1. Introduction.

Factorial experiments are in use in their present form since Fisher obtained them in 1926 and Yates (1933, 1935, 1937) extended them by introducing devices like confounding and supplying suitable methods for their analysis. The objective behind such experiments has been mainly investigations of main effects and interactions of the factors under study. Such an objective by itself seems somewhat inadequate when particularly the factors are quantitative, as in such cases a study of the relationship between the response and the levels of the factors is likely to be more useful. There have been, however, occasions when such dose-response relation has been investigated, though usually for each factor separately, through fitting of Mitscherlich and quadratic curves. Such factor by factor investigation does not evidently take into account the interactions of the factors for which it is necessary to take into account all the factors simultaneously.

The first concerted effort for obtaining experimental designs suitable for the investigation of response surface seems to have been made by Box and his co-workers. Box and Wilson (1951) introduced the method of steepest ascent for the exploration of response surface, such that with a minimum number of experiments an optimum operating set of conditions within the region of interest in a k-dimensional factor space, can be found. Box (1952) discussed the properties and construction of multifactorial first order designs. Following some further studies on the exploration of response surface, Box and Hunter (1957)
finally introduced rotatable designs. They defined the rotatable designs as follows.

Let there be $v$ variates each at $s$ levels. If a design be formed with $N$ of the $s^v$ treatment combinations, it can be written as the following $N \times W$ matrix where the elements $x_{iu} (i = 1,2,.. v, u = 1,2,..N)$ denotes the level of the $i$-th factor in the $u$-th treatment combination on a suitably chosen origin and scale:

$$
\begin{bmatrix}
  x_{11} & x_{21} & \cdots & x_{v1} \\
  x_{12} & x_{22} & \cdots & x_{v2} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{1N} & x_{2N} & \cdots & x_{vN}
\end{bmatrix}
$$

Such a matrix will be called a design matrix and the treatment combinations in it, the design points. The point $(0 \ 0 \ \cdots \ 0)$ defines the centre of the design and will be called the central point. For convenience, let us associate the variate $x_{1}$ with the $i$-th factor for denoting its levels. According to Box and Hunter (1957) a design of the above form will be a rotatable design of order $d$ if a polynomial response surface,

$$
y = b_0 + \sum_{i} b_i x_{i} + \sum_{i} b_{ij} x_{i} x_{j} + \sum_{i} b_{ijk} x_{i} x_{j} x_{k} + \cdots,
$$

of order $d$ of the response $y$ obtained from the treatments, on the variates $x_{i}$ ($i = 1,2,\ldots,v$) can be so fitted that the variance of the estimated response of any treatment combination is a function of the sum of squares of the levels of the factors.
in that treatment combination. In other words, the variance of the estimated response at any point is a function of the distance of the point from the centre of the design so that the variances of estimated responses at points equidistant from the centre are the same.

Box and Hunter (1957) used geometrical configurations in forms of regular and semi-regular solid figures for the construction of several second order rotatable designs. Afterwards, Gardiner and others (1959) constructed several third order designs for 2, 3 and 4 factors. Bose and Draper (1959) obtained several second order designs adopting a different procedure. Draper (1960 a,b) constructed an infinite series of second order designs in three and more factors and then obtained some third order designs in 3 and 4 factors. Box and Behnken (1960 a) obtained a class of second order designs from first order designs and subsequently (1960 b) gave a method of construction of second order designs by using balanced incomplete block designs with \( r = 3 \). The present author also constructed both second and third order designs through factorial and balanced incomplete block designs with equal as also unequal block sizes (Das, 1961, 1962, 1963). This work which has since been published together with some further investigations on the topic, has been included in the present thesis. Construction and analysis of second order rotatable
designs have been discussed first and this has been followed by a similar discussion regarding third order rotatable designs.

1.2. Second order rotatable designs.

Second order and other rotatable designs serve two main purposes - one is that through them the response surface can be fitted with least involvement in the solution of the normal equations obtainable through least squares technique for the estimation of the parameters in the equation of the surface and the other ensures the equivariance of the responses estimated from the surface fitted through the design at points equidistant from the centre of the design.

Denoting a second degree polynomial response surface by

\[ y = b_0 + \sum b_i x_i + \sum b_{ij} x_i x_j \]

the normal equations for fitting the surface through least squares, will be the least involved if the design points are so chosen as to satisfy the following relations:

Relation A: \[ \sum x_{iu} = 0, \sum x_{iu} x_{ju} = 0, \sum x_{iu} x_{ju} x_{ku} = 0, \sum x_{iu} x_{ju} x_{ku} x_{lu} = 0 \]

Relation B: (i) \[ \sum x_{iu}^2 = \text{Constant} = N \lambda 2. \]

(ii) \[ \sum x_{iu}^4 = \text{Constant} = cN \lambda 4. \]
Relation C: \[ \sum_{u} x_{iu}^2 x_{ju}^2 = \text{Constant} = N \lambda_4. \]

From relations B(\(\text{ii}\)) and C it will be seen that
\[ \sum_{u} x_{iu}^4 / \sum_{u} x_{iu}^2 x_{ju}^2 = c, \text{ a constant.} \]

In future, we shall denote the summations like \( \sum_{u} x_{iu} \) simply by \( \sum x_1 \) meaning thereby that such summations are over the design points and not over the factors.

Under these restrictions the variance of an estimated response \( y_o \) at the point \((x_{10}, x_{20}, \ldots, x_{v0})\) comes out as
\[
\text{Var} (y_o) = \text{Var} (b_0) + d^2 \left\{ \text{Var}(b_i) + 2 \text{Cov}(b_0, b_{ij}) \right\} + d^4 \text{Var} (b_{ij}) + \frac{c-3}{c-1} \text{Var} (b_{ij}) \sum_{i \neq j} x_{i0} x_{j0}^2
\]
where \( \sum x_{i0}^2 = d^2 \), \( \text{Var}(b_o) = \frac{\lambda_4 (v+1-c-1) \sigma^2}{N \{ \lambda_4 (v+c-1) - v \lambda_2^2 \}} \),
\[
\text{Cov} (b_0, b_{ij}) = \frac{-\lambda_2 \sigma^2}{N \{ \lambda_4 (v+c-1) - v \lambda_2^2 \}} ,
\]
\[
\text{Var} (b_{ij}) = \frac{\lambda_4 (v+c-2) - (v-1) \lambda_2^2}{(c-1)N \lambda_4 \{ \lambda_4 (v+c-1) - v \lambda_2^2 \}} \sigma^2 .
\]
\[
\text{Var} (b_i) = \frac{\sigma^2}{N \lambda_2} \quad \text{and} \quad \text{Var} (b_{ij}) = \frac{\sigma^2}{N \lambda_4} .
\]

It will be seen that the variance of \( y_o \) will depend only on \( d^2 \) and on no other function of the co-ordinates of the
point \((x_{10}, x_{20}, \ldots, x_{v0})\) when \(c = 3\).

Thus, the further restriction viz.

Relation D: \(\sum x_i^4 = 3 \sum x_i^2 x_j^2\)

makes the design rotatable when the other relations are already satisfied.

Under this restriction the variance of an estimated response at a point whose distance from the centre of the design is \(d\), becomes

\[
\frac{N \text{Var}(y_0)}{\sigma^2} = \frac{\lambda_4(v + 2)}{\lambda_4(v+2)-v \lambda_2^2} + d^2 \left\{ \frac{1}{\lambda_2} - \frac{2 \lambda_2}{\lambda_4(v+2)-v \lambda_2^2} \right\} + \frac{\lambda_4(v+1) - (v-1) \lambda_2^2}{2 \lambda_4 \left\{ \lambda_4(v+2)-v \lambda_2^2 \right\}}
\]

It will be seen from the expression of \(\text{Var}(b_0)\) that \(\frac{\lambda_4(v+2)-v \lambda_2^2}{\lambda_4(v+2)-v \lambda_2^2}\) must be positive. Hence, the design points must satisfy the further restriction viz.

Relation E: \(\frac{\lambda_4}{\lambda_2^2} > \frac{v}{v+2}\)

It has been shown by Box and Hunter (1957) that \(\lambda_4(v+2) = v \lambda_2^2\) when the design points are equidistant from the centre. Hence by adding at least one central point to the design, this relation, E can always be satisfied. It will be
noticed that by adding central points to the design none of the other relations are affected but only \( N \) is increased.

Further in order to have equal dispersion of the transformed levels of each of the factors, \( \lambda^2 \) is usually taken as unity.

### 3. Method of construction of Second order Rotatable designs.

As rotatable designs are essentially incomplete factorials, each point in a design is a combination of the levels of different factors. We thus propose first to take some unknown levels to be denoted by \( a, b, c \), etc excepting that some of them can be zero also, and to get a factorial design in \( v \) factors out of these unknown levels. Thus, if there are four factors each at two levels denoted by \( a \) and \( b \), the 16 combinations will be of the form

\[
\begin{align*}
    &a \quad a \quad a \quad a \\
    &a \quad b \quad b \quad b \\
    &b \quad a \quad a \quad b \\
    &a \quad a \quad b \quad b \\
    &\vdots & \quad \vdots & \quad \vdots \\
    &\vdots & \quad \vdots & \quad \vdots \\
    &\text{etc.}
\end{align*}
\]

Next, we shall have another design in \( v \) factors of the form \( 2^v \) where the two levels of each factor are +1 and -1. We now get one more set of combinations when any combination of the first design is associated with any combination of the second design, \( 2^v \), by multiplying the corresponding entries, that is, the levels of the same factor in the two combinations and writing the products in the same order. This method of association of any two combinations of the two designs will
hereafter be called multiplication.

For example, in the design with four factors if the combination $abbb$ be multiplied by each of the $2^4$ combinations of the levels +1 and -1, we shall get the following 16 combinations.

- $abbb$ when multiplied by $1111$
- $a b - b b$ when multiplied by $11-11$
- $a b b - b$ when multiplied by $111-1$
- $a b - b - b$ when multiplied by $11-1-1$

etc.

If one of the unknown levels, say, $a$ be zero, all the 16 combinations will not be distinct, but only 8 of them will be distinct, as by associating +1 and -1 with zero we get the same thing. We shall consider, in future, only those combinations which are distinct unless otherwise mentioned. Thus, by multiplying any combination of the first design with each of the combinations of the second design $2^p$, we shall get $2^p$ distinct combinations where $p$ denotes the number of the non-zero unknown levels in the combination considered of the first design. As a matter of fact if there be only $p$ unknowns in a combination together with some zeros, we have to multiply only the non-zero levels in the combination with each of the $2^p$ combinations of +1 and -1.
We have by now come across three types of combinations, namely, (i) factorial combinations of the unknown levels $a, b, \ldots$ together with zero; (ii) factorial combinations of levels $+1$ and $-1$ and (iii) combinations when each of $o, a, b$ etc. is associated with $+1$ and $-1$ through multiplication.

The first type of factorial combinations will be called combinations of unknown levels and the second will be called associate combinations. The third combinations will actually constitute the design points and hence they will be referred to as the design points.

Though originally we had the unknown levels as $a, b, c, \ldots$, the unknown levels in the design points will be $a, -a, b, -b, c, -c, \ldots$ together with zero. It can be seen easily that if a design be formed by including all the distinct points which are got by multiplying any combination of the unknown levels with all the associate combinations, these points will always satisfy the relation $A$. As a matter of fact if we have only $n$ associate combinations from the design $2^v$ such that no main effect is confounded in these $n$ combinations then the sum of products like $\sum x_1, \sum x_1^2 x_2, \sum x_1 x_2 x_3$ etc will be zero for each factor when the design points are obtained by multiplying any combination of the unknown levels with these $n$ associate combinations. Similarly if no two-factor interaction is confounded in the $n$ associate combinations, all sum of products like $\sum x_1 x_2$,
\[ \sum_{i} x_i x_j x_k \text{ and } \sum_{i} x_i^3 x_j \text{ will be zero. If again no 3-factor interaction is confounded in them, all sum of products } \sum_{i} x_i x_j x_k \text{ will be zero. Finally, if no 4-factor interaction is confounded in the } n \text{ associate combinations, all sum of products } \sum_{i} x_i x_j x_k x_l \text{ will be zero. Thus, when } v > 4 \text{ or } p > 4, \text{ the relation A will also be satisfied when a suitable fraction of } 2^v \text{ or } 2^p \text{ associate combinations, as the case may be, are so chosen for multiplication that no interaction with less than five factors is included in the identity group of interactions for obtaining the fractional replicate.}

It will be seen that if only one combination of the unknown levels be taken, though relation A will be satisfied by the design points obtained from it through multiplication by the requisite associate combinations, all the other relations cannot be satisfied by them. If the design points be obtained through a homogeneous combination like \((a \ a \ \ldots \ a)\) that is, with the same level for each factor, these points will satisfy relations B and C but not D and E.

In general relations B have to be satisfied by suitably choosing one or more combinations of unknown levels, so that each of the unknown levels in the design points occurs against each factor in equal numbers. For example, in the case of 2 factors if we choose the two combinations of unknown levels \((1)\ ab \text{ and } (11)\ ba\), relations A and B will be
satisfied by the design points obtained from them. Though sometimes relations C and D can be satisfied by suitably choosing combinations of unknown levels, it is not necessary that they have to be satisfied by proper choice of combinations of unknown levels. Where they are not satisfied by choice of unknown combinations which in some cases may not also be possible, they provide equations in terms of the unknown levels by virtue of these relations such that the solution of these equations provide values for the unknown levels for which alone the relations hold. Thus, if we take the combinations (a b)and(b a) relation C does not arise as there are only two factors but we have \[ \sum x_1^4 = 4(a + b^4), \text{ and } \sum x_1^2 x_j^2 = 8 a^2 b^2. \]

Hence relation D requires that \( 4(a^4 + b^4) = 3 \times 8 a^2 b^2 \) and this provides an equation involving the unknown levels a and b. Another equation involving the unknown levels can always be obtained from the relation \( B(1) \), namely \( \sum x_1^2 = N. \)

One or more equations are obtained from relation C when \( \sum x_1^2 x_j^2 \) takes two or more values for different pairs of factors. If the number of unknown levels be equal to the number of equations thus obtained, they are evaluated through the solution of these equations. If, again, the number of unknown levels be more than the number of equations, some of the unknown levels have to be fixed arbitrarily, thus providing an infinite series of solutions, leading to as many designs.
1.4. Some concepts and notations.

The set of $v$ combinations involving one unknown level $a$ viz.

\[
\begin{array}{c}
  a \circ \circ \cdots \circ \\
  \circ a \circ \cdots \circ \\
  \circ \circ a \cdots \circ \\
  \cdots \cdots \\
  \circ \circ \circ \cdots a
\end{array}
\]

can be obtained from the initial combination $(a \circ \circ \cdots \circ)$ by cyclically changing the unknown levels over the factors. The $v$ combinations, obtained in this way, will in future, be denoted by writing only the initial combination $(a \circ \circ \cdots \circ)$ and will be termed combinations of the type $(a \circ \circ \cdots \circ)$.

The design points obtainable through such combinations always satisfy all relations excepting $D$ and $E$. In general, given any combination of some unknown levels, a number of other $(v-1)$ combinations of the same unknown levels occurring in the same sequence can be obtained from the initial one by cyclically changing the unknown levels over the factors. Thus, if we have the initial combination $(a \ b \ c \ d)$, we can obtain three other combinations viz. $(b \ c \ d \ a)$, $(c \ d \ a \ b)$ and $(d \ a \ b \ c)$ by cyclically changing the unknown levels over the factors. This procedure of obtaining combinations from an initial combination will be called rotation and the set of combination obtained by rotation will be denoted by its initial combination. It will be seen easily that rotation ensures that the design
points obtainable from all the combinations got by rotation will satisfy relations B.

1.5. **Some choices of combinations of unknown levels leading to second order designs through rotation and multiplication.**

Several choices of combinations of unknown levels for obtaining second order rotatable designs have been discussed below.

(i) **Designs with three levels.**

A second order design in \( v \) factors can always be obtained from the following combinations of unknown levels:

1. \((a \ a \ a)\)
2. \((a \ o \ o \ o)\)

provided the second set of combinations is repeated \( n \) times where \( n \) is the number of associate combinations used for multiplying the combination \((a \ a \ ... \ a)\) for generating the design points. For these sets:

\[
\sum x_i^2 = 3na^2
\]
\[
\sum x_i^4 = 3na^4
\]
and \(\sum x_i^2x_j^2 = na^4\).

Hence, all the relations including \( D \) are automatically satisfied. The number of points in the design is \( N = (2v+1)n\).

The value of \( a \) can be obtained from the relation \( \sum x_i^2 = 3na^2 = N \).

As \( n \), the number of associate combinations required for multiplication in which no interaction with less than 5 factors is confounded varies with \( v \) and there is no
definite known relation connecting \( n \) and \( v \), the problem of determining the \( n \) requisite associate combinations remains. A solution to this problem can be obtained from the knowledge of the maximum number of factors each at 2 levels that can be accommodated in a given block size without confounding any interaction with less than 5 factors. It is known from existing literature that the values of \( n \) for different values of \( v \) up to 12 are as shown below:

\[
\begin{array}{cccccccc}
v & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
n & 2^4 & 2^5 & 2^6 & 2^7 & 2^7 & 2^7 & 2^8 \\
\end{array}
\]

From the investigation presented in the third section of this thesis containing an alternative approach for the construction of confounded symmetrical factorial designs, it has been established that when \( v \) or \( p \) varies from 12 to 17, the number of associate combinations to be taken is \( 2^8 \). Hence fractional replicates of the orders \( 1/4 \) to \( 1/9 \) have to be used when the number of factors increases from 12 to 17. The identity group of interactions for obtaining the fractional replicate \( \frac{1}{2^9} \left( 2^{17} \right) \) can be obtained from the generalised interactions of the following 9 independent interactions involving 17 factors: \( ABCDI, ABEFJ, ACEGK, BDFGL, AFGHM, DEGHN, BCFHO, BCDHOP \) and \( ACDEFGHQ \).

The \( 2^8 \) associate combinations for any value of \( v \) or \( p \) from 12 to 17 can be obtained from the \( 2^8 \) associate combinations
of the $2^{17}$ design by retaining from the beginning as many columns as there are factors. For example, for 17 factors we shall have a $(256 \times 17)$ matrix of elements 0 and 1 from the 256 treatments each containing the levels of 17 factors. (From such combinations written by using 0 and 1 as the levels, the associate combinations can be obtained by replacing 0 by -1.) By retaining only the first 13 columns, we shall get 256 treatment combinations out of 13 factors which will be a $1/2^5(2^{13})$ fractional replicate in the identity group of which there is no interaction with less than 5 factors.

To simplify matters further we have given below the 8 independent treatment combinations of 17 factors from which the $2^8$ treatment combinations of the fractional replicate can be generated by following the usual method.

```
1 0 0 0 0 0 0 0 1 1 1 0 1 0 0 1 1
0 1 0 0 0 0 0 0 1 0 1 0 1 0 1 1 1
0 1 0 0 0 0 0 1 0 0 1 0 0 1 1 0
0 0 1 0 0 0 0 1 0 1 0 0 0 1 1 0
0 0 0 1 0 0 0 0 1 0 0 1 0 1 0 1 1 1
0 0 0 0 1 0 0 0 0 1 1 0 1 0 1 1 1 1
0 0 0 0 0 1 0 0 0 1 1 1 0 0 1 1 1 1
0 0 0 0 0 0 1 0 1 0 0 0 1 1 1 1 1 1
```

Coming back to our original discussion on the construction of rotatable designs with 3 levels of each factor we have given below the number of points required for some of the rotatable designs showing side by side the number of points in the corresponding factorials of which the rotatable design is a fraction.


<table>
<thead>
<tr>
<th>v</th>
<th>No. of points in the rotatable design</th>
<th>No. in the corresponding factorials</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>56</td>
<td>27</td>
</tr>
<tr>
<td>4</td>
<td>144</td>
<td>81</td>
</tr>
<tr>
<td>5</td>
<td>126</td>
<td>243</td>
</tr>
<tr>
<td>6</td>
<td>416</td>
<td>729</td>
</tr>
<tr>
<td>7</td>
<td>960</td>
<td>2187</td>
</tr>
<tr>
<td>8</td>
<td>1088</td>
<td>6561</td>
</tr>
<tr>
<td>9</td>
<td>2432</td>
<td>19683</td>
</tr>
<tr>
<td>10</td>
<td>2688</td>
<td>59049</td>
</tr>
<tr>
<td>11</td>
<td>2944</td>
<td>177147</td>
</tr>
</tbody>
</table>

This shows that though for 3 and 4 factors the number of points in the rotatable designs with 3 levels are greater than those in the corresponding factorials, for larger number of factors the rotatable designs in 3 levels are just fractional replicates of the corresponding factorials.

(2) Designs with two unknowns a and b.

(1) It is always possible to get second order rotatable designs for v factors from the set of combinations obtainable through rotation from \((a \ b \ b \ \ldots \ b)\). These designs points obtainable from them satisfy all relations excepting D and E. Taking \(n\) to be the number of associate combinations for multiplying each of the combinations of unknown levels, we get

\[
\sum_{i=1}^{4} x_i^4 = n a^4 + (v-1)n b^4
\]

and

\[
\sum_{i=1}^{4} x_i^2 x_j^2 = 2 n a^2 b^2 + (v-2)n b^4
\]

Hence, relation D gives the equation

\[
a^4 + (v-1)n b^4 = 6 n a^2 b^2 + 3(v-2)n b^4.
\]
Putting \( \frac{a^2}{b^2} = s \), the equation reduces to
\[
s^2 + v-1 = 6s + 3(v-2)
\]
Hence \( s = 3 \pm \sqrt{4+2v} \).

As all these points are equidistant from the centre, at least one central point has to be added to the design. The number of points in the design is \( vn \) plus the number of central points. From relation B(i), \( b \) comes out as
\[
b^2 = \frac{N}{s + v - 1} n.
\]
These designs have actually five levels, \(-b, -a, 0, a\) and \(b\).
This series of designs was obtained by Box and Hunter (1957) through geometrical methods.

(ii) Another series of designs with 5 levels in \( v \) factors can be obtained from the combinations (l) \((a a ... a)\) and (ii) \((b, o, ... o)\)

All the relations excepting D will be satisfied by the design points obtainable from the above combinations. Here
\[
\sum x_1^4 = na^4 + 2b^4
\]
and \( \sum x_1^2 x_j^2 = na^4 \).
Hence relation D gives the equation
\[
na^4 + 2b^4 = 3 na^4,
\]
which gives \( \frac{a^2}{b^2} = s = 1/\sqrt{n} \).

In such designs the number of points is \( N = n + 2v \) and no central point need be added for satisfying relation E excepting when \( v = 2 \) and \( 4 \), as in these two cases the
distances of all the points from the centre become equal. The value of \( b^2 \) in this case is \( N/(ns+2) \). These designs are called central composite designs and were first obtained by Box and Hunter (1957) through geometrical methods.

(iii) One more series of designs with 5 levels can be obtained for \( v \) factors from the combinations

(i) \((a \ a \ \ldots a)\)

and (ii) \((b \ b \ \ldots o)\)

when the second set is permuted over the factors in all possible ways giving \( v(v-1)/2 \) combinations in all. The number of non-central points in the design is \( n+2v(v-1) \).

Relation \( D \) gives the equation

\[
na^4 + 4(v-1)b^4 = 3 na^4 + 12b^4.
\]

i.e. \( s = a^2/b^2 = 2(v-4)/n \).

When \( v = 4 \), \( a \) becomes zero and hence a design in 4 factors is available from the second set only with 24 points together with at least one central point. The value of \( b \) can be obtained from \( N = b^2 \{ns + 4(v-1)\} \).

(3) Some specific designs with smaller number of points.

(i) A design in 3 factors with 12 non-central points can be obtained from the set of combination \((oab)\) through rotation and multiplication. Relation \( D \) gives here the equation

\[
8(a^4 + b^4) = 12 a^2 b^2.
\]

Taking \( s = a^2/b^2 \), it becomes

\[
s^2 + 1 = 3s
\]

and hence \( s = 3 \pm \sqrt{5}/2 \).
As there are two positive values of \( s \), two designs \( x_i \) are possible. The design has 12 non-central points with at least one central point.

(ii) A design in 4 factors with 32 non-central points can be obtained from the set of combinations \((oacb)\) through rotation and multiplication. For this design
\[
\sum x_i^4 = 8(a^4+b^4+c^4)
\]
and
\[
\sum x_i^2 x_j^2 = 8a^2c^2 + 8b^2c^2
\]
\[
= 16a^2b^2,
\]
as \( \sum x_i^2 x_j^2 \) takes two types of values for different pairs of factors.

Hence, from relation C we get the equation
\[
a^2c^2 + b^2c^2 = 2a^2b^2
\]
and from relation D,
\[
a^4 + b^4 + c^4 = 6a^2b^2.
\]
Putting \( s = a^2/c^2 \), and \( t = b^2/c^2 \), these reduce to
\[
s + t = 2st
\]
and \( s^2 + t^2 + 1 = 6st \).

Solving them we get \( s = 3.137 \) and \( t = .595 \).

The unknown \( c \) has to be obtained from the relation
\[
8c^2(s + t + 1) = N.
\]

Here \( N \) contains at least one central point in addition to the 32 non-central points.
(iii) By generating the design points from the set of unknown combinations (ooacb) in the case of 5 factors, a design can be obtained in 40 non-central points with at least one central point.

For this design
\[ \sum x_i^4 = 8(a^4 + b^4 + c^4) \]
\[ \sum x_i^2 x_j^2 = 8(a^2 c^2 + b^2 c^2) \]
\[ = 8 a^2 b^2 \]

Hence from relations C and D we get the equations
\[ a^2 c^2 + b^2 c^2 = a^2 b^2 \]
\[ a^4 + b^4 + c^4 = 3 a^2 b^2 \]

Putting \( s = a^2/c^2 \) and \( t = b^2/c^2 \)
we get \( s + t = st \)
and \( s^2 + t^2 + 1 = 3 st \)
whence \( s = 3.3706 \) and \( t = 1.4206 \).

The value of \( c \) can be obtained from
\[ 8c^2 (s + t + 1) = N \]

(iv) A design in 6 factors can be obtained from the set of combinations (ocoaobc).

For this design relation C gives the equation
\[ a^2 b^2 = b^2 c^2 = 2a^2 c^2 \]
and relation D,
\[ a^4 + b^4 + c^4 = 6 a^2 c^2 \]
Putting $s = a^2/c^2$ and $t = b^2/c^2$, they become

\[ st = t = 2s \]

and

\[ s^2 + t^2 + 1 = 6s \]

The solutions are $s = 1$ and $t = 2$.

The design has 48 non-central points and at least one central point is necessary. As $s=1$, we get $a = c$ and hence the design will have only 5 levels.

(v) A design for 7 factors with 56 non-central points can be obtained from the set of combinations $(oooaboc)$.

For this design relation C gives the equation

\[ a^2 b^2 = b^2 c^2 = c^2 a^2 \]

and relation D,

\[ a^4 + b^4 + c^4 = 3 a^2 c^2 \]

Putting $s = a^2/c^2$, $t = b^2/c^2$, they become

\[ st = t = s \]

and

\[ s^2 + t^2 + 1 = \frac{2}{3} 3s \]

The solutions are $s = t = 1$.

This design has 56 points and at least one central point is necessary. The solutions $s = t = 1$, implies $a = b = c$ and hence there are only three levels of each of the factors in this design.

(vi) A design in 8 factors can be obtained from the following unknown combinations.

(i) $(oooabocd)$

and (ii) $(2d/p)$ ooooooo
Relations C and D give the equations

(i) \[ a^2b^2 + b^2c^2 = a^2c^2 + c^2d^2 = b^2d^2 = 2a^2d^2 \]
and (ii) \[ a^4 + b^4 + c^4 + d^4 = 2p^2d^4 = 6a^2d^2 \]

Putting \[ \frac{a^2}{d^2} = s, \quad \frac{b^2}{d^2} = t, \quad \frac{c^2}{d^2} = u, \]
they become

\[ st + tu = su + u = t = 2s \]
and \[ s^2 + t^2 + u^2 + 4p^2 = 6s. \]

The solutions are \( s = (\sqrt{2} - 1), \quad t = 2(\sqrt{2} - 1), \quad u = \sqrt{2}(\sqrt{2} - 1) \)
and \( p = .363. \)

This design has 144 points and no central point is necessary.

1.6. **Second order designs through balanced incomplete block designs.**

Let there be a balanced incomplete block design with parameters \( (v, b^*, r, k, \gamma) \). As the symbol \( b \) has been used to denote an unknown level of the factors, the symbol \( b^* \) has been used here to denote the number of blocks in the B.I.B design. Let us write the design in the form of \( a(b^* \times v) \) matrix, the elements of which are 0 and \( a \). If in any block a particular treatment occurs, the element in that block corresponding to that treatment will be \( a \), otherwise zero. Each row of the matrix corresponding to a block of the B.I.B design can be considered to give a combination containing the unknown level \( a \) and zero.
By multiplying each of these $b^*$ combinations thus obtained through the B.I.B design(s) with 2, since $p = k$ here, or a suitable fraction of the associate combinations, we shall get a number of design points less than or equal to $b^* \times 2^k$. These points which we shall denote as $a=(v, b^*, r, k, \lambda) x 2^k$ (or a suitable fraction of $2^k$) will satisfy all relations except D and E, as constancy of replication of any treatment will satisfy relation B and that of any pair of treatments will satisfy relation C.

This method of construction of rotatable design was included in a paper a reprint for which has been appended with this thesis as paper No. 2. After the manuscript of this paper was submitted for publication, the author's attention was drawn to the work of Box and Behnken (1960 b) who also used B.I.B designs with $r = 3\lambda$ for the construction of second order rotatable designs. They did not, however, present the method of construction of rotatable designs using any B.I.B. designs.

If in a B.I.B design, $r = 3\lambda$, relation D is automatically satisfied. As in such a design all the points are equidistant from the centre, at least one central point has to be taken for satisfying relation E.

When $r \neq 3\lambda$ in any B.I.B design, we can always get a second order rotatable design through it by taking some
more combinations involving one more unknown level, b and then by multiplying them with the requisite number of associate combinations. The combinations to be taken are either the \( v \) combinations \((b, o, ... o)\) or the homogeneous combination \((b, b ... b)\) according as \( r < 3 \lambda \) or \( r > 3 \lambda \). The design points obtainable through the combinations \((b, o, o, ... o)\) and combination \((b, b ... b)\) after multiplication by the requisite associate combinations will hereafter be denoted respectively as \((b, o, o, ... o) \times 2^1\) and \((b, b ... b) \times \) suitable fraction of \( 2^\lambda \), though this type of notation has not been used earlier.

In the above designs, \( \sum x_1^4 \) and \( \sum x_i^2 \) will be functions of \( a \) and \( b \). From the relation \( \sum x_i^4 = 3 \sum x_i^2 \), we shall get an equation connecting \( a \) and \( b \). This equation will always give a positive solution of \( a^2/b^2 \) provided that the extra combinations are suitably chosen taking into account which of \( r < 3 \lambda \) or \( r > 3 \lambda \) holds. For determining the unknown levels \( a \) and \( b \), we have as before another equation, viz., \( \sum x_i^2 = N \), where \( N \) is the total number of points including the central points.

A number of second order rotatable designs could be obtained by using B.I.B designs. Some of these designs with up to 16 factors have been presented in appendix I of paper No. 2 appended with this thesis.
Second order designs through B.I.B designs with unequal block sizes. (Das, 1963)

Let \((v, b^*, r, k_1, k_2, \ldots k_m, \gamma)\) be the parameters of a B.I.B design with unequal block sizes corresponding to an arrangement of \(v\) treatments in \(b^*\) blocks such that

(i) \(b_1^*\) of them are of size \(k_1 (l = 1, 2, \ldots m)\),

(ii) the replication of each treatment in the set of \(b_1^*\) blocks is \(r_1\) so that \(r = \sum r_1\) gives the total number of replications in the design,

and (iii) any two treatments occur together in a block \(\gamma\) times in the whole design.

Thus, for this design, \(b^* = \sum b_1^*\), \(r = \sum r_1\) and \(vr_1 = b_1^* r_1\) \((l=1, 2, \ldots m)\). This type of B.I.B design with unequal block sizes may be called block-wise divisible B.I.B designs and all the B.I.B designs with unequal block sizes presented subsequently refer to such designs only. Let us now take a block of this design and a row showing the numbers of the \(v\) treatments written in order. By writing an unknown quantity, say \(a_1\), if the block taken be of size \(k_1 (l = 1, 2, \ldots m)\) below those treatment numbers in the row which occur in the block and zero elsewhere, we shall get a combination of the unknown level \(a_1\) in which \(a_1\) occurs \(k_1\) times and zero \((v-k_1)\) times. This method of obtaining combinations of unknown levels through the blocks of a B.I.B design will also be referred to in
future as associating unknown levels to the blocks.

Obtaining such combinations of unknown levels out of each of the $b$ blocks, we shall get $b$ combinations such that $b_1$ of them obtained from the blocks of size $k_1$, will contain the unknown level $a_1$, $b_2$ of them obtained from the blocks of size $k_2$ will contain the unknown level $a_2$, and so on, there being in all $m$ unknown levels $a_1, a_2, \ldots, a_m$ in these combinations. When the design points are obtained from these combinations through multiplication by the requisite number of associate combinations, they will satisfy both relations A and B. Relation B will be satisfied as the replication of each treatment in the set of blocks of any particular size is constant, though it may vary from size to size.

Through these design points relation $C$ will not be satisfied in general, but $\sum x_i^2 x_j^2$ may take several values involving the unknown levels for different pairs of $i$ and $j$ instead of being constant. If the number of values of $\sum x_i^2 x_j^2$ does not exceed $m$, relations $C$ can always be satisfied by obtaining $(m-1)$ equations from the $m$ values of $\sum x_i^2 x_j^2$ by equating them mutually. These equations will involve the unknown levels which can be evaluated through Solution. Thus only for the set of unknown values obtained from this solution relations $C$ will be satisfied.
If the number of values of $\sum x_i^2 x_j^2$ be less than $m$, there will be infinite number of solutions for the unknown levels which will satisfy relations $C$.

It will be shown later on that if the relation $r = 3\lambda$ is satisfied in a B.I.B design with unequal blocks, relation D will automatically be satisfied and no further sets need be taken outside those obtained from the B.I.B design for constructing the rotatable design. In this case the $m$ unknown levels will be solved out from the $(m-1)$ equations obtained through relations $C$ and one more equation, $\sum x_i^2 = N$. If the relation $D$ is not satisfied automatically, that is, without solution of the equation obtained from $\sum x_i^4 = 3\sum x_i^2 x_j^2$, one more unknown level has to be introduced by taking one of the following two sets of combinations of unknown levels viz. (i) $(a a \ldots a)$ in terms of the unknown level $a$ or (ii) the $v$ sets $(a o o \ldots o)$.

Set (i) or (ii) will be taken according as $r > \lambda$ or $\lambda$. In such cases one equation will be obtained through the relation $D$ and hence there will be $(m + 1)$ unknowns with as many equations.

For the design points obtained through the B.I.B design described earlier

$$\sum x_i^4 = \sum_{l=1}^{m} 2^{pl} r_i a_i^4.$$
where \( 2_{P_t} \) denotes the number in a suitable fraction of \( 2 \) associate combinations required for multiplying the set of unknown levels involving \( a_t \):

\[
\sum x_i x_j = \sum_{l=1}^{m} 2_{(\lambda_{ij})_l} a_t^4
\]

where \( (\lambda_{ij})_l \) denotes the number of blocks of size \( k \) in each of which both the \( i \) and \( j \) treatments occur together.

### 1.3. Methods of construction of B.I.B designs with unequal block sizes.

Several methods of obtaining B.I.B designs with unequal blocks have been described below. The method of obtaining second order rotatable designs through such designs has also been indicated briefly together with illustrations.

(i) Construction of B.I.B designs with two block sizes through B.I.B designs with equal blocks.

If from a B.I.B design \((v, b^*, r, k, \lambda)\), a particular treatment is omitted, we shall get another B.I.B design in blocks of sizes \( k \) and \( k-1 \) with parameters \((v-1, b^*, r, k, k-1, \lambda)\) such that in \( r \) blocks each of size \((k-1)\) each treatment will be replicated \( \lambda \) times while in the remaining \((b^*-r)\) blocks each of size \( k \), each treatment will be replicated \((r-\lambda)\) times, (Kishen, 1940).

Of all such designs we shall consider for the purpose of construction of second order rotatable designs only those
designs in which any pair of treatments occurs together in a block either (i) \(\lambda_1\) times in the blocks of size \(k\) and \(\lambda - \lambda_1\) times in the remaining blocks of size \(k-1\), or (ii) \(\lambda_2\) times in the blocks of size \(k\) and \(\lambda - \lambda_2\) times in the remaining blocks of size \(k-1\), excluding all other designs, if any at all.

When the design points are obtained through such designs, associating \(a_1\) to blocks of size \(k\) and \(a_2\) to those of size \(k-1\), \(\sum x_i^2 x_j^2\) will have the following two values:

\[
\begin{align*}
(1) & \quad 2p_1 \lambda_1 a_1^4 + 2p_2 (\lambda - \lambda_1) a_2^4 \\
(2) & \quad 2p_1 \lambda_2 a_1^4 + 2p_2 (\lambda - \lambda_2) a_2^4.
\end{align*}
\]

Hence, relation C will be satisfied if

\[
2p_1 \lambda_1 a_1^4 + 2p_2 (\lambda - \lambda_1) a_2^4 = 2p_1 \lambda_2 a_1^4 + 2p_2 (\lambda - \lambda_2) a_2^4.
\]

i.e. if \(2p_1 a_1 = 2p_2 a_2\).

As \(\sum x_i^4 = 2p_1 (r - \lambda) a_1^4 + 2p_2 \lambda a_2^4\), relation D will be satisfied if

\[
2p_1 (r - \lambda) a_1^4 + 2p_2 \lambda a_2^4 = 3\left(2p_1 \lambda a_1^4 + 2p_2 (\lambda - \lambda_1) a_2^4\right)
\]

i.e. if \(r = 3\lambda\).
If the relation \( r = 3 \lambda \) is not satisfied, we have to take further sets as indicated earlier.

For illustration we have presented the construction of a design in 6 factors.

By omitting treatment 7 from the B.I.B design:
\( v = b^* = 7, r = k = 3, \lambda = 1 \), we get the following design in 6 factors.

\[
\begin{array}{cccccc}
1 & 2 & 4 \\
2 & 3 & 5 \\
3 & 4 & 6 \\
5 & 6 & 1 \\
4 & 5 \\
6 & 2 \\
1 & 3 \\
\end{array}
\]

The 7 combinations of unknown levels will thus be

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
a_1 & a_1 & 0 & a_1 & 0 & 0 & \\
0 & a_1 & a_1 & 0 & a_1 & 0 & (From blocks of size 3.) \\
0 & 0 & a_1 & a_1 & 0 & a_1 & \\
a_1 & 0 & 0 & 0 & a_1 & a_1 & \\
\hline
0 & 0 & 0 & a_2 & a_2 & 0 & \\
0 & a_2 & 0 & 0 & 0 & a_2 & (From blocks of size 2.) \\
a_2 & 0 & a_2 & 0 & 0 & 0 & \\
\end{array}
\]
When the first block is multiplied by 2, associate combinations we get the following 8 design points:

\[
\begin{align*}
1 & & 1 & & 0 & & 1 & & 0 \\
1 & & 1 & & 0 & & -1 & & 0 \\
1 & & -1 & & 0 & & 1 & & 0 \\
1 & & -1 & & 0 & & -1 & & 0 \\
-1 & & 1 & & 0 & & 1 & & 0 \\
-1 & & 1 & & 0 & & -1 & & 0 \\
-1 & & -1 & & 0 & & 1 & & 0 \\
-1 & & -1 & & 0 & & -1 & & 0 \\
\end{align*}
\]

Generating design points similarly from the other combinations, we shall get in all 44 design points. As in this design, \( r = 3 \), no further set need be added. In this design, \( \sum x_i^2 x_j^2 \) takes the following two values:

(i) \( 2^2 a_4^2 \) for the pairs (4,5), (2,6) and (1,3).

(ii) \( 2^3 a_4^4 \) for the remaining possible pairs.

Hence, relation C will be satisfied if

\[
2 a_1^3 a_4 = 2 a_2^2 a_4
\]

i.e., if \( 2 a_1^4 = a_2^4 \).

As \( \sum x_i^4 = 2 \times 2^3 a_1^4 + 2^2 a_2^4 \), relation D will be satisfied if

\[
2 \times 2 a_1^4 + 2^2 a_2^4 = 3 \times 2^3 a_1^4 .
\]

Substituting for \( a_2^4 = 2 a_1^4 \), the left hand side is \( 3 \times 2^3 a_1^4 \) which is identical with the right hand side.
As \( \sum x_1^2 = 2 \times 2^3 a_1^2 + 2^2 a_2 = 44 \) we get another equation or obtaining \( a_1 \) and \( a_2 \).

Similarly by omitting from the B.I.B design

\[ \nu = 9, \ b^* = 12, \ r = 4, \ k = 3 \text{ and } \lambda = 1 \]

treatment 9, we can likewise obtain a second order rotatable design in 8 factors. As in this design \( r > 3 \lambda \), the sets (a a a a a a a) have to be taken and we have to multiply by \( 2^6 \) associate combinations which are 1/4 replicate of the design \( 2^3 \) selected as indicated earlier. The number of points in this design will be

(1) \( 8 \times 2^3 = 64 \) from blocks of size 3.

(ii) \( 4 \times 2^2 = 16 \) from blocks of size 2.

(iii) \( 2^6 = 64 \) from the set \( (a \ a \ \ldots \ a) \).

\underline{Total} \quad 144

(ii) B.I.B designs with three block sizes obtained through B.I.B designs of the series: \( \nu = s^2, \ b^* = s^2 + s, \ r = s + 1, \ k = s, \ \lambda = 1 \) where \( s \) is a prime or a prime power.

By writing the \( s^2 \) treatment numbers in the form of an \( s \times s \) square, we can obtain the B.I.B design of the above series by taking as blocks the rows and columns of the square together with another \( s(s-1) \) blocks which can be
obtained by super-imposing on the square each of \((s-1)\) mutually orthogonal latin squares, and including in a block all those treatments which fall under a letter in a square.

Through these designs a B.I.B design in blocks of sizes \(s\), 2 and \(s^2 - s\) can be obtained as below.

Type (i) blocks: All \(s(s-1)\) blocks each of size \(s\) obtained from the \((s-1)\) orthogonal latin squares.

Type (ii) blocks: A number of \(s^2(s-1)/2\) blocks each of size 2 obtained by taking in a block all possible pairs of treatments out of those treatments which occur in each of the \(s\) blocks of the original B.I.B design formed of the columns of the \(s \times s\) square.

Type (iii) blocks: A number of \(s\) blocks of size \(s^2 - s\) each, which are complementary to the blocks of the B.I.B design formed out of the rows.

Each treatment is replicated \((s-1)\) times in each of the three types of blocks.

In the blocks of type (iii) any two treatments occurring in the same row of the square will occur together in a block \((s-1)\) times, while all other pairs of treatments will occur together \(s^2\) in a block \((s^2-2)\) times. In the blocks of type (i) all possible pairs of treatments not occurring
together in the same row or column, will occur together in
a block once. Again, in the blocks of type (ii) all
treatment pairs occurring together in the same column
will occur together in a block once.

Thus, in all the blocks of the three types any
pair of treatments will occur together in a block \((s-1)\)
times, that is, \(\gamma = s-1\) in this design. As \(r = 3(s-1)\),
the relation \(r = 3\gamma\) will always be satisfied for such
designs.

For design points obtained through such designs
\[\sum x_i^2 x_j^2\] will take the following values when the levels
\(a_1, a_2\) and \(a_3\) are associated respectively with the three
types of blocks in the order of their presentation.

\[\begin{align*}
(1) & \quad (s-1)^2 a_3^4 \\
(\text{ii}) & \quad (s-2)^2 a_3^4 + 2^2 a_2^4 \\
(\text{iii}) & \quad (s-2)^2 a_3^4 + 2^2 a_2^4
\end{align*}\]

and (i) \((s-1)^2 a_3^4\)
(ii) \((s-2)^2 a_3^4 + 2^2 a_2^4\)
(iii) \((s-2)^2 a_3^4 + 2^2 a_2^4\)

for pairs occurring together in
any block obtained from the \(s\)-orthogonal
latin squares.

Relation \(C\) will be satisfied

\[\begin{align*}
& \quad \text{if } (s-1)^2 a_3^4 = (s-2)^2 a_3^4 + 2^4 a_2^4 = (s-2)^2 a_3^4 + 2^4 a_2^4 \\
& \quad \text{i.e. if } 2 a_3 = 2 a_2 = 2 a_1.
\end{align*}\]

Again, relation \(D\) will be satisfied if
A little scrutiny shows that this relation always holds.
For example, if we take $s = 3$, we get the following designs:

(i) $(1, 6, 8), (2, 4, 9), (3, 5, 7), (1, 5, 9), (2, 6, 7), (3, 4, 8)$

(ii) $(1, 4), (1, 7), (4, 7), (2, 5), (2, 8), (5, 8), (3, 6), (3, 9), (6, 9)$

(iii) $(4, 5, 6, 7, 8, 9), (1, 2, 3, 7, 8, 9), (1, 2, 3, 4, 5, 6)$

Relation C gives for this design

$$\sum_{i=1}^{s} a_i = 2a_1 + 2a_2 + 2a_3 = 3s - 1 \times 2a_3 \cdot$$

This design will have $6 \times 2^3 + 9 \times 2^2 + 3 \times 2^5 = 180$ points.

(iii) B.I.B designs with two block sizes obtained through P.B.I.B designs with two associate classes.

Let there be a P.B.I.B design with parameters $(v, b^*, r_1, k, \lambda_1, \lambda_2, n_1, n_2)$ such that $\lambda_1 > \lambda_2$ no matter whether the requirement of secondary parameters are satisfied or not. By putting together in a block two treatments $i$ and $j$ such that the two treatments are second associates, we shall get $vn_2 / 2$ blocks each of size 2 and in these blocks each treatment will be replicated $n_2$ times.

If $\lambda_1 - \lambda_2 = d$, the $b^*$ blocks of the P.B.I.B design together with all the $vn_2 / 2$ blocks of size 2 each repeated
d times will form a B.I.B design with blocks of sizes k and 2, \( \lambda = \lambda_1 \) and \( r = r_1 + n_2 \). Bose, Shrikhande and Parker (1960) obtained such designs through a similar method when \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \). For design points obtained through these designs \( \sum x_i^2 x_j^2 \) will take the following two values when \( a_1 \) is associated with the blocks of the P.B.I.B design and \( a_2 \) with the other blocks:

(i) \( 2^{p_1} \lambda_1 a_1^4 \) for first associate pairs

and (ii) \( 2^{p_1} \lambda_2 a_1^4 + 2^q a_2^4 \) for second associate pairs.

Thus, relations C will be satisfied if

\[
2^{p_1} a_1^4 = 2^q a_2^4.
\]

Again, relation D will be satisfied if

\[
\sum x_i^2 x_j^2 = 2^{p_1} r a_1^4 + 2^q n a_2^4 = 3 \cdot 2^{p_1} \lambda_1 a_1^4,
\]

i.e. if \( r + n_2 d = 3 \lambda_1 \),

i.e. \( r = 3 \lambda \) for the B.I.B design.

Instead of taking all the \( d \) repetitions of the blocks of size 2, if we take \( q \) repetitions, \( \sum x_i^2 x_j^2 \) will have values

(i) \( 2^{p_1} \lambda_1 a_1^4 \)

and (ii) \( 2^{p_1} \lambda_2 a_1^4 + 2^q a_2^4 \).
Hence, relation C will be satisfied if

\[
\frac{p_1 (\lambda_1 - \lambda_2)}{q} a_1 = 2^2 a_2.
\]

As in this case \( \sum x_i = 2^p r_1 a_1 + n_2 q^2 a_2 \), relation D will be satisfied automatically if

\[
2^p r_1 a_1 + n_2 q^2 a_2 = 3 x 2^p \lambda_1 a_1
\]

i.e. if \( r_1 + n_2 = 3 \lambda_1 \) which is the same condition as in the case with \( d \) repetitions.

This shows that if relation D be satisfied automatically by taking the blocks of size 2 repeated \( d \) times so that \( \lambda_2 + d \) becomes equal to \( \lambda_1 \), the relation will also be satisfied automatically if these blocks of size 2 be taken any number of times, particularly once, a case which will lead to a smaller number of design points.

If the relation \( r = 3 \lambda \) does not hold, one more unknown level will have to be introduced through fresh sets as indicated earlier. In such design also the set of \( v n_2/2 \) blocks of size 2 each has to be taken once. This method
has been illustrated through the following examples:

Example 1. Let us first take the following P.B.I.B design with parameters: \( v = 10, r = 4, k = 5, b^* = 8, \lambda_1 = 2, \lambda_2 = 0, n_1 = 8 \) and \( n_2 = 1 \)

\[
\begin{align*}
1, & \ 2, \ 3, \ 4, \ 5 \\
6, & \ 7, \ 8, \ 9, \ 5 \\
1, & \ 2, \ 8, \ 9, \ 10 \\
6, & \ 7, \ 3, \ 4, \ 10 \\
1, & \ 7, \ 3, \ 9, \ 5 \\
6, & \ 2, \ 8, \ 4, \ 5 \\
1, & \ 7, \ 8, \ 4, \ 10 \\
6, & \ 2, \ 3, \ 9, \ 10
\end{align*}
\]

This design can be converted to a B.I.B design with blocks of sizes 6 and 2, by taking each of the following blocks twice.

\[
\begin{align*}
1, & \ 6 \\
2, & \ 7 \\
3, & \ 8 \\
4, & \ 9 \\
5, & \ 10
\end{align*}
\]

These blocks have been obtained by putting in a block two treatments which are second associates.

After the second set of blocks is taken twice, the value of \( r \) becomes 6 with \( \lambda = 2 \). Hence the relation \( r = 3\lambda \) is
satisfied and no further set need be taken. As stated earlier, the relation D will remain satisfied if the set of blocks of size 2 be taken only once. Under this circumstance the total number of points in the design will be

\[ 8 \times 2^4 + 5 \times 2^2 = 148. \]

Relation C will be satisfied in these design points if

\[
p_1 \left( \frac{\lambda_1 - \lambda_2}{\lambda_1} \right) a_1 = 2^a_2\n\]

i.e. if \( 2^4 \times 2a_1 = 2^2a_2 \), since \( q = 1, \lambda_1 - \lambda_2 = 2 \) and \( p_1 = 4 \).

Example 2. Again, with help of the design

\[ v = 12, r_\underline{2} = 4, k = 6, b^* = 8, \lambda_1 = 2, \lambda_2 = 0, n_1 = 10, n_2 = 1 \]

of which the plan is

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 5 & 6 \\
1 & 2 & 9 & 10 & 11 & 12 \\
7 & 8 & 3 & 4 & 11 & 12 \\
10 & 5 & 12 & 1 & 8 & 3 \\
4 & 5 & 12 & 7 & 2 & 9 \\
4 & 11 & 6 & 1 & 8 & 9 \\
10 & 11 & 6 & 7 & 2 & 3 \\
\end{array}
\]
we get a B.I.B design by adding twice the following blocks.

1, 7
2, 8
3, 9
4, 10
5, 11
9, 12

As in this design \( r = 6 \) and \( g = 2 \), relation D will be satisfied with the design points obtained through the above blocks. Hence we can get a rotatable design without adding any further set even when the set of blocks of size 2 is taken only once. In the latter case the number of points in the design will be \( 3 \times 2^5 + 6 \times 2^2 = 280 \).

The above two designs have been given by Bose, Clatworthy and Shrikhande (1954) in their tables for P.B.I.B designs under reference Nos. SR(16) and SR(22) respectively.

Similarly converting the P.B.I.B designs under references C₁, R24 and R27 in the above tables of P.B.I.B designs into B.I.B designs with two block sizes, second order rotatable designs in 13, 14 and 15 factors can be obtained with 516, 508 and 556 points respectively. In each of these B.I.B designs \( r > 3 \) and hence a set of
the form \((a a \ldots a)\) will have to be taken in each case.

1.9. **B.I.B** designs with two block sizes obtained from **P.B.I.B** designs with more than two associate classes.

Let there be a **P.B.I.B** design with parameters

\[ v, b^*, r_1, k, \lambda_1, \lambda_2, \ldots, \lambda_t, n_1, n_2, \ldots, n_t \]

where \(\lambda_1 > \lambda_2 > \ldots > \lambda_t\) no distinction being made between equal \(\lambda\)'s. As described in the previous section \(v n_l/2\) blocks \((l = 2, 3, \ldots, t)\) each of size 2 can be obtained by including in a block all possible pairs of treatments which are mutually \(l\)-th associates. We shall call such blocks containing \(l\)-th associate pairs as the blocks of \(l\)-th set. If \(\lambda_1 - \lambda_l = w_l\) all the blocks of the \(l\)-th set taken \(w_l\) times for all \(l\)'s will form a **B.I.B** design together with the blocks of the **P.B.I.B** design. The parameters of this design will be

\[ v, b + \sum_1^t \frac{v n_l w_l}{2}, r_1 + \sum_1^t w_l n_l, k, 2, \lambda \]

We have discussed upto the last section **B.I.B** designs with 2 or 3 block sizes associated with as many unknown levels. We restricted ourselves to **B.I.B** designs for which \(\sum x_i^2 \frac{1}{x_j^2}\) does not take more than \(m\) values for
different pairs of \( i \) and \( j \) when there are blocks of \( m \) sizes. But for B.I.B designs obtained through P.B.I.B designs with more than two associate classes by taking further blocks of size 2 as described above, may take more than two values even though there are only two sizes of blocks, if the unknown levels are associated with the blocks as described earlier. Actually for design points obtained through such designs \( \sum x_i^2 x_j^2 \) will take as many values as there are associate classes. Hence for obtaining rotatable designs through such B.I.B designs, it is necessary to associate the unknown levels to the blocks in a different way as described below.

Instead of associating only one unknown level to the blocks of size 2 as done earlier, we have to associate as many unknown levels as there are associate classes excluding the first one, the blocks in the \( l \)-th set being associated with \( a_l \) \((l=2, 3, \ldots t)\).

If in such B.I.B designs \( r_1 \sum_{i=2}^{t} n_i w_i = 3 \gamma_1 \) the relation \( D \) will automatically be satisfied when the design points are obtained through it and as in the previous case the same relation will remain satisfied even by taking only once each set of blocks of size 2 instead of \( w_i \) times \((l=2, 3\ldots t)\) and obtaining the design points from the thus reduced design.
For the design points obtained through this reduced design $\sum_{i,j} x_{ij}^2$ will take the following values.

(i) $2a_1^4 + 2p_1 \lambda_1 a_1^4$ for pairs which are $\ell$-th associates,

(ii) $2p_1 \lambda_1 a_1^4$ for pairs which are first associates.

Hence relations C will be satisfied if

$2p_1(\lambda_1 - \lambda_\ell)a_1^4 = 2a_\ell^4$ for $\ell = 2, 3, \ldots t$.

For example, if we take a circular design in blocks of size 3 with $v$ factors obtainable by developing the initial block (1, 2, 3) mod ($v$), it will be a P.B.1.B design with 3 distinct $\gamma$-values viz $\gamma_1 = 2$, $\gamma_2 = 1$, and $\gamma_3 = 0$ and $n_1 = 2$, $n_2 = 2$ and $n_3 = v-5$.

This design can be converted to a B.I.B design by adding (i) $v$ blocks each of size 2 formed out of pairs of treatments which are second associates and (ii) $v(v-5)/2$ blocks of size 2 repeated twice formed of the treatment pairs which are third associates.

For the purpose of constructing rotatable designs through such designs it is not necessary to take twice each block formed of the third associate pairs but only once. For such designs relation D will not be satisfied automatically for any $v$ greater than 5 and the set $(a a \ldots a)$ will have to be taken for all factors greater than 5.
1.10 Second order rotatable designs with blocking.

No attempt has so far been made for dividing the totality of design points into two or more blocks. It is well known that blocking is desirable in many situations, as it increases the precision of estimates. It has been shown by Box and Hunter (1957) that for blocking the design points should satisfy the following further conditions so that the parameters involved in the equation of the surface can be estimated free from block effects:

1. \[ \sum x_i = 0, \sum x_i x_j = 0 \] within each block.

2. \[ \sum x_i^2 = \text{constant} \] within each block.

and

3. \[ \frac{\sum_m x_i^2}{\sum_l x_i^2} = \frac{n_m}{n_l} \] for each pair of blocks,

where \( \sum_m \) and \( \sum_l \) denote respectively the summations over the design points in the m-th and l-th blocks, and \( n_m \) and \( n_l \) are respectively the sizes of the m-th and l-th blocks.

Box and Hunter (1957) obtained some second order designs with blocking but their block sizes were unequal. As it is desirable, particularly for agricultural experimentation, that blocks should be equal in size, an attempt has been made to obtain second order designs split into blocks of equal size. Cochran and Cox (1957) divided the
central composite designs with 4 into two equal blocks.

Box and Behnken (1960 b) obtained a second order design in 4 factors in 3 blocks of 9 plots each.

Several designs with blocking have been presented below.

(i) A design in 3 factors split into 3 blocks of 8 plots each, has been presented below by writing down each of the design points.

<table>
<thead>
<tr>
<th>Block 1</th>
<th>Block 2</th>
<th>Block 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>-a -a 0</td>
<td>-a 0 -a</td>
<td>0 -a -a</td>
</tr>
<tr>
<td>-a a 0</td>
<td>-a 0 a</td>
<td>0 -a a</td>
</tr>
<tr>
<td>a -a 0</td>
<td>a 0 -a</td>
<td>0 a -a</td>
</tr>
<tr>
<td>a a 0</td>
<td>a 0 a</td>
<td>0 a a</td>
</tr>
</tbody>
</table>

Design points

- o o a
- o o -a
- o o a
- o o -a

If some central points have to be added, they must be added in equal numbers in each of the blocks. Each factor has 3 levels in this design.

(ii) The central composite designs also can be split into blocks of equal size through the following procedures.
The central composite designs are obtained through the following combinations of unknown levels:

(i) \((a \ a \ ... \ a)\)

(ii) \((b \ o \ ... \ o)\)

Let \(n\) be the number of design points from the first combination \((a \ a \ ... \ a)\) obtainable by multiplying it by \(n\) associate combinations such that the identity group of interactions for choosing these associates does not contain any interaction with less than 5 factors. Let us now divide these \(n\) associate combinations into \(2^n_1\) groups of 2 each (so that \(n = 2^{n_1} + 2^{n_2}\)) such that for this grouping no two-factor interaction or main effect is confounded. If a block is now obtained by multiplying the combination \((a \ a \ ... \ a)\) by the associate combinations in any of the groups, the design points in the block will satisfy the relations (i) and (ii) required for blocking.

From the \(2^n_1\) groups we shall thus get \(2^{n_1}\) blocks each of size, \(2^{n_2}\). Let us now obtain a block containing all the design points obtainable through the combinations \((b \ o \ ... \ o)\) and repeat this block \(x\) times where \(x\) is an unknown to be determined. The size of each of these \(x\) blocks is \(2^n\), while the size of each of the \(2^{n_2}\) blocks obtained from the combination \((a \ a \ ... \ a)\) is \(2^n\). If \(2^n > 2^{n_2}(2^n - 2^{n_2})\)
central points have to be included in each of the \(2^{n_1}\) blocks of size \(2^n\). If, again, \(2v < 2^n\), \((2^n - 2v)\) central points have to be included in each of the \(x\) blocks of size \(2^v\). The sizes of all the blocks can thus be made equal.

Now, \(\sum x_1^2 = 2^{n_2} a^2\) for each block size \(2^n\)

\[= 2^b \] for each block of size \(2^v\).

Hence from relation (iii) for blocking

\[\frac{n_2}{2} \frac{b}{a} = 2 b^2\]

i.e. \(4 = 2n_2 - 2 a^4\).

Again, we have for the whole design including \(2^{n_1} + x\) blocks,

\[\sum x_1^4 = 2^{n_1 + n_2} a^4 + 2x b^4\]

and \[\sum x_1^2 x_2^2 = 2^{n_1 + n_2} a^4\].

Hence, from relation D, \(2^{n_1 + n_2} a^4 + 2x b^4 = 3 \cdot 2^{n_1 + n_2} a^4\)

i.e. \(x b^4 = 2^{n_1 + n_2} a^4\).

But for blocking \(b^4\) should be equal to \(2^{n_2 - 2} a^4\).

Hence both relation D and requirement for blocking will be satisfied if \(x \cdot 2^{n_2 - 2} a^4 = 2^{n_1 + n_2} a^4\)

i.e. \(x = 2^{n_1 - n_2 + 2}\).
Thus, by repeating the blocks of size $2^v$

$$n_1 = n_2 + 2$$

$2^1 = n_2$ times, a central composite design can be split into blocks of equal size. It will be seen that for every possible values of $n_1$ and $n_2$ blocking is not possible as $x$ is necessarily a positive integer. The following table shows some of the possible sets of values of $n_1$, $n_2$ and $x$ for different values of $n$.

<table>
<thead>
<tr>
<th>Design No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>64</td>
<td>64</td>
<td>64</td>
<td>64</td>
</tr>
<tr>
<td>$n_1$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$n_2$</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$x$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>No. of factors</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

Design No. 1 was obtained by Cochran and Cox (1957).

1.11 Analysis of second order designs.

The first step in the analysis of the data collected through a second order design without blocking consists of the application of the analysis of variance technique with the following sub-divisions.
<table>
<thead>
<tr>
<th>Sources</th>
<th>d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fitted constants</td>
<td>( \frac{v(v + 3)}{2} )</td>
</tr>
<tr>
<td>Lack of fit</td>
<td>( N' - \frac{(v + 1)(v + 2)}{2} )</td>
</tr>
<tr>
<td>Error</td>
<td>( N - N' )</td>
</tr>
<tr>
<td>Total</td>
<td>( N - 1 )</td>
</tr>
</tbody>
</table>

Here \( N' \) stands for the number of distinct design points and \( \frac{(v+1)(v+2)}{2} \) is the number of parameters in the second degree surface with \( v \) factors.

The S.S. due to the fitted constants can be obtained as usual from

\[
G b_0^2 + \sum_i b_i \sum_u x_{iu} y + \sum_i b_{ii} \sum_u x_{iu}^2 + \sum_{i<j} b_{ij} \sum_u x_{iu} x_{ju} - \text{C.F.}
\]

Here \( G \) stands for the grand total of \( y \).

Next, the error S.S. is obtained from a comparison of the observation against these design points which are repeated. The remaining component of S.S due to lack of fit can be obtained by subtraction as these three components of S.S. add to the total corrected sum of squares.
When the design is with blocking, the analysis needs some modification. The components of S.S. due to blocks and fitted constants are to be obtained as usual.

The error S.S has to be obtained from a comparison of the observations belonging to the design points repeated in the same block and adding up the S.S. obtained from such comparisons over the different blocks. This will give what is known as pure error. Another component of error S.S. can be obtained from the observations coming from the same design points but belonging to different blocks. This component can be obtained by first sorting out the design points each of which occur in each of a number of blocks. From an analysis of these observations when classified as block x design points (here block may be even a part of the original block), we can get an error component as block x design point interaction in so far as this group of design points is concerned. Like this there may be several mutually exclusive groups of design points, each of which has to be analysed separately as above. All these components of S.S. when added together will supply another component of error S.S.

Let us take the following design for illustration: —

which is another new design:
In this design the last six design points are repeated but remaining in different blocks. From the interaction between block and these six design points a component of error S.S. with 5 d.f is available. No other component of error S.S. is available from this design. As 5 d.f is too small, it may be advisable to repeat the design to get 4 blocks. In this design with 4 blocks the interaction between the last six treatments and the 4 blocks, will give an error component with 15 d.f. Again, as there are two blocks viz. blocks I and III with the first four treatments of Block I, these will give another component of error with
3 d.f. Another component of error S.S. with 3 d.f can be obtained similarly from the first four design points of block II. From a comparison between average of the first four observations and the average of the rest six observations in block I and block III which is the repetition of block I, we can get another component of S.S. with 1 d.f. Similarly from blocks II and IV we can get another component with 1 d.f. Thus, the error will be in all 25 d.f for error when the design is repeated.

The above analysis supplies an estimate of error variance and also a test to indicate the adequacy of the fit. Once this preliminary analysis is over, it can be extended to examine the response surface, estimate the optimum combinations of the levels of the factors, etc., through standard techniques.

1.12 Transformation of coded levels to actual levels.

Coded levels on suitably chosen origin and scale have so far been used as levels of the factors. As actual levels are necessary while laying at the experiment, it is necessary to work out the relation connecting the two types of levels.

Let $M_u$ and $m_u$ denote the known minimum and maximum actual levels of any factor within which its other levels have to be taken. Let now the coded doses in the ascending
order of magnitude be \(-c -b \ldots \ldots \ldots -a o a \ldots b c\), so that \(-c\) and \(c\) are respectively the minimum and maximum coded levels.

If \(X\) stands for the actual level and \(x\), the coded level, evidently the relation between \(X\) and \(x\) is linear, say, \(X = mx + i\). The two constants \(m\) and \(i\) can be obtained from the two known points on the line viz. \((M_l - c)\) and \((M_u, c)\). From this relation any actual level \(M_a\) corresponding to the coded level \(a\), is given by

\[
M_a = \frac{M_u + M_l}{2} + \frac{a}{c} \left(\frac{M_u - M_l}{2}\right)
\]

1.13 Third order rotatable designs.

We have already seen that a set of design points will form a second order rotatable design if the design points satisfy certain relations which have been designated as relations A, B, C, D and E. It has been shown by Gardiner et al (1969) that the following further relations should also be satisfied by the design points forming a third order rotatable design.

Relation A1: Each of the sums of powers or products of powers of \(x_{iu}\) 's up to sixth degree in which at least one power is odd, is zero.
Relation C₁: \[ \sum x_i^4 x_j^2 = \text{constant} \]

Relation D₁: 

(i) \[ \sum x_i^4 = 5 \sum x_i^4 x_j^4 \]

(ii) \[ \sum x_i^4 x_j^4 = 3 \sum x_i^4 x_j^4 x_k^2 \]

Relation E₁: \[ \frac{\lambda_2 \lambda_6}{\lambda_4^2} \geq \frac{v + 2}{v + 4} \]

The summations in these relations are over the design points. Relation E₁ is not satisfied when the design points are all equidistant from the centre. By adding central points this relation cannot be satisfied like relation E. While the choice of combinations of unknown levels will be discussed afterwards for individual cases, the associate combinations required for multiplication have to be so chosen that no interaction with less than seven factors is included in the identity group of interactions to be used for obtaining this group of say \( m \) associate combinations.

Third order rotatable designs can be both non-sequential and sequential. If the design points satisfy all the relations given under both second and third order designs.
and are tried in one occasion, they form a non-sequential third order rotatable design. Alternatively, Gardiner et al (1959) have shown that an experiment in a third order design can be performed sequential by dividing all the design points into two groups, each forming the contents of a block. If the design points in the first block form a second order design and the inclusion of the additional points in the second block makes the whole a third order design, the design with the points in both the blocks is called a sequential third order rotatable design. The design points in the first block are tried first in one occasion and the design points in the second block will be tried subsequently when the fit as obtained from the first block happens to be inadequate.

For the estimation of the third degree polynomial coefficients independently of block effects, there is a further condition to be satisfied viz.

$$\frac{\sum_1 x_1^2}{\sum_2 x_1^2} = \frac{n_1 + n_{10}}{(n_2 + n_{20})}$$

where $n_1$ and $n_2$ are the numbers of non-central points in the two blocks; $n_{10}$ and $n_{20}$ are the numbers of central points in the two blocks and $\sum_1$ and $\sum_2$ denote respectively the summations over the design points in the first and
second blocks. As $\sum x_1^2$ and $\sum x_1$ are functions of the levels of the factors which can be evaluated from the other relations in the design, the above relation can always be satisfied by suitably choosing $n_{10}$ and $n_{20}$.

2.14. **Some specific non-sequential third order rotatable designs.**

Some specific non-sequential third order designs have been obtained below by suitably choosing several combinations of unknown levels.

(1) A design in 2 factors.

A design in 2 factors can be obtained from the combinations of unknown levels (i) (a b), (ii) (a a) and (iii) (c o) by obtaining the design points from them through rotation and multiplication. This design has 16 points.

Here, $\sum x_1^4 = 4a^4 + 4b^4 + 2c^4$

$\sum x_1^2 x_j^2 = 4a^2 + 8a^2 b^2$

$\sum x_1^6 = 8a^6 + 4b^6 + 2c^6$

$\sum x_1^4 x_j^2 = 4a^6 + 4a^4 b^2 + 4a^2 b^4$.

From relations D and $D_1$, we get the equations

$8a^4 + 4b^4 + 2c^4 = 12a^4 + 24a^2 b^2$

$8a^6 + 4b^6 + 2a^6 = 20 a^6 + 20 a^4 b^2 + 20 a^2 b^4$. 
Putting \( s = a^2/b^2 \) and \( t = c^2/b^2 \), they become
\[
\begin{align*}
\quad t^2 + 2 &= 12s + 2s^2 \\
\text{and} \\
\quad t^3 + 2 &= 10(s^2 + s) + 6s^3.
\end{align*}
\]
Solving these equations
\[
\begin{align*}
\quad s &= 0.25538 \\
\text{and} \\
\quad t &= 1.09320.
\end{align*}
\]
The value of \( b \) can be obtained from the relation
\[
b^2(8s + 2t + 4) = N.
\]
As the points are not all equidistant from the centre, relations \( B \) and \( B_1 \) are satisfied.

(ii) Third order designs in three and more factors.

It has been seen that with the help of the combinations (i) \((a \ a \ ... \ a)\) and (ii) \((b \ o \ o \ ... \ o)\) second order central composite design can be obtained. But no third order design can be obtained from them, as relations \( D_1 \) cannot be satisfied. But if one more combination of the type \((c \ c \ o \ o \ ... \ o)\) permuted in all possible ways to give \( v(v-1)/2 \) combinations, be taken together with the above types of combinations, some designs can be obtained.

When the design points are obtained from the above three types of combinations, we have
From relations D and D₁ we get the equations

\[ \sum_{i} x_i^4 = ma^4 + 2b^4 + 4(v-1)c^4 \]
\[ \sum_{i,j} x_i^2 x_j^2 = ma^4 + 4 c^4. \]
\[ \sum_{i} x_i^6 = ma^6 + 2b^6 + 4(v-1)c^6 \]
\[ \sum_{i,j} x_i^4 x_j^2 = ma^6 + 4 c^6 \]
\[ \sum_{i,j,k} x_i^2 x_j^2 x_k^2 = ma^6 \]

where \( m \) denotes the number of associate combinations required for multiplying \((a \ a \ \cdots \ a)\) in which no interaction with less than 7 factors is confounded.

Putting \( s = a^2/c^2 \), \( t = b^2/c^2 \), these become

\[ t^2 = ms^2 + 2(4-v) \]
\[ s^3 = 2/m \]
\[ t^3 = 2(3-v) \]

As there are three equations and two unknowns, combinations with one more unknown is to be introduced. We can take
these combinations as one of the three types already
taken out involving a different unknown. There will thus
be the following three cases according as the type of the
added combinations is (d o o ... o), (d d d ...d) or
(d d oo ... o).

Case 1. If a set of combinations (d o o ... o) be taken
and \( d^2/c^2 = u \), the equations for the design based on this
set and the previous three sets of combinations will be

\[
\begin{align*}
    t^2 + u^2 &= ms^2 + 2(4 - v) \\
    s^3 &= 2/m \\
    t^3 + u^3 &= 2(8 - v)
\end{align*}
\]

Case 2. If, instead, the combination (d d ... d)
be taken and \( d^2/c^2 = u \), the equation will be

\[
\begin{align*}
    m(s^2 + u^2) &= t^2 - 2(4 - v) \\
    s^3 + u^3 &= 2/m \\
    t^3 &= 2(8 - v)
\end{align*}
\]

Case 3. Lastly, if the set of \( v(v-1)/2 \) combinations
(d d o o ... o) be taken, the equations will be

\[
\begin{align*}
    t^2 &= ms^2 + 2(4 - v)(1 + u^2) \\
    s^3 &= 2(1 + u^3)/m \\
    t^3 &= 2(8 - v)(1 + u^3)
\end{align*}
\]

It will be seen that in cases 1 and 2, one of the equations
involves only one unknown. The other two equations are

of the form,

\[
\begin{align*}
    t^2 + u^2 &= A \\
    t^3 + u^3 &= B
\end{align*}
\]
It can be shown that in such equations if
\[ A^2 \leq B^2 \leq A^3 \]
holds and A and B are positive, there will be a positive real solution of \( t \) and \( u \) and one of these two will lie between \( \sqrt{A} \) and \( \sqrt{A}/2 \).

Some of the designs from such combinations for different values of \( v \) upto 8 have been presented below.

(i) A design in 3 factors.

The following sets of combinations (i) (a a a) (ii) (b o o), (iii) (c c o) and (iv) (d o o) give a design which belongs to case 1 above. This design has 32 points and no central point is necessary.

The equations for this design are
\[ 3^2 = 0.25 \text{ i.e. } s = 0.62996 \]
\[ t^2 + u^2 = 8s^2 + 2 = 5.174797 \]
\[ 3^3 + u^2 = 10 \]

The solutions of \( t \) and \( u \) are
\[ t = 2.1090 \]
\[ u = 0.8526 \]

(ii) A design in 4 factors.

The combinations (i) (a a a a) (ii) (b o o o) (iii) (c c o o) and (iv) (d d o o) give a design which evidently belongs to case 3 above. The equations for this design are
\[ t^2 = 16s^2 \]
\[ s^3 = 143/8 \]
\[ t^3 = 8(1 + 43) \]
As the third equation follows from the first two, whatever \( u \) may be, there will be a design with these sets for each value of \( u \). The values of \( t \) and \( s \) can be obtained as soon as \( u \) is fixed. When \( u \neq 0 \), there will be 72 points in the design. If \( u = 0 \), there will be 48 points, but in this design all the points will be equidistant from the centre and hence relation \( E_1 \) cannot be satisfied even by adding central points.

(iii) A design in 5 factors.

The combinations (i) \((a \ a \ a \ a \ a)\), (ii) \((b \ o \ o \ o \ o)\), (iii) \((c \ c \ o \ o \ o)\) and (iv) \((d \ d \ d \ d \ d)\) will give a design in factors with 114 points. The design belongs to case 2 and the equations are

\[
\begin{align*}
    s^2 + u^2 &= (t^2 + 2)^{1/2}/32 \\
    s^3 + u^3 &= 1/16 \\
    t^3 &= 6
\end{align*}
\]

The solutions are \( t = 1.81712 \)
\[
    u = .9991
\]
\[
    s = .3948
\]

(iv) A design in 6 factors.

Exactly similar sets as in the design with 5 factors give a design in 6 factors with 200 points. The equations for this design are
\[ t^3 = 4 \]
\[ s^2 + u^2 = \frac{(t^2 + 4)}{64} = 0.10187 \]
\[ s^3 + u^3 = 1/32 \]

Solving these equations

\[ t = 1.58740 \]
\[ s = 0.31446 \]
\[ u = 0.05466 \]

(v) A design in 7 factors.

Again, similar sets as in the designs with 5 and 6 factors give a design in 7 factors with 226 points.

In this design \( m = 64 \) and not 128.

The equations are

\[ t^3 = 2 \]
\[ s^2 + u^2 = \frac{(t^2+6)}{64} = 0.11855 \]
\[ s^3 + u^3 = 1/32 \]

The solutions are

\[ t = 1.2599 \]
\[ u = 0.1777 \]
\[ s = 0.2949 \]

(vi) A design in 8 factors.

In this case if we take the sets

(i) \( a a a a a a a a \)

and (ii) \( r r o o o o o o \),

the equations come out by taking \( m = 128 \)

\[ s^2 = \frac{8}{m} = 1/16 \]
\[ s^3 = \frac{2}{m} = 1/64 \]
It is found that \( s = 1/4 \) is a solution and hence a rotatable arrangement is possible with these combinations. With this solution for \( s \), we find \( 8s^2 = 2c^2 \) and hence all the points are equidistant from the centre. This indicates that relation \( E_1 \) cannot be satisfied with these points even by adding central points. Under this situation a design is not possible even by adding other sets of combinations as in that case no positive solution for each of \( s, t \) and \( u \) will be available.

1.15 Third order sequential designs.

The third order designs presented earlier cannot be fitted into a sequential programming of experimentation as they are. For this purpose it is necessary to divide the design points into two groups to form the contents of two blocks so that the design points in one block can be tried in one occasion while that in the other can be tried on a subsequent occasion, if necessary. For such division, in addition to the relations which have to be satisfied by the third order non-sequential rotatable designs, one more relation viz. \( \sum x_1^4 = 3 \sum x_1^2 x_j^2 \) must be satisfied for a design to be sequential, as this relation along with relation \( D \) will ensure that each block is a second order rotatable design. We shall call this relation as relation \( F \). A number of sequential designs for 3, 4, 5 and 6 factors have now been presented below.
(1) A design in 3 factors.

<table>
<thead>
<tr>
<th>Block contents</th>
<th>No. of design points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block I</td>
<td></td>
</tr>
<tr>
<td>(i) (c c c o)</td>
<td></td>
</tr>
<tr>
<td>(ii) (d o o)</td>
<td>n_1 = 18</td>
</tr>
<tr>
<td>Block II</td>
<td></td>
</tr>
<tr>
<td>(i) (a a a)</td>
<td></td>
</tr>
<tr>
<td>(ii) (b o o)</td>
<td>n_2 = 22</td>
</tr>
<tr>
<td>(iii) (w w w)</td>
<td></td>
</tr>
</tbody>
</table>

From equations obtainable from relations D, D_1 and F, we get the unknowns as

\[
\frac{a^2}{c^2} = 0.6, \quad \frac{b^2}{c^2} = 1.92849
\]

\[
\frac{d^2}{c^2} = 1.41421 \quad \text{and} \quad \frac{w^2}{c^2} = 0.32390
\]

As \( \sum_1 x_1^2 = c^2 (8 + 2q) \)

\[
\sum_2 x_1^2 = c^2 (8s + 8u + 2t)
\]

\( n_1 = 18, \quad n_2 = 22 \)

where \( s = \frac{a^2}{c^2}, \quad u = \frac{w^2}{c^2}, \quad t = \frac{b^2}{c^2} \) and \( q = \frac{d^2}{c^2} \)

the relation \( \frac{\sum_1 x_1^2}{\sum_2 x_1^2} = \frac{n_1 + n_{10}}{n_2 + n_{20}} \)

can be satisfied by suitably choosing \( n_{10} \) and \( n_{20} \) so that

\[
\frac{8 + 2q}{8s + 8u + 2t} = \frac{18 + n_{10}}{22 + n_{20}}
\]
The value of \( c \) has to be obtained from

\[
c^2(8 + 2q + 8s + 8u + 2t) = N
\]

where \( N = n_1 + n_2 + n_{10} + n_{20} \)

(ii) A design in 4 factors.

<table>
<thead>
<tr>
<th>Block contents</th>
<th>No. of design points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block I (i) (a a a a) (ii) (b o o o)</td>
<td>( n_1 = 24 )</td>
</tr>
<tr>
<td>Block II (i) (c c o o) (ii) (d d o o)</td>
<td>( n_2 = 48 )</td>
</tr>
</tbody>
</table>

As this design is the same as the non-sequential design in 4 factors presented earlier, the values of the unknown levels also remain the same.

(iii) A design in 5 factors.

<table>
<thead>
<tr>
<th>Block content</th>
<th>No. of design points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block I (i) (a a a a a) (ii) (b o o o o) (iii) (d o o o o)</td>
<td>( n_1 = 52 )</td>
</tr>
<tr>
<td>Block II (i) (w w w w w) (ii) (c c o o o)</td>
<td>( n_2 = 72 )</td>
</tr>
</tbody>
</table>

Putting \( s = a^2/c^2 \), \( t = b^2/c^2 \), \( u = w^2/c^2 \) and \( q = d^2/c^2 \), the solutions come out as.

\[
\begin{align*}
u &= 0.25 \\
s &= 0.36056 \\
t &= q = 1.44225.
\end{align*}
\]
(iv) A design in 6 factors.

<table>
<thead>
<tr>
<th>Block content</th>
<th>No. of design points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block I</td>
<td></td>
</tr>
<tr>
<td>(i) (w w w w w)</td>
<td></td>
</tr>
<tr>
<td>(ii) (c o o o o)</td>
<td>124</td>
</tr>
<tr>
<td>Block II</td>
<td></td>
</tr>
<tr>
<td>(i) (a a a a a)</td>
<td></td>
</tr>
<tr>
<td>(ii) (b o o o o)</td>
<td></td>
</tr>
<tr>
<td>(iii) (d o o o o)</td>
<td>136</td>
</tr>
</tbody>
</table>

Here the solutions are

\[ s = 0.4660, \ t = 2.4636, \ q = 1.3990 \]

and \( u = 0.25 \) where \( s, t, u \) and \( q \) have the same definition as in the design for 5 factors.

1.16. Third order rotatable designs through doubly B.I.B designs.

The a-combinations chosen through B.I.B designs for the construction of second order rotatable designs do not usually satisfy the relations \( C_1(ii) \) together with \( D_1(i) \) and \( D_1(ii) \) and \( E \) as required for third order rotatable designs. If the B.I.B design happens to be doubly balanced i.e. in addition to pairs of treatments occurring a constant number of times, the triplets of treatments also occur a constant number of times, in the blocks (Calvin, 1954), the relation \( C_1(ii) \) is also satisfied. For satisfying the other relations not yet satisfied, viz. \( D_1, D_1(i) \) and \( D_1(ii) \) and \( E \), we have to introduce combinations involving
fresh unknowns which can be evaluated by solving the equations obtained through $D_1, D_1(1)$ and $D_1(i)$. For example, each of the following designs are doubly balanced. For the sake of convenience only the numerical values of the parameters $(v, k, r, b, \lambda, \gamma)$ of the doubly $B_1B$ designs have been shown in brackets in the same order in which they are written above. In the future, the parameters of $B_1B$ designs also will be presented similarly except for $\gamma$.

(1) (3, 2, 2, 3, 1, 0) (vi) (9, 3, 28, 84, 7, 1) (vii) (10, 4, 12, 30, 4, 1)
(ii) (4, 3, 3, 4, 2, 1)
(iii) (5, 3, 6, 10, 3, 1) (viii) (11, 5, 15, 33, 6, 2)
(iv) (6, 3, 10, 20, 4, 1) (ix) (12, 6, 11, 22, 5, 2)
(v) (8, 4, 7, 14, 3, 1)

With the help of each of these designs which will supply us the a-combinations as described earlier for second order rotatable designs, third order designs, both sequential and non-sequential, can be obtained by taking further one or more of the combinations of the type (b 0 0 ...0), (c 0 0 0 ...0), (d, d...d) involving fresh unknown levels b, c, d and multiplying them with the associate combinations as earlier. The combinations (c 0 0 0...0) will give $\binom{v}{2}$ combinations when permuted over all the
v factors and these \( v(v-1)/2 \) combinations will hereafter be called combinations of type \( (c c 0 0 \ldots 0) \). The design points obtained from the combinations of type \( (c c 0 0 \ldots 0) \) after multiplying each of them with the \( 2^2 \) associate combinations will be denoted as \( (c c 0 \ldots 0) \times 2^2 \). The other two types of combinations have been described earlier. Sometimes it becomes necessary to include in the same design more than one set of the same type for getting positive solutions for all the levels.

As an example, we can get a third order non-sequential rotatable design in 9 factors with the help of the following design points:

(i) 672 points from \( a = (9, 3, 28, 84, 7, 1) \times 3^3 \)

(ii) 256 points from \( (b b \ldots b) \times 1/2 \text{ repl. } 2^9 \)

(iii) 256 points from \( (c c \ldots c) \times 1/2 \text{ repl. } 2^9 \)

(iv) 18 points from \( (d 0 \ldots 0) \times 2 \)

The equations for solving the unknowns come out as

From D: \( (28 \times 8)a^4 + 256(b^4 + c^4) + 2d^4 = (21x8)a^4 + 3x256(b^4 + c^4) \)

From D (i): \( (28 \times 8)a^6 + 256(b^6 + c^6) + 2d^6 = (35x8)a^6 + 5 \times 256(b^6 + d^6) \)

From D (ii): \( (7 \times 8)a^6 + 256(b^6 + c^6) \) = \( 3 \times 8a^6 + 3x256(b^6 + c^6) \).

Solving these equations we get

\[
\frac{b^2}{a^2} = 0.392768
\]
\[
\frac{c^2}{a^2} = 0.122376
\]
\[
\frac{d^2}{a^2} = 3.914868
\]
The value of \( a \) can be obtained from \( \Sigma x_i^2 = N \). This design contains 1202 points.

Sequential third order designs can be obtained with the help of the same types of combinations, viz., a-combinations through the doubly B.I.B designs, together with one or more of the types of combinations \((b 0 0 ... 0), (c c 0 ... 0)\) and \((d d ... d)\). For example, we can get a sequential third order rotatable design in 8 factors with the help of the following design points:

**Block I**
- (i) 128 points of \((d d ... d) \times (1/2 \text{ replicate of } 2^3)\)
- (ii) 16 points of \((e 0 0 ... 0) \times 2\)

**Block II**
- (iii) 224 points of \(a-(8, 4, 7, 14, 3, 1) \times 2^4\)
- (iv) 112 points of \((c c 0 ... 0) \times 2^2\)

The design relations will lead to the following equations.

From relations

\[(D): 112a^4 + 28c^4 + 128d^4 + 2e^4 = 144a^4 + (3 \times 128)d^4 + 12c^4\]

\[(D_1)(i): 112a^6 + 28c^6 + 128d^6 + 2e^6 = 240a^6 + (5 \times 128)d^6 + 20c^6\]

\[(D_1)(ii): 48a^6 + 128d^6 + 4c^6 = 48a^6 + (3 \times 128)d^6\]

There is one more relation to make each block a second order rotatable design. This relation gives \(2e^4 + 128d^4 = (3 \times 128)d^4\).

Putting \(a^2/d^2 = s, c^2/d^2 = u, e^2/d^2 = t\), the equations become

\[8u^2 + t^2 = 16s^2 + 128\]
\[4u^3 + t^3 = 64s^3 + 256\]
\[4u^3 = 2 \times 128\]
\[t^2 = 128,\]
whence \( u = 4 \), \( t = 128^{\frac{1}{4}} \), and \( s = 8^{\frac{1}{4}} \). The value of \( d \) can be obtained from \( \sum x_i^2 = N \). The number of central points to be added to the two blocks to ensure estimation of the polynomial coefficients independently of block effects will be determined from

\[
\left( \frac{\sum_1 x_1^2}{\sum_2 x_1^2} \right) = \frac{(144 + n_{10})}{(336 + n_{20})},
\]

where \( \sum_1 x_1^2 \) is summed over the points in the first block and \( \sum_2 x_1^2 \) is summed over the points in the second block.

As \( \sum_1 x_1^2 \) and \( \sum_2 x_1^2 \) are functions of the unknown levels, which have been obtained by solving the equations gotten from the different relations to be satisfied, \( n_{10} \) and \( n_{20} \), the number of central points to be added to the first and second block respectively, can be obtained from the above relation. Actually, \( \sum_1 x_1^2 = 128a^2 + 2e^2 \) and \( \sum_2 x_1^2 = 112a^2 + 28c^2 \).

Substituting for \( s, u \) and \( t \) obtained earlier, \( n_{10} \) and \( n_{20} \) can be obtained from

\[
\frac{(64 + t)}{(56s + 14u)} = \frac{(144 + n_{10})}{(336 + n_{20})}.
\]

Thus, we get a sequential third order rotatable design for 8 factors in 480 non-central points.

1.17. Third order designs obtained through complementary B.I.B. Designs. A B.I.B design, not necessarily doubly balanced, is taken together with its complementary B.I.B design, repeated
if necessary, for generating the a-combinations as before. We can now get points through these a-combinations which will satisfy $C_{1}$ (ii), as $\gamma$ will be a constant in the combined B.I.B designs, together with all the other relations excepting $D$, $D_{1}$ (i) and $D_{1}$ (ii), E. For satisfying these relations we have to take one or more of the types of combinations $(b \ 0 \ 0 \ ... \ 0)$, $(c \ c \ 0 \ ... \ 0)$ and $(d \ d \ ... \ d)$ involving fresh unknowns.

For example, a non-sequential third order rotatable design in 10 factors can be obtained with the following points:

(i) $(18 \times 32)$ points of $a-(10, 5, 9, 18, 4) \times 2$,

(ii) $(18 \times 32)$ points of $a-(10, 5, 9, 18, 4) \times 2^5$, the design in (ii) being the complementary B.I.B design of the design in (i).

(iii) 20 points of $(b \ 0 \ ... \ 0) \times 2$,

(iv) 180 points of $(d \ d \ 0 \ ... \ 0) \times 2^2$,

(v) 20 points of $(c \ 0 \ ... \ 0) \times 2$.

Here $\gamma = 3$ in the combined designs.

The relations $D$, $D_{1}$ (i) and (ii) give the equations,

\[(18 \times 32) a^4 + 2b^4 + 2c^4 + 36d^4 = (24 \times 32) a^4 + 12d^4.\]

\[(18 \times 32) a^6 + 2b^6 + 2c^6 + 36d^6 = (40 \times 32) a^6 + 20d^6.\]

\[(8 \times 32) a^6 + 4d^6 = (9 \times 32) a^6.\]
Putting $\frac{b^2}{a^2} = s$, $\frac{c^2}{a^2} = t$, $\frac{d^2}{a^2} = u$, we get $u = 2$, $s^2 + t^2 = 48$, and $s^3 + t^3 = 288$. Solving, we get $s = 6.494905$, $t = 2.411955$. Thus, we get a non-sequential third order rotatable design in 1372 points.

Sequential third order designs can also be constructed with the help of the complementary B.I.B designs together with three other types of combinations involving fresh unknowns.

For example, with the following points we can get a sequential third order design for 7 factors.

Block I (i) 112 points of $a-(7, 4, 4, 7, 2) \times 2$

(ii) 14 points of $(b 0 0 ...) \times 2$

Block II (iii) 112 points from $a-(7, 3, 3, 7, 1) \times 2^3$, the design in (iii) being the complementary design of the B.I.B design in (i). Each of the 56 points in (iii) is to be repeated once more.

Here, $\gamma = 1$ in the combined designs. The complementary B.I.B design in (iii) has to be repeated once more as each $a$-combination from the first B.I.B design gives 16 design points on "multiplication" with the associate combinations and each $a$-combination from the complementary B.I.B design gives only 8 combinations on "multiplication" with
the associate combinations. Hence unless all the points obtained from the \(a\)-combinations of the complementary design be repeated once, \(\sum x_i^2 x_j^2 x_k^2\) will not be constant for all \(i, j, k\) such that \(i \neq j \neq k\).

In the case of the above design points condition \(D_1(\text{ii})\) is satisfied, as and the requirement that each block is a second order rotatable design is satisfied as \(r = 3\) in block II.

Relations \(D\) and \(D_1(\text{i})\) give the equations,
\[
112a^4 + 2b^4 = (3 \times 48) x a^4.
\]
\[
112a^6 + 2b^6 = (5 \times 48) a^6.
\]
Putting \(b^2/a^2 = t\), these equations become \(t^2 = 16\) and \(t^3 = 64\). Hence \(t = 4\). \(a\) can be obtained from the equation \(\sum x_i^2 = N\).

Numbers of central points to be added to the two blocks are given by the relation,
\[
48a^2/(64a^2 + 2b^2) = (112 + n_{10})/(126+n_{20}),
\]
i.e.
\[
48/(64+2t) = (112 + n_{10})/(126+n_{20}).
\]
Thus, we get a sequential third order rotatable design for 7 factors in 238 non-central points, with some central points to be added.

Appendices II and III of paper No.2 appended with this thesis present respectively non-sequential and
sequential third order rotatable designs up to 15 factors obtained by utilizing doubly balanced incomplete block designs or by a B.I.B design together with its complementary B.I.B design.
1.8. References

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