through the method of Box and Behnken (1960 b) if the asterisks used by them in the B.I.B. designs only, be replaced by $a$ and then the sign + or — of 1 presented as column, be associated with $a$ in the same manner described therein.

4. Second order rotatable designs. Box and Behnken (1960 b) have shown that the points obtained through a B.I.B. design with $r = 3\lambda$, together with at least one central point, will always give a second order rotatable design in $v$ factors. Their designs will not involve $a$, but the design obtained through the present method will involve $a$, which has to be obtained from the relation $\sum a_i^2 = N$, where $N$ denotes the total number of design points.

There are four B.I.B. designs up to 16 variates satisfying the relation $r = 3\lambda$. Two of these designs, viz., for 4 and 7 variates, are presented in the above reference, while the other two designs for 10 and 16 variates, are presented in Appendix I for the sake of completion. These two designs have also been reported by Box, G. E. P. and Behnken, D. W. (1958). We have described below the method of obtaining second order rotatable designs through any B.I.B. design where $r \geq 3\lambda$.

1 As the symbol $b$ has been used in this paper to denote an unknown level of the factors together with $a, c, \ldots$, etc., the symbol $b^*$ has been used in this paper to denote the number of blocks in a balanced incomplete block design.
If the relation \( r = 3X \) does not hold in any B.I.B. design, we can always get a second order rotatable design through it by taking some more combinations involving one more unknown level, \( b \), and then by "multiplying" them with the requisite number of associate combinations. The combinations to be taken are either the \( \nu \) combinations, obtained from the combination \((b, 0 0 \ldots 0)\) by permuting over the different factors, or the combination \((b, b \ldots b)\) according as \( r < 3X \) or \( r > 3X \).

We have so far used two types of combinations, viz., one involving the unknown level \( a \) and the other involving \( b \). The combinations obtained through B.I.B. design will hereafter be called \( a \)-combinations, while the \( v \) combinations obtained from \((b, 0 0 \ldots 0)\) will be called combinations of the type \((b, 0 \ldots 0)\).

The design points obtained by the combinations of type \((b, 0 \ldots 0)\) and combination \((b, b \ldots b)\) after "multiplication" with the requisite associate combinations will hereafter be denoted respectively as \((b, 0 \ldots 0) \times 2^r\) and \((b, b \ldots b) \times \) suitable fraction of \( 2^r \).

In the above designs \( \sum x_i^2 \) and \( \sum x_i^2 \) will be functions of \( a \) and \( b \). From the relation \( \sum x_i^2 = 2 \sum x_i^2 \), we shall get an equation connecting \( a \) and \( b \). This equation will always give a positive solution of \( a^2/b^2 \), provided that the extra combinations are suitably chosen, taking into account which of \( r < 3X \) or \( r > 3X \) holds. For determining the unknown levels \( a \) and \( b \), we have one more equation, viz., \( \sum x_i^2 = N \), where \( N \) is the total number of points including the central points.

For example, in the design \( v = 8 \), \( k = 2 \), \( r = 7 \), \( b^* = 28 \), \( \lambda = 1 \), \( r > 3 \lambda \), and hence the combination \((b, b \ldots b)\) has to be taken together with the 28 \( a \)-combinations given by the B.I.B. design in 8 factors. The design points will be (i) \((a, v = 8, k = 2, r = 7, b^* = 28, \lambda = 1) \times 2^7\), and (ii) \((b, b \ldots b) \times \) \( \frac{1}{2} \) replicate of \( 2^8 \).

In this design we have \( \sum x_i^2 = 28a^2 + 64b^2 \), \( \sum x_i^2 x_j^2 = 4a^4 + 64b^4 \). Hence, relation \( D \) gives the equation

\[
28a^4 + 64b^4 = 3(4a^4 + 64b^4),
\]

whence

\[
b^4/a^4 = \frac{1}{2}.
\]

The above equation, together with \( \sum x_i^2 = 28a^2 + 64b^2 = N \), will completely determine the two unknowns \( a \) and \( b \). The number of points in this design will be 112 + 64 = 176. No central points are necessary in this design, though they may be added if otherwise necessary.

By properly choosing the B.I.B. designs the number of design points can be...
reduced. A list of second order rotatable designs, together with the additional type of combinations, when necessary, to be taken for the construction of such designs up to 16 factors, is presented in Appendix I, together with relevant details. Designs for larger number of factors can, however, be obtained on the same lines. It will be seen that all the designs obtainable through B.I.B. designs have either 3 or 5 levels, according as the extra combinations with $b$ are taken or not.

5. Third order rotatable designs. Third order rotatable designs can be both non-sequential and sequential. If the design points satisfy all the requirements mentioned in Section 2 and are tried in one occasion, they form a non-sequential third order rotatable design. Alternatively, Gardiner et al. (1959) have shown that a third order rotatable design can be performed sequentially by dividing all the design points into two groups, each forming the contents of a block. If the design points in the first block form a second order design and the inclusion of the additional points of the second block makes the whole a third order design, this design with the points in both the blocks is called a sequential third order rotatable design. The points in the second block will be tried when the fit as obtained from the first block happens to be inadequate.

For the estimation of the polynomial coefficients independently of block effects through such designs, there is a further condition to be satisfied, viz.,

$$\frac{\sum x_i^2}{\sum x_i^4} = \frac{\sum_1 n_1 + n_{20}}{\sum_1 n_2 + n_{20}}$$

where $n_1$ and $n_0$ are the numbers of treatments excluding the central points in the two blocks respectively; $n_{20}$ and $n_{20}$ denote the central points which may have to be added to the two blocks and $\sum_1$ and $\sum_2$ denote the summation over the points in the first and second blocks respectively.

As $\sum x_i^2$ and $\sum x_i^4$ are functions of the levels of the factors, which can be evaluated from the other relations of the design, the above relation can always be satisfied by suitably choosing $n_0$ and $n_{20}$.

The $a$-combinations chosen through B.I.B. designs for the construction of second order rotatable designs do not usually satisfy the relations $C_1$ (ii) together with $D_1$, $D_1$ (i) and $D_1$ (ii) and $E$ as required for third order rotatable designs. If the B.I.B. design happens to be doubly balanced i.e. in addition to pairs of treatments occurring a constant number of times, $\lambda$, the triplets of treatments also occur a constant number of times, $\mu$, in the blocks (Calvin, 1954), the relation $C_1$ (ii) is also satisfied. For satisfying the other relations not yet satisfied, viz. $D_1$, $D_1$ (i) and $D_1$ (ii) and $E$, we have to introduce combinations involving fresh unknowns which can be evaluated by solving the equations obtained through $D_1$, $D_1$ (i) and $D_1$ (ii). For example, each of the following designs are doubly balanced. For the sake of convenience only the numerical values of the parameters $\nu$, $k$, $\phi$, $\phi^*$, $\lambda$ and $\mu$ of the doubly B.I.B. designs have been shown in brackets in the same order in which they are written above. In the future, the parameters of B.I.B. designs also will be presented similarly except for $\mu$. 

...
With the help of each of these designs which will supply us the \(a\)-combinations as described earlier for second order rotatable designs, third order designs, both sequential and non-sequential, can be obtained by taking further one or more of the combinations of the type \((b \ 0 \ 0 \cdots 0)\), \((c \ c \ 0 \cdots 0)\), \((d \ d \cdots d)\) involving fresh unknown levels \(b, c, d\) and multiplying them with the associate combinations as earlier. The combination \((c \ c \ 0 \cdots 0)\) will give \(c^{v/2}\) combinations when permuted over all the \(v\) factors and these \(v(v - 1)/2\) combinations will hereafter be called combinations of type \((c \ c \ 0 \cdots 0)\). The design points obtained from the combinations of type \((c \ c \ 0 \cdots 0)\) after multiplying each one of them with the \(2^v\) associate combinations will be denoted as \((c \ c \ 0 \cdots 0) \times 2^v\). The other two types of combinations have been described earlier. Sometimes it becomes necessary to include in the same design more than one set of the same type for getting positive solutions for all the levels.

As an example, we can get a third order non-sequential rotatable design in 9 factors with the help of the following design points:

(i) 672 points from \(a-(9, 3, 28, 84, 7, 1) \times 2^v\)
(ii) 256 points from \((b \ b \cdots b) \times \frac{1}{2} \text{ repl.} \ 2^v\)
(iii) 256 points from \((c \ c \cdots c) \times \frac{1}{2} \text{ repl.} \ 2^v\)
(iv) 18 points from \((d \ 0 \cdots 0) \times 2\).

The equations for solving the unknowns come out as

From \(D\): \((28 \times 8)a^4 + 256(b^5 + c^5) + 2d^6 = (21 \times 8)a^4 + 3 \times 256(b^4 + c^4)\)

From \(D_1(i): (28 \times 8)a^6 + 256(b^5 + c^5) + 2d^6 = (35 \times 8)a^6 + 5 \times 256(b^4 + c^4)\)

From \(D_1(ii): (7 \times 8)a^6 + 256(b^5 + c^5) = 3 \times 8a^6 + 3 \times 256(b^4 + c^4)\).

Solving these equations we get

\[
\frac{b^5}{a^5} = 0.392708 \\
\frac{c^5}{a^5} = 0.122376 \\
\frac{d^6}{a^5} = 3.914868.
\]

The value of \(a\) can be obtained from \(\sum x_i^2 = N\). This design contains 1202 points.

Sequential third order designs can be obtained with the help of the same types of combinations, viz., \(a\)-combinations through the doubly B.I.B. designs, together with one or more of the types of combinations \((b \ 0 \ 0 \cdots 0)\), \((c \ c \ 0 \cdots 0)\) and \((d \ d \cdots d)\). For example, we can get a sequential third order rotatable design in 8 factors with the help of the following design points:
Block I (i) 128 points of \((d d \cdots d) \times (\frac{1}{2} \text{ replicate of } 2^d)\)
(ii) 16 points of \((e 0 0 \cdots 0) \times 2^d\)
Block II (iii) 224 points of \((-8, 4, 7, 14, 3, 1) \times 2^d\)
(iv) 112 points of \((c c 0 \cdots 0) \times 2^d\)

The design relations will lead to the following equations. From relations

\begin{align*}
(D) : & \quad 112a^4 + 28c^4 + 128d^4 + 2e^4 = 144a^4 + (3 \times 128)d^4 + 12c^4 \\
(D_1) (i) : & \quad 112a^6 + 28c^6 + 128d^6 + 2e^6 = 240a^6 + (5 \times 128)d^6 + 20c^6 \\
(D_2) (ii) : & \quad 48a^6 + 128d^6 + 4c^6 = 48a^6 + (3 \times 128)d^6
\end{align*}

There is one more relation to make each block a second order rotatable design. This relation gives \(2e^4 + 128d^4 = (3 \times 128)d^4\). Putting \(a^2/d^2 = s, \ c^2/d^2 = u, \ e^2/d^2 = f\), the equations become

\begin{align*}
8a^2 + t^2 = 16s^2 + 128 \\
4a^2 + t^2 = 64s^2 + 256 \\
4a^2 = 2 \times 128 \\
t^2 = 128,
\end{align*}

whence \(u = 4, \ t = 128^2, \) and \(s = 8^3\). The value of \(d\) can be obtained from \(\sum x_i^4 = N\). The number of central points to be added to the two blocks to ensure estimation of the polynomial coefficients independently of block effects will be determined from

\[
\frac{(\sum_1 x_i^4)}{(\sum_2 x_i^4)} = \frac{144 + n_{10}}{336 + n_{20}},
\]

where \(\sum_1 x_i^4\) is summed over the points in the first block and \(\sum_2 x_i^4\) is summed over the points in the second block.

As \(\sum_1 x_i^2\) and \(\sum_2 x_i^2\) are functions of the unknown levels, which have been obtained by solving the equations gotten from the different relations to be satisfied, \(n_{10}\) and \(n_{20}\), the number of central points to be added to the first and second block respectively, can be obtained from the above relation. Actually, \(\sum_1 x_i^2 = 128d^2 + 2e^2\) and \(\sum_2 x_i^2 = 112a^2 + 28c^2\).

Substituting for \(s, u\) and \(t\) obtained earlier, \(n_{10}\) and \(n_{20}\) can be obtained from

\[
(64 + 0)/(56s + 14u) = (144 + n_{10})/(336 + n_{20}).
\]

Thus, we get a sequential third order rotatable design for 8 factors in 480 non-central points.

6. Third order designs obtained through complementary B.I.B. Designs. A B.I.B. design, not necessarily doubly balanced, is taken together with its complementary B.I.B. design, repeated if necessary, for generating the \(a\)-combinations as before. We can now get points through these \(a\)-combinations which will satisfy \(D_1 (ii)\), as \(a\) will be a constant in the combined B.I.B. designs, together with all the other relations excepting \(D, D_1 (i)\) and \(D_1 (ii), E.\) For satisfying these
relations we have to take one or more of the types of combinations \((b \ 0 \ 0 \ \cdots \ 0)\), 
\((c \ 0 \ \cdots \ 0)\) and \((d \ d \ \cdots \ d)\) involving fresh unknowns.

For example, a non-sequential third order rotatable design in 10 factors can be obtained with the following points:

(i) \((18 \times 32)\) points of \(a-(10, 5, 9, 18, 4) \times 2^3\),
(ii) \((18 \times 32)\) points of \(a-(10, 5, 9, 18, 4) \times 2^3\), the design in (ii) being the complementary B.I.B. design of the design in (i).
(iii) \(20\) points of \((b \ 0 \ \cdots \ 0) \times 2^2\),
(iv) \(180\) points of \((d \ d \ 0 \ \cdots \ 0) \times 2^3\),
(v) \(20\) points of \((c \ 0 \ \cdots \ 0) \times 2^2\).

Here \(\mu = 3\) in the combined designs.

The relations \(D, D_1(i)\) and (ii) give the equations,
\[(18 \times 32) a^4 + 2b^4 + 2c^4 + 36d^4 = (24 \times 32) a^4 + 12d^4.\]
\[(18 \times 32) a^6 + 2b^6 + 2c^6 + 36d^6 = (40 \times 32) a^6 + 20d^6.\]
\[(8 \times 32) a^6 + 4d^6 = (9 \times 32) a^6.\]

Putting \(b^2/a^2 = s, c^2/a^2 = t, d^2/a^2 = u,\) we get \(u = 2, s^2 + t^2 = 48,\) and \(s^2 + t^2 = 288.\) Solving, we get \(s = 6.494805, t = 2.411955.\) Thus, we get a non-sequential third order rotatable design in 1372 points.

Sequential third order designs can also be constructed with the help of the complementary B.I.B. designs together with three other types of combinations involving fresh unknowns.

For example, with the following points we can get a sequential third order design for 7 factors:
Block I (i) \(112\) points of \(a-(7, 4, 4, 7, 2) \times 2^4\)
(ii) \(14\) points of \((b \ 0 \ 0 \ \cdots \ 0) \times 2^4\)

Block II (iii) \(112\) points from \(a-(7, 3, 3, 7, 1) \times 2^3,\)
the design in (iii) being the complementary design of the B.I.B. design in (i).

Each of the \(56\) points in (iii) is to be repeated once more.

Here, \(\mu = 1\) in the combined designs. The complementary B.I.B. design in (iii) has to be repeated once more as each \(a\)-combination from the first B.I.B. design gives 16 design points on “multiplication” with the associate combinations and each \(a\)-combination from the complementary B.I.B. design gives only 8 combinations on “multiplication” with the associate combinations. Hence unless all the points obtained from the \(a\)-combinations of the complementary design be repeated once, \(\sum x_i^2 x_j^2 x_k^2\) will not be constant for all \(i, j, k, i \neq j \neq k.\)

In the case of the above design points condition \(D_1(i)\) is satisfied, as \(\lambda = 3\mu\) and the requirement that each block is a second order rotatable design is satisfied as \(r = 3\lambda\) in block II.

Relations \(D\) and \(D_1(i)\) give the equations,
\[112a^4 + 2b^4 = (3 \times 48) a^4.\]
\[112a^6 + 2b^6 = (5 \times 48) a^6.\]
Putting $b^2/a^2 = t$, these equations become $t^2 = 16$ and $t^3 = 64$. Hence $t = 4$.

Numbers of central points to be added to the two blocks are given by the relation,

$$
\frac{48a^2}{(64a^2 + 2b^2)} = \frac{112 + n_2}{126 + n_2},
$$

i.e.

$$
\frac{48}{64 + 2t} = \frac{112 + n_2}{126 + n_2}.
$$

Thus, we get a sequential third order rotatable design for 7 factors in 238 non-central points, with some central points to be added.

Appendices II and III present respectively non-sequential and sequential third order rotatable designs up to 15 factors obtained by utilizing doubly balanced incomplete block designs or by a B.I.B. design together with its complementary B.I.B. design. Some of the designs in the appendices, particularly for small number of factors, have been obtained by others through other methods.