2. An alternative approach for construction of symmetrical factorial designs and obtaining maximum number of factors.

2.1. Introduction.

Bose and Kishen (1940) and subsequently, Bose (1947) developed a theory for the construction of confounded symmetrical factorial designs through hyper-space projective geometry. Both Fisher (1942, 1945) and Bose (1947) discussed the problem of finding the maximum number of factors that can be accommodated in a block of given size without confounding any interaction up to a given order. Fisher showed that when each of the factors has $s$ levels where $s$ is a prime power, the maximum number of factors that can be accommodated in a block of size $s^r$ without confounding any main effect and two factor interaction is $(s^r-1)/(s-1)$. Bose (1947) in addition to obtaining the results of Fisher showed through hyper-space projective geometry that when $s = 2$, the maximum number of factors that can be accommodated in a block of size $2^r$ without confounding any interaction with less than four factors is $2^{r-1}$ together with some other results including several inequality relations fixing limits for such maximum number of factors in the general case when interactions up to any order are saved. Rao (1947) also obtained independently the inequality relations. The purpose of the present paper is to put forward an alternative approach for the construction of confounded symmetrical factorial designs. Through this
technique ultimately leads to all the results of Bose (1947), it is straightforward and very simple. It consists in first obtaining the independent treatment combinations in the key block and next the other requirements of a design are obtained therefrom. It eliminates to a very great extent the trial and error procedure for saving desired interactions. Further through this method the maximum number of factors that can be accommodated in a block of given size such that no interaction up to a given order is confounded can be obtained in a straightforward way.

2. The method

Let there be \( n \) factors each at \( s \) levels where \( s \) is either a prime or a prime power. Let the \( s \) elements of a Galois field denote the \( s \) levels of each of the factors. We shall first describe the method of construction of confounded factorial designs of the type \( s^n \) when \( s = 2 \) and then the other cases will follow.

For the construction of the design \( 2^n \) in blocks of size \( 2^r \) we shall first take the following \( r \) independent treatment combinations of \( r \) factors each at 2 levels denoted by 0 and 1.

### Basic factors

<table>
<thead>
<tr>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
<th>( \ldots )</th>
<th>( A_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Independent treatment combinations 1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>\ldots</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>\ldots</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\ddots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( r )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
</tr>
</tbody>
</table>
It is well known that \(2^r-1\) treatment combinations can be generated out of these \(r\) independent treatment combinations by adding \(\text{mod}(2)\) the corresponding elements, that is, those belonging to the same factor, in these treatment combinations in all possible ways by taking the treatment combinations two, three, etc., at a time. The \(2^r-1\) treatment combinations obtained in this way along with the control treatment will give all the \(2^r\) treatment combinations obtainable from the \(r\) independent treatment combinations.

The \(r \times r\) square given above formed of the \(r\) independent treatment combinations of \(r\) factors can be extended by introducing \((n-r)\) further columns such that each column is a combination of \(r\) factors each at levels 0 and 1 to form a scheme of \(r\) rows and \(n\) columns. Writing the factor notations \(A_{r+1}, A_{r+2}, \ldots, A_n\) above these \((n-r)\) columns, the rows of this scheme give \(r\) independent treatment combinations of \(n\) factors each at two levels and from these combinations a total of \(2^r-1\) treatments combinations of \(n\) factors can be obtained exactly in the same way as described earlier in the case of \(r\) factors. These \(2^r-1\) treatment combinations along with the control treatment form the key block of the design \(2^n\) in blocks of size \(2^r\). We shall call, in future, the \(r\) factors \(A_1, A_2, \ldots, A_r\) as the basic factors and the \((n-r)\) factors \(A_{r+1}, A_{r+2}, \ldots, A_n\), the added factors.

We shall now find out the \((n-r)\) independent interactions which are confounded as a result of obtaining the key block in this manner. Denoting any added factor by \(A_j\) \((j = r+1, r+2, \ldots, n)\)
if there is unity in the column below this factor at the \( p \) th position \( (p \leq r) \) then an interaction containing \( A_j \), and all factors, \( A_p \), will be the independent interaction confounded due to the introduction of this column. For example, if \( r = 4 \), we have the 4 independent treatment combinations as given below.

<table>
<thead>
<tr>
<th>Basic factors</th>
<th>Added factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>( A_j )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( A_4 )</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>( A_j )</td>
</tr>
</tbody>
</table>

| \( \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \) | \( 1 \) | \( 1 \) | \( 0 \) | \( 1 \) |

If now we take a column below the factor \( A_j \) as shown beyond the line, there is unity in this column in positions 1, 2 and 4 and hence \( p \) takes the values 1, 2 and 4. The independent interaction confounded due to the introduction of this column is therefore \( A_1 A_2 A_4 A_j \). That this should be confounded is evident from the fact that in the independent combinations of the factor \( A_j \) and all the basic factors, there will be always two 1's in those combinations which contain the level 1 of \( A_j \) and no 1 below \( \Delta_{A_j} \) the factors in the interaction in the other combinations. Hence all these independent treatment combinations must be solutions of the equation for obtaining the key block when \( A_1 A_2 A_4 A_j \) is confounded.

When \( j \) varies from \( r + 1 \) to \( n \) we shall get \( n - r \) interactions from the \( n - r \) columns which are confounded in the key block of size \( 2^r \) involving \( n \) factors. As each of these
interactions involves a fresh factor, \( A_j^{(j = r+1, r+2, \ldots, n)} \), they must be independent. The rest of the \( 2^{n-r} - 1 \) interactions confounded can be obtained from these \( n-r \) independent interactions through the usual method. They can also be got from the columns obtainable from the \( (n-r) \) columns corresponding to the factors \( A_j^{(j = r+1, r+2, \ldots, n)} \) by adding them two by two, three by three and so on upto all the \( (n-r) \) columns added together. The method is the same as described earlier for obtaining the interaction corresponding to the column of any factor, \( A_j \) excepting that instead of taking \( A_j \) in the interaction for the column generated by addition, we have to take the product of these \( A_j \)'s the columns for which have been added together to give this column. By addition of column means addition of their corresponding elements in the G.F. For example, if we have the following scheme of 4 basic factors

<table>
<thead>
<tr>
<th>Basic factor</th>
<th>Added factors</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 \ A_2 \ A_3 \ A_4 )</td>
<td>( A_5 \ A_6 )</td>
<td>( A_5 \ A_6 )</td>
</tr>
<tr>
<td>Treatment combinations</td>
<td>0 1 0 0</td>
<td>1 1 0</td>
</tr>
<tr>
<td></td>
<td>0 0 1 0</td>
<td>0 1 1</td>
</tr>
<tr>
<td></td>
<td>0 0 0 1</td>
<td>0 0 0</td>
</tr>
</tbody>
</table>

the independent interactions confounded due to the introduction of the factors \( A_5 \) and \( A_6 \) are \( A_1 A_2 A_5 \) and \( A_1 A_2 A_3 A_6 \). The generalised interaction can be obtained from the sum of the columns under \( A_5 \) and \( A_6 \) as shown under the product \( A_5 A_6 \). As this column has 1 in the third position for treatment, \( p = 3 \) and hence the
interaction corresponding to it is \( A^2A \).

The other blocks of the design can be obtained through the usual method of obtaining them from the key block.

2.3. Confounded designs saving main effects and two factor interactions.

It is evident that if in an introduced column there is only zero, the main effect of the factor corresponding to the column will be confounded, as the whole of the key block of size \( 2^r \) when generated through the method described earlier, cannot contain any level of this factor other than zero. Actually all main effects can be saved if no column containing all zeros be taken. Again, if two columns be identical in each of the independent treatment combinations in the key block, the levels of the two factors heading these two columns will be the same in all the combinations in the block. Now, we know that if any two-factor interaction is confounded, the levels of these two factors in each of the treatment combinations in the key block are identical, as these combinations are (00) and (11). Thus, if the columns be so chosen that no two columns are identical, then the confounded design obtainable through such columns will not have any two factor interaction confounded.

When the block size is \( 2^r \) and thus each column contains \( r \) elements, there can be the maximum of \( 2^r \) such different columns, each being a separate combination of \( r \) factors each at 2 levels. Excluding the column involving zero everywhere and the initial \( r \) columns corresponding to the basic factors, the remaining \( 2^r - r - 1 \) columns give the maximum number of columns.
that can be added so that no two columns are identical and there is no column with zero everywhere. Thus, the maximum number of factors that can be accommodated in a block of size $2^r$ without confounding any main effect and two factor interaction is $2^r - 1$ consisting of the $r$ basic factors and $2^r - r - 1$ added factors.

For example, to get the key block of size $2^3$ of a confounded design we have to start with three basic factors and 3 independent treatment combinations as shown below the factors $A_1, A_2$ and $A_3$.

<table>
<thead>
<tr>
<th>Basic factors</th>
<th>Added factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$A_2$</td>
</tr>
</tbody>
</table>

Independent treatments

| 1 0 0 | 1 1 0 1 |
| 0 1 0 | 1 0 1 1 |
| 0 0 1 | 0 1 1 1 |

For obtaining the independent treatment combinations in the key block for 4 factors such that no main effect and 2-factor interaction is confounded, all that we have to do is to take a further column consisting of 0 and 1 as shown below the factor $A_4$ such that this column does not contain only zero and that it is not identical with any of the previous columns. The 3 rows each consisting of 4 elements give the 3 independent treatments. Similarly by taking further columns in any manner subject to the above two restrictions (shown below the factors $A_5, A_6$ and $A_7$), we shall get from the rows the independent treatments in the key block for the design with 5, 6 or 7 factors when the first 5, 6 or 7 columns are taken. As no further fresh column exists, it is not possible to get any confounded design with more than 7 factors without confounding any main effect and two factor interaction.
The above scheme actually gives row-wise the independent treatment combinations in the key block and columnwise the independent confounded interactions as indicated earlier. The interaction confounded in the $2^4$ design is $A_1 A_2 A_4$. The independent interactions confounded in the $2^5$ design are $A_1 A_2 A_4$ and $A_1 A_3 A_5$ and the other confounded interactions can be obtained from the totals of added columns. The following scheme illustrates how the independent and generalised confounded interactions can be obtained from the columns under the added factors and their totals.

<table>
<thead>
<tr>
<th>Basic factors</th>
<th>Added factors</th>
<th>Totals of columns under added factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$ $A_2$ $A_3$ $A_4$ $A_5$ $A_6$ $A_7$</td>
<td>$A_4$ $A_5$ $A_4$ $A_5$ $A_4$ $A_5$ $A_4$ $A_4$ $A_5$ $A_4$ $A_5$ $A_4$</td>
<td>$A_5$ $A_5$ $A_5$ $A_7$ $A_7$ $A_5$ $A_6$ $A_6$ $A_5$ $A_6$ $A_5$ $A_6$ $A_7$ $A_7$ $A_7$ $A_7$ $A_7$ $A_7$ $A_7$ $A_7$ $A_7$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Independent treatments</th>
<th>1 0 0 1 1 0 1</th>
<th>0 1 1 0 0 1 0 0 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 1 0 1 0 1 1 1</td>
<td>1 0 1 0 0 1 0 1 0 1</td>
</tr>
<tr>
<td></td>
<td>0 0 1 0 1 1 1 1</td>
<td>1 1 0 0 0 0 1 0 1 1</td>
</tr>
</tbody>
</table>

Interactions confounded:

(i) Independent (from $A_1$ to $A_7$ columns)

$A_1$ $A_2$ $A_4$, $A_1$ $A_3$ $A_7$, $A_2$ $A_3$ $A_6$, $A_1$ $A_2$ $A_3$ $A_7$

(ii) Generalised (from totals of columns)

$A_2$ $A_3$ $A_4$, $A_1$ $A_3$ $A_4$ $A_6$, $A_1$ $A_2$ $A_3$ $A_6$, $A_4$ $A_5$ $A_6$, $A_3$ $A_4$ $A_7$, $A_2$ $A_5$ $A_7$, $A_1$ $A_6$ $A_7$, $A_1$ $A_4$ $A_5$ $A_7$, $A_2$ $A_4$ $A_6$ $A_7$, $A_3$ $A_5$ $A_6$ $A_7$, $A_1$ $A_2$ $A_3$ $A_4$ $A_5$ $A_6$ $A_7$
2.4. Confounded designs without confounding any interaction with less than 4 factors.

When a three factor interaction is confounded, the key block contains each of the combinations 0 0 0 so far as these three factors are concerned. These combinations can be obtained from a solution of the equation \( x_1 + x_2 + x_3 = 0 \), mod 2, where each of \( x_1, x_2 \) and \( x_3 \) takes values 0 and 1. An examination of these combinations as also of the equation shows that the sum of the corresponding entries in any two of the three columns gives the corresponding entries in the third column. Or, in other words, we may say that the sum of any two columns gives the third column. Thus, for saving all three factor interactions in any confounded design the columns should be so chosen that the sum of no two columns should be equal to a third column. On similar arguments this result generalises to that when all interactions of order \( \gamma \) are to be saved, the columns are to be so chosen that the sum of any \( p(\gamma) \) of the columns should not be the same as the sum of another \( (d-p) \) columns.

Now, while choosing columns if we take those columns which have only odd number of 1's avoiding repetition of columns, then in the confounded design obtainable through these \( r \) independent treatment combinations given by the basic and added columns, no main effect, two factor interaction and three factor interaction will be confounded. This is so because addition of any two of these columns will always give a column with an even number of 1's and hence the sum cannot be equal to any of the columns taken. Further
from all the columns having only odd number of 1's including those corresponding to the basic factors, each of the columns with even number of 1's can be obtained by adding the formers in two's. Thus, the maximum number of factors that can be accommodated in a block of size $2^r$ without confounding any interaction with less than 4 factors is
\[ n_1 \cdot n_2 \cdot n_3 + \ldots = 2^{n-1} \]
as this gives the total number of columns possible with odd number of 1's.

This result has been obtained by Bose (1947) but in a different way.

2.5. Maximum number of factors saving interactions of higher orders.

The treatment combinations in the key block only of a confounded design are taken for experimentation in a fractionally replicated design. The identity group of interactions consists of all those interactions which are confounded to obtain the key block. In such designs all interactions in an alias subgroup of interactions can be said to be mutually confounded. Now, a case of interest is that a main effect or a two factor interaction should not be confounded with another main effect or two factor interaction and for achieving this, the identity group must not contain any interaction with less than 5 factors. Thus, the confounded designs in which no interaction with less than 5 factors is confounded have a special interest in so far as they are used to obtain fractionally replicated designs. Such fractionally replicated designs are also necessary for the construction of second order rotatable designs involving
more than 4 factors. Unfortunately, the investigation for obtaining such designs is not as straightforward. In this case enumeration of columns such that the sum of no two columns is the same as the sum of another two columns, is necessary. It has been possible to enumerate up to 39 columns with block size $2^{12}$ so that no interaction with less than 5 factors are confounded. The results will be published subsequently.

Bose (1947) obtained upper limits to the maximum number of factors in this and other cases of higher order interactions. Through the present approach also it is possible to obtain the same upper limit as indicated below.

Let there be a block of size $2^p$ and $m$ distinct columns (factors) including the basic $r$ columns such that in the design obtainable through these columns as indicated earlier no interaction with less than 5 factors is confounded. These columns have to be such that the sum of no two columns is equal to either another column or the sum of another two columns. Now, by adding two columns in all possible ways we shall get $m^2$ further columns and the total number of columns will be $m + m^2$ when the previous columns are also included. Now, if $m + m^2 > 2^p - 1$ then all these $m + m^2$ columns are not distinct and some columns must be repeated as $2^p - 1$ is the maximum number of columns possible, excluding the one with zero everywhere. As the initial $m$ columns are distinct, a repetition must be due to the fact that the sum of any two columns gives either one of the $m$ initial columns or the sum of two other columns. In the first situation a three-factor interaction gets confounded, while in the second situation a four-factor interaction gets confounded. Hence if $m$ is the
maximum number of factors that can be accommodated in a block of size $2^r$ such that no interaction with less than 5 factors is confounded, then

$$m + m_{c_2}^2 \leq 2^{r-1}.$$ 

Again, if any 5 factor interaction is also not to be confounded in addition to the previous requirements, the sum of any three columns must not give rise to a column which is the sum of another two columns. Now, if we leave out one of the initial $m$ columns and add the rest two by two we shall get $m-1_{c_2}$ further columns. When each of these $m-1_{c_2}$ columns is added to the column left over, we shall get $m-1_{c_2}$ more columns each of which is the sum of three columns. We have thus a total of $m + m_{c_2} + m-1_{c_2}$ columns, $m$ being the number of initial columns, $m_{c_2}$ columns are obtained by adding these columns two by two and the rest $m-1_{c_2}$ are obtained as the sum of three columns as described above.

Here also, if $m + m_{c_2} + m-1_{c_2} > 2^{r-1}$, then all the columns are not distinct but some must be repeated. If the repetition be due to the addition of the $m$ columns two by two then as indicated earlier either a 3-factor or a 4-factor interaction will be confounded. If, again, any of the $m-1_{c_2}$ columns each of which is the sum of three of the $m$ initial columns, is the same as any of the $m$ initial columns, then either a two or a four factor interaction will be confounded according as the repetition is of the left over column or not. Again, if a column exists both among $m_{c_2}$ and $m-1_{c_2}$ columns which are sums of two and three columns respectively then either a 5-factor or a 3-factor interaction or a main effect will be confounded according as
the two columns summed up in one case (i) has nothing in common, (ii) has one column common or (iii) has both the columns common with the three columns summed up in the other case. Thus, if all interactions up to 5 factors are to be saved, \( m \) must be such that

\[ \sum_{i=1}^{m} c_i^2 + \sum_{i=1}^{m-1} c_i^2 \leq 2^k - 1. \]

Arguing on identical lines the general result obtained by Bose (1947) in regard to other upper limit of the maximum number of factors so as not to confound any interaction up to a given order, can be obtained for the case \( s = 2 \).

2.6. Designs with factors each at three levels.

When the factors are at 3 levels each and the block size is \( 3^r \), the \( r \) independent treatment combinations from which \( 3^r - 1 \) combinations of \( r \) factors each at 3 levels can be generated remains the same as in the case of \( s = 2 \), viz

<table>
<thead>
<tr>
<th>Basic factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_1 A_2 A_3 \ldots \ldots A_r</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Independent treatment combinations</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>\ldots</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>\ldots</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>\ldots</td>
</tr>
<tr>
<td></td>
<td>r</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

As a matter of fact this scheme remains the same for any \( s \).

The other treatment combinations can be generated through the well-known method of adding corresponding elements of 2, 3, \ldots, \( r \) rows, some rows being doubled and reducing the sum mod 3. For example, when two independent treatment combinations are taken, one more
can be obtained simply by adding the corresponding elements
in them and writing them in the same order reducing mod 3.
Three more treatment combinations can be obtained by adding
(i) any element of the first to twice the corresponding element
of the second (ii) twice any element of the first to the
 corresponding element of the second and (iii) twice the element
of one with twice the corresponding element of the other. Again,
by doubling every element of the two starting treatment
combinations we shall get two more combinations. These will
give all the 8 treatment combinations that can be obtained from
the two combinations.

By taking further n-r columns each consisting of
the elements 0, 1 and 2 in any manner below the added factors.
\[ A_{n+1}, A_{n+2}, \ldots, A_n \], we shall get from the rows r independent
combinations of n factors. Out of these r independent
combinations \(3^r\) combinations can be obtained as described
earlier and these will form the key blocks of the design \(3^n\)
in blocks of size \(3^r\). If the element in the column under the
added factor \(A_j\) be \(c_{ij}\) in the ith row, then the interaction
confounded due to the introduction of this column will consist
of \(A_j^2\) together with all the basic factors \(A_{i}^{c_{ij}} (i = j, z, \ldots, n)\),
where \(A_{i}^{c_{ij}}\) means \(A_i\) to the power \(c_{ij}\) and \(A_i^3\) is to be treated
as 1.

Thus, for each column added, one independent interaction
will be confounded and this can be obtained as indicated
above. From these n-r independent interactions obtained
from the n-r added columns, all the \((3^{n-r} - 1)\) interactions
can be obtained through the usual procedure or through the
method of column addition. In this case if \( c_1 \) and \( c_2 \) denote two columns, they have to be totaled as \( c_1 + c_2 \) and \( c_1 + 2c_2 \) and the interaction corresponding to the second total should contain square of the factor corresponding to \( c_2 \).

Just like the case of \( s = 2 \), here also if a column consists of zero only, the main effect of the added factor against that column will be confounded.

When a two factor interaction is confounded i.e. either the component \( AB \) or \( AB^2 \), the combinations that occur in the key block are (1) 00 or (2) 00

\[
\begin{array}{ll}
12 & 11 \\
21 & 22 \\
\end{array}
\]

An examination of the two columns against the two factors shows that when a two factor interaction is confounded, the two columns below these two factors are either equal or one is double the other. The elements in a column can be taken to form a column vector and the addition of two or more columns or the product of a column by an element will therefore mean the corresponding operation on the column vectors. Thus, twice a column means a column consisting of each of these elements in the column multiplied by two.

Thus, for saving all two factor interactions, the columns should be so chosen that no two of them are equal and no column is the double of another. Now, if we take the first non-zero element of each added column as unity in all possible ways without repeating any column, then both the requirements for saving...
all two factor interactions will be satisfied by all these columns. When the column with zero everywhere is omitted, we can get \( \frac{3^r - 1}{3 - 1} \) combinations from all possible combinations of \( r \) factors each at 3 levels such that the first non-zero element of each combination is unity. By using these combinations as the added columns we shall have \( \frac{3^r - 1}{3 - 1} \) columns and no more as all other possible columns can be obtained by doubling these columns. Hence the maximum number of factors that can be accommodated in a block of size \( 3^r \) without confounding any main effect and two factor interaction is \( \frac{3^r - 1}{3 - 1} \).

Through exactly corresponding arguments it follows that such maximum number of factors in the general use of \( s \) levels of each factor is \( \frac{s^r - 1}{s - 1} \), the only point to note in such cases is that instead of doubling a column whenever necessary, the columns are to be multiplied by all the non-zero elements of the Galois field other than 1.

2.7. Choice of column for saving interactions with less than 4 factors.

For saving all three factor interactions, columns have to be so added that from any two columns no third column can be obtained either by adding (i) the two columns or (ii) any one with twice another, or (iii) twice of both the columns. Accordingly, no column in which there are only one or two non-zero elements can be taken. Further elements occupying corresponding positions in two columns should be different for at least two positions. There cannot be three columns in which elements in two corresponding positions only are different. For example,
in the following scheme for $3^4$,

\[\begin{array}{cccccc}
A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 2 & 2 \\
0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\
\end{array}\]

If a column like one below $A_5$, with only two non-zero elements be taken, the three factor interaction $A_1 A_2 A_5$ will be confounded. Again, if another column as below $A_6$ be taken such that in columns 5 and 6 the elements only in the 4th position are different, the interaction $A_1 A_2 A_5$ will be confounded.

Again, if we take three columns as below $A_6$, $A_7$ and $A_8$ in which the corresponding elements in the 3rd and 4th positions only are different, the interaction $A_1 A_2 A_5$ will be confounded as by summing any two the third one can always be obtained in such situations.

When the columns are chosen subject to the above restrictions a maximum of the following 15 columns could be added for blocks of $3^5$.

<table>
<thead>
<tr>
<th>Basic factors</th>
<th>Added factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1 A_2 A_3 A_4 A_5$</td>
<td>$A_6 A_7 A_8 A_9 A_{10} A_{11} A_{12} A_{13} A_{14} A_{15} A_{16} A_{17} A_{18} A_{19} A_{20}$</td>
</tr>
<tr>
<td>1 1 0 0 0 0 1 1 1 0 1 1 1 1 1 0 0 0 0 1</td>
<td>1 1 1 0 1 1 1 1 1 0 0 0 0 0 0 0 0 0 1</td>
</tr>
<tr>
<td>2 0 1 0 0 0 1 1 0 1 2 2 2 0 0 1 1 0 1 1 1</td>
<td>1 1 0 1 2 2 2 0 0 1 1 0 1 1 1 1 0 1 1</td>
</tr>
<tr>
<td>3 0 0 1 0 0 2 0 1 1 1 2 0 1 0 .1 0 2 0 1 2 1 1</td>
<td>2 0 1 1 1 2 0 1 0 .1 0 2 0 1 2 1 1</td>
</tr>
<tr>
<td>4 0 0 0 1 0 0 2 1 1 2 1 0 0 1 0 2 1 1 2 1</td>
<td>0 0 1 0 2 1 1 2 1 0 0 1 0 2 1 1 2 1</td>
</tr>
<tr>
<td>5 0 0 0 0 1 0 0 0 0 0 0 0 1 1 1 2 2 2 1 1 1</td>
<td>0 0 0 0 0 0 0 1 1 1 2 2 2 1 1 1</td>
</tr>
</tbody>
</table>

Through this scheme a $3^{20}$ design in blocks of $3^5$ can be obtained without confounding any interaction with less than 4 factors.
Those columns which have zero in the fifth position give the maximum number of columns (factors) for block size $3^4$ when the fifth row is omitted. Thus, the added columns $A_6$ to $A_{11}$ with the basic factors $A_1$ to $A_4$ give the maximum number of factors when the block size is $3^4$. In the case of blocks of size $3^6$, the maximum number of factors has been enumerated to be 40, while for blocks of size $3^5$ and $3^4$ it is 20 and 10 respectively. These results indicate that probably such maximum number of factors is $2^{r-1} + 2^{r-3}$ when the block size is $3^r$.

2.8. General case of $s$ levels.

In the general case if we are to get a design in which all interactions with up to $d$ letters are to be saved, each of the added columns must contain at least $d$ non-zero levels. Taking any $d$ column the sum of any $(d-1)$ columns either as they are or when multiplied by the elements of the G.F in any manner should not give rise to the remaining column. Elements in at least $(d-1)$ corresponding positions should be unequal in any two columns. The independent interaction confounded due to the introduction of any $j$-th column in which the element in the $i$-th position is denoted by $c_{ij}$ can be obtained from $\sum_i a_i^j a_j^{p-1}$ where $s = p^n$.

The upper limits of the maximum number of factors for the general case can be obtained from the same arguments as given in the same arguments as given in the case of $s = 2$, excepting that the total number of columns that can be added is to be taken as $(s^r-1)/(s-1)$ for similar reasons as advanced in the case of $s = 3$ while obtaining the columns for saving all two factor interactions and main effects.

Though all possible treatment combinations each consisting of the non-zero level of one of the factors and zero level of the rest
of the initial factors have so far been used as the basic independent treatment combinations, as a matter of fact any set of \( r \) independent treatment combinations of the initial factors can be used instead. In this case the interaction confounded due to the introduction of any column has to be obtained differently while the other procedures remain the same. When a column is added, it is always possible to find out a linear combination of the initial columns so as to give the introduced column. If this column \((A_j)\) is given by \((A_j) = \sum_{i=1}^{n} \omega_i (A_i)\) then the interaction confounded due to \(A_j\) is \(\sum_{i=1}^{n} \omega_i (A_i - A_j)\) where \((A_j)\) denotes the column vector corresponding to the factor \(A_j\).

2.6. Designs for subsequent introduction of factors.

Pearce (1953) said that for perennial experiments it becomes sometimes necessary to introduce fresh factors after an experiment has run for one or more years. The present method of construction of confounded factorial designs can be used conveniently for obtaining such designs. For this purpose a description as to how the blocks other than the key block can be obtained from the key block is necessary.

We have described earlier how the treatments in the key block for a confounded design can be obtained. Let these treatments be written in some order as rows. If there be \(2^a\) block per replication in a design with factors each at two levels, any of these \(2^a\) blocks can be obtained from the key blocks by adding any of the \(2^a\) combinations of factors each at two levels of the last \(a\) elements of each of the treatments in the key block keeping the other \((n-a)\) elements in these treatments unaltered. This procedure establishes
a correspondence among the treatments in the different blocks of a replication, viz. those treatments occupying the same position (row) in the different blocks when they are obtained as described above are corresponding treatments.

For introducing a fresh factor at any stage the number of blocks that will be taken initially has to be a power of 2, so that the number of replications that can be formed out of them for any number of factors is either a power of 2 or a fractional replicate. These replications are then grouped into pairs of replications. Now we know the position in which the level of the newly introduced factor has been added in the key block of one of the pair of replications. We have now to introduce the level 1 of this factor in the corresponding treatments in each of the blocks of this replication and the level zero has to be added to the other treatments in these blocks. Writing the treatments in the same order in each of the blocks of the other replication as in the first replication, we have to introduce zero as the level of the introduced factor in those rows in which level 1 was added. Thus out of each pair of replications a design with, say, n factors, we shall get one in the first replication and vice versa. Now a stage may reach when all the blocks constitute only one replication. At this stage introduction of levels 1 and zero has to be done as in the first of the pair of replications described above, and it will be a fractionally replicated design, the identity group containing that interaction which is confounded due to the column added for this factor. Further factors can be added likewise and the identity group will contain all the interactions that can be generated from the independent interactions obtained from the excess factors.
Though the method described is for factors at two levels, it can be extended on similar lines when the factors have other numbers of levels.

2.10 References


