PART - II
A ring of endomorphisms of an additive commutative group plays a very important role in many parts of mathematics. The property of a ring itself is also clarified when we consider it as a ring of endomorphisms of an additive commutative group; but if we consider a set of homomorphisms of an additive commutative group to another additive commutative group the set is closed under addition and subtraction defined naturally but we cannot define natural multiplication of two homomorphisms in it. However, if we consider two additive commutative groups A and B and the additive commutative group M consisting of all homomorphisms from A to B then we can define the product of three elements $f_1, g$ and $f_2$ where $f_1$, $f_2$ are the members of M and $g$ is a homomorphism from B to A. In this case the product $f_1 \cdot g \cdot f_2$ is also an element of M. Thus we can define multiplication in M using $\Gamma'$ where $\Gamma'$ is the additive commutative group of all homomorphisms from B to A. Similarly we can define a multiplication in $\Gamma'$ using M. Also we have

$$(f_1 \cdot g_1 \cdot f_2) \cdot f_3 = f_1(g_1 \cdot f_2 \cdot g_2) \cdot f_3 = f_1g_1(f_2 \cdot g_2 \cdot f_3)$$

where $f_1$, $f_2$, $f_3$ are the members of M and $g_1$, $g_2$ are the members of $\Gamma'$. 
Again we know that the ring of all square matrices over a division ring plays a vital role in classical ring theory. However, if we consider the set $M$ of all rectangular matrices of the type $m \times n (m \neq n)$ then $M$ is an additive commutative group; there appears to be no natural way of introducing a binary multiplication into it. Various authors like Lister [19] and Hestenes [8] tried to offset this difficulty by considering a natural ternary multiplication in the set of rectangular matrices. Their investigations led to the respective notions of associative triple systems of first kind (ternary rings) and of second kind. These structures provide a suitable setting for the study of rectangular matrices.

Now if $M$ is the additive commutative group of all rectangular matrices of the type $m \times n (m \neq n)$ and $\Gamma$ is that of all rectangular matrices of the type $n \times m$ and also if $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ then $a \alpha b \in M$ and $\alpha a \beta \in \Gamma$ and also $a \alpha (b \beta c) = a(\alpha b \beta) c = a \alpha (b \beta c)$.

Noting this fact N. Nobusawa [25] in the year 1964 defined $\Gamma$-ring as follows:

Let $M$ and $\Gamma$ be two additive abelian groups. If for all $a, b, c \in M$ and $\alpha, \beta, \gamma \in \Gamma$ the conditions

1. $a \alpha b \in M$ and $\alpha a \beta \in \Gamma$
2. $(a + b) \alpha c = a \alpha c + b \alpha c$, $a(\alpha + \beta)b = a \alpha b + a \beta b$, $a \alpha (b + c) = a \alpha b + a \alpha c$.
(3) \( a \times (b \otimes c) = a(\alpha \beta) c = (a \alpha b) \beta c \)

(4) \( a \gamma b \equiv 0 \) for all \( a, b \in M \) implies \( \gamma = 0 \) are satisfied then \( M \) is called a \( \Gamma' \)-ring.

Later in the year 1966 Barnes slightly weakened the defining conditions for Nobusawa's \( \Gamma' \)-ring and defined \( \Gamma' \)-ring as follows: Let \( M \) and \( \Gamma \) be two additive abelian groups. If for all \( a, b, c \) in \( M \) and \( \alpha, \beta \) in \( \Gamma \) the following conditions are satisfied

1. \( a \times b \in M \)
2. \( (a + b) \alpha c = a \alpha c + b \alpha c, \quad a(\alpha + \beta) b = a \alpha b + a \beta b, \quad a \alpha (b + c) = a \alpha b + a \alpha c, \)
3. \( (a \alpha b) \beta c = a \alpha (b \beta c) \)

then \( M \) is called a \( \Gamma' \)-ring.

Usually, the former is called a \( \Gamma' \)-ring in the sense of Nobusawa and the latter merely a \( \Gamma' \)-ring.

The class of \( \Gamma' \)-rings contains not only all rings but also all Hestenes ternary rings \((2)\).

In recent years there has been a remarkable growth of \( \Gamma' \)-ring. Many interesting results have come out during the past twenty years indicating the likelihood of still further fruitful results.

The role of ideals and radicals in the study of classical ring theory is well recognised.
The study of ideals and radicals have played such a central role in the development of \( F' \)-ring and their importance cannot be overemphasized. A lot of interesting research is centered around the ideals and radicals of \( F' \)-rings. Also, lot of interesting concepts and theorems have resulted in the attempts to study the properties of simplicity and semisimplicity of \( F' \)-rings.

In \([25]\) Nobusawa introduced the notions of simplicity and semisimplicity of \( F' \)-rings and obtained analogue of the Wedderburn's theorem for \( F' \)-rings with minimum condition on one-sided ideals, which was as follows: A simple \( F' \)-ring which satisfies the chain condition for left and right ideals is the set \( D_{n,m} \) of all rectangular matrices of type \( n \times m \) over some division ring \( D \) and \( F' \) is \( D_{m,n} \) of type \( m \times n \). The product \( a \cdot b \) is the usual matrix product of three elements \( a, \gamma \) and \( b \) of \( D_{n,m}, D_{m,n} \) and \( D_{n,m} \). This is a generalization of the theorem of Wedderburn on simple rings. Subsequently he proved that a semi-simple \( F' \)-ring \( M \) satisfying the chain condition for left and right ideals was a direct sum of simple \( F' \)-rings where

\[
F' = F'_1 + F'_2 + \cdots + F'_n \quad \text{(direct)},
\]

\[
M = M_1 + M_2 + \cdots + M_n \quad \text{(direct)},
\]

where \( M_1 \) are simple \( F' \)-rings and \( M_i F' M_j = 0 \) if \( i \neq j \) and \( M_i F' j M_i = 0 \) if \( i \neq j \).
In [1] Barnes introduced the notions of \( f' \)-homomorphism, prime and (right) primary ideals, \( m \)-systems and the radical of an ideal for \( f' \)-rings where the defining conditions for a \( f' \)-ring had been slightly weakened, as stated earlier, to permit defining residue class \( f' \)-rings.

He defined radical \( r(A) \) of an ideal \( A \) in a \( f' \)-ring \( M \) and proved that the radical \( R \) of a \( f' \)-ring \( M(\text{i.e.} \ r(0)) \) was an ideal of \( M \) and the radical of \( M/R \) was zero, by methods similar to those of McCoy. Also he obtained the classical Noether-Lasker theorems concerning primary representation of ideals for \( f' \)-rings.

In [20] J. Luh extended the concept of primitivity to \( f' \)-rings and characterised the primitive \( f' \)-rings in the sense of Nobusawa having minimal one-sided ideals, by means of certain \( f' \)-rings of continuous semilinear transformations. This characterisation generalised the well-known structure theorem for primitive \( f' \)-rings given by Jacobson as well as the result of Nobusawa for simple \( f' \)-rings. Also in [21] J. Luh extended the notions of simplicity and complete primeness to \( f' \)-rings. The definition of simple \( f' \)-rings given by J. Luh differs slightly from Nobusawa's original definition. However, the two concepts are identical for a \( f' \)-ring in the sense of Nobusawa with minimum condition on one-sided ideals. The author also studied the relations among simplicity, primeness, primitivity and complete primeness for \( f' \)-rings.
Much of the developments was analogous to the corresponding part of the classical ring theory. He also defined socles for $\mathcal{P}$-rings and studied their basic properties. Lastly, he proved an analogue of the Likoff theorem for simple $\mathcal{P}$-rings having minimal one sided ideals.

Coppage - Luh [3] introduced the notions of Jacobson radical, Levitski nil radical, nil radical and strongly nilpotent radical for $\mathcal{P}$-rings. They also studied Barnes' prime radical and obtained inclusion relations for these radicals. Lastly, they have proved that all these radicals coincide in case of a $\mathcal{P}$-ring which satisfied the descending chain condition on one-sided ideals.

the $V$-ring in the sense of Nobusawa with right and left unities. He proved that the lattices of all ideals of $M$, $\Gamma$, $R$, $L$ were isomorphic to one another where $R$ and $L$ denoted respectively the right and the left operator ring of $M$. He also defined the residue class $V$-ring for a $V$-ring in the sense of Nobusawa. By noting that $(R, \Gamma, M, L)$ was a Morita context, the form of any ideal in the ring

$$\left( \begin{array}{c} R \\ \Gamma \\ M \\ L \end{array} \right) \left( \begin{array}{c} r \\ \gamma \\ m \\ \lambda \end{array} \right), \quad r \in R, \ m \in M, \quad \gamma \in \Gamma, \quad \lambda \in L$$

with respect to the matrix addition and multiplication, had been decided. Using this he obtained the mutual relations of different types of radicals among $M$, $\Gamma$, $R$, $L$. In [15] he introduced the notion of commutativity to a $V$-ring in the sense of Nobusawa and defined $V$-field. He also characterised commutative $V$-rings. In [16] S. Kyuno studied Nobusawa's $V$-ring with minimum condition and obtained a generalisation of Nobusawa's result corresponding to the Wedderburn - Artin's theorem in the classical ring theory. In [17] and [12] S. Kyuno studied the Jacobson radical of Nobusawa's $V$-ring and also of Barnes' $V$-ring. He proved that the right and the left Jacobson radicals coincided for Nobusawa's $V$-ring with right and left unities and also for Barnes' $V$-ring without unities. In [18] he studied prime ideals in a Barnes' $V$-ring and obtained a correspondence between the set of prime ideals of a $V$-ring $M$ and the right (left) operator ring $R(L)$ of $M$. Lastly he obtained a relation between the prime radical of a $V$-ring $M$ and the prime radical of the $V_{n,m}$-ring $M_{n,m}$. 
In [23] K. Murata, Y. Kurata and H. Marubayashi introduced the notion of a prime ideal, called \( f \)-prime ideal which was a generalisation of Van der Walt's \( s \)-prime ideal [27] and also introduced \( f \)-primary ideal and \( f \)-radical of an ideal \( A \). They obtained some results which were analogous to those of prime ideals in the ring theory.

In [27] B. F. Hau extended the notions of \( f \)-prime and \( f \)-primary ideals, called \( g \)-prime and \( g \)-primary, to \( \Gamma \)-rings in the sense of Barnes and obtained some results analogous to the results of \( f \)-prime and \( f \)-primary ideals in the classical ring theory. He also obtained an analogue for \( \Gamma \)-rings of primary ideals and the classical Noether–Lasker theorem concerning primary decomposition of ideals.

In [26] T. S. Ravisankar and U. S. Shukla introduced the notion of a module, called \( R \)-module over a \( \Gamma \)-ring and studied some properties of it. The notions of primitivity and Jacobson radical were introduced via \( R \)-module. They proved that the usual properties of the Jacobson radical in the classical ring theory also held good for the radical in \( \Gamma \)-ring also. They defined in this paper a weak \( \Gamma_n \)-ring and characterised the Jacobson radical of the weak \( \Gamma_n \)-ring in different ways. It was shown that a ring, ternary rings [19], [8] and associative triple system [24] could be considered as a weak \( \Gamma_n \)-ring and any weak \( \Gamma_n \)-ring \( R \) could be embedded into a suitable associative ring \( A \) and simplicity and semi-simplicity of \( R \) and \( A \) were related. They obtained a result which
was a generalisation of the classical Wedderburn - Artin theorem for rings to \( P' \)-rings and which characterised the strongly simple, strongly right Artinian weak \( P'_n \)-rings as the \( P' \)-rings of rectangular matrices over division rings.

From the above discussion, we may conclude that in the area of \( P' \)-rings, most of the works have been done involving different types of ideals and radicals viz. prime radical, Jacobson radical, Levitzki nil radical, nil radical, strongly nilpotent radical etc. Some authors have dealt with \( R' \)-modules.

The object of this part of the thesis is to study some more properties of \( P' \)-rings. We open our discussion of the thesis by introducing a characterisation of \( P' \)-ring in the sense of Barnes.

In chapter 1 we have given a simple characterisation of Barnes' \( P' \)-ring. We have shown that if \( M \) is a \( P' \)-ring and \( A, B \) are two additive abelian groups such that \((M, P')\) acts faithfully on \((A, B)\) then the \( P' \)-ring \( M \) is isomorphic to a \( P'_1 \)-ring \( M_1 \) such that the each element of \( M_1 \) is a homomorphism of \( A \) into \( B \) and each element of \( P'_1 \) is a homomorphism of \( B \) into \( A \). In the classical ring theory, we have seen that an additive abelian group \( M \) is an \( R \)-module if and only if there is a ring morphism \( f: R \rightarrow \text{End}(M) \) where \( R \) is a unitary ring. We have generalised this in the theory of \( P' \)-rings. We have obtained the following result:
Let \((A, B)\) be a pair of additive abelian groups and \(R_1\) be a \(\Gamma'\)-ring; then \(A\) will be a right \(R_1, \Gamma'\)-module and \((R_1, \Gamma')\) acts on \((B, A)\) if and only if there exists a homomorphism \((f_1, f_2)\) from the \(\Gamma'_1\)-ring \(R_1\) into the \(\Gamma'\)-ring \(R\) where \(\Gamma' = \text{Hom}(A, B)\) and \(R = \text{Hom}(B, A)\).

In chapter 2 we have introduced the notions of regularity and biregularity in \(\Gamma'\)-rings and studied the properties of regular and biregular \(\Gamma'\)-rings. In §3 of chapter 2 we have shown that the centre of a regular \(\Gamma'\)-ring is also regular. Also we have obtained different necessary and sufficient conditions for a \(\Gamma'\)-ring to be regular. In the classical ring theory we have seen that in a regular ring the sum of two principal right (left) ideals is a principal right (left) ideal. We have generalised this result in a \(\Gamma'\)-ring in which idempotents commute with one another. Lastly we have shown that every regular \(\Gamma'\)-ring is semi-simple. In §4 we have studied biregular \(\Gamma'\)-rings. We have generalised some results of biregular rings in biregular \(\Gamma'\)-rings. We have defined right (left, two-sided) annihilator of an ideal \(A\) in a \(\Gamma'\)-ring and have shown that in a biregular \(\Gamma'\)-ring the two annihilators of an ideal \(A\) coincide. Lastly with these notions we have proved the following theorem:

If \(M\) is a biregular \(\Gamma'\)-ring in which every maximal ideal has a nonzero annihilator then
i) the ideals of \( M \) form a Boolean algebra.

ii) each nonzero ideal \( A \) of \( M \) contains a minimal ideal.

iii) \( M \) is the discrete direct sum of its minimal ideals.

In the classical ring theory we have seen that a commutative ring \( R \) will be a field if and only if for every nonzero element \( a \) and for every pair of nonzero elements \( b, c \) there exists an element \( a' \) such that \( a' b c = x \) for all \( x \in M \). Similarly we have seen in chapter - 3 that a commutative \( \Gamma' \)-ring \( M \) in which for every nonzero element \( a \) of \( M \) and for every pair of nonzero elements \( \gamma_1, \gamma_2 \) of \( \Gamma' \) there exists an element \( a' \) in \( M \) such that \( \gamma_1 a' \gamma_2 b = b \) for all \( b \in M \) satisfies most of the properties which are analogous to those of field in the classical ring theory. We call such a commutative \( \Gamma' \)-ring a \( \Gamma' \)-field. In this chapter we have studied some properties of \( \Gamma' \)-field. Also we have introduced the notions of subdirect sum and direct sum of commutative \( \Gamma' \)-rings (\( \Gamma' \) being the same for all of them) and obtained the following results:

1) A commutative \( \Gamma' \)-ring \( M \) is isomorphic to a subdirect sum of \( \Gamma' \)-fields if and only if the intersection of all maximal ideals of \( M \) is zero.

2) A commutative semisimple \( \Gamma' \)-ring is a subdirect sum of \( \Gamma' \)-fields.
CHAPTER 1

A characterisation of $\Gamma'$-ring.

Introduction 1.

In this chapter we have given two simple characterisations of $\Gamma'$-ring (in the sense of Barnes) and $R\Gamma'$-modules.

Preliminaries.

2.1 Definition. Let $M$ and $\Gamma'$ be two additive abelian groups. If for all $a, b, c \in M$ and $\alpha, \beta, \gamma \in \Gamma'$ the conditions

1. $a \cdot b \in M$
2. $(a + b) \cdot c = a \cdot c + b \cdot c$, $a(\alpha + \beta) = a \cdot \alpha + a \cdot \beta$, $a \cdot (b + c) = a \cdot b + a \cdot c$.
3. $(a \cdot b) \cdot \gamma = a \cdot (b \cdot \gamma)$ are satisfied then following Barnes [1] $M$ is called a $\Gamma'$-ring.

2.2 Definition. If $M_i$ is a $\Gamma_i'$-ring for $i = 1, 2$ then an ordered pair $f = (f_1, f_2)$ of mappings is called a homomorphism of $M_1$ into $M_2$ if

1) $f_1$ is a group homomorphism from $M_1$ into $M_2$,
2) $f_2$ is a group homomorphism from $\Gamma_1'$ into $\Gamma_2'$,
3) for every $x, y \in M_1$, $\gamma \in \Gamma_1'$, $f_1(x \cdot \gamma \cdot y) = f_1(x) f_2(\gamma) f_1(y)$.

If $f_1$ and $f_2$ are both group isomorphisms then $f$ is called an isomorphism of $M_1$ onto $M_2$ [16].
2.3 Definition. Let $R$ be a $\Gamma'$-ring. An additive abelian group $M$ is called a right $R\Gamma'$-module if there exists a map $\phi:M \times R \times R \to M$ satisfying ($\phi(m, \alpha, x)$ will be denoted by $m \alpha x$ in short)

1) $(m + n) \alpha x = m \alpha x + n \alpha x$,
2) $m \alpha (x + y) = m \alpha x + m \alpha y$,
3) $m \beta (x \alpha y) = (m \beta x) \alpha y$ for all $x, y$ in $R$, $\alpha, \beta$ in $\Gamma'$ and $m, n$ in $M \{26\}$.

Section 3.

Let $A, B$ be two additive abelian groups. $M = \text{Hom}(A, B)$ denotes the set of all homomorphisms of $A$ into $B$, $\Gamma' = \text{Hom}(B, A)$ denotes the set of all homomorphisms of $B$ into $A$. Let $x, y \in M$ and $\alpha, \beta \in \Gamma'$. If $x \alpha y$ is the usual composite map, then it can be shown easily that $M$ is a $\Gamma'$-ring. Define the mappings $(x, a) \to x a$ from $M \times A \to B$ and $(b, \alpha) \to b \alpha$ from $B \times \Gamma' \to A$ for all $a \in A, b \in B, x \in M$ and $\alpha \in \Gamma'$ where $x a$ and $b \alpha$ denote the images of $a$ and $b$ under $x$ and $\alpha$ respectively. It can be shown that the above two mappings will satisfy the following conditions

i) $(x + y) a = x a + y a$, $x(a_1 + a_2) = xa_1 + xa_2$

ii) $b (\alpha + \beta) = b \alpha + b \beta$, $(b_1 + b_2) \alpha = b_1 \alpha + b_2 \alpha$

iii) $y((xa) \alpha) = (y \alpha x) a$ for all $x, y \in M$, $\alpha, \beta \in \Gamma'$, $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$. 
Also we note that for any \((0 \neq x)\) in \(M\), \(x \neq 0\) and for any \((0 \neq \alpha)\) in \(\Gamma\), \(B \neq 0\).

Following the above discussion, let us write the following definition.

3.1. **Definition.** Let \((A, B)\) be two additive abelian groups and \(M\) be a \(\Gamma\)-ring. \((M, \Gamma)\) is said to act on \((A, B)\) if there are mappings \(M \times A \rightarrow B\) and \(B \times \Gamma \rightarrow A\) satisfying

i) \((x + y) a = x a + y a, x(a_1 + a_2) = x a_1 + x a_2\),

ii) \(b(\alpha + \beta) = b \alpha + b \beta, (b_1 + b_2) \alpha = b_1 \alpha + b_2 \alpha\),

iii) \(y((x a) \alpha) = (y \alpha x) a\) for all \(a, a_1, a_2 \in A, b, b_1, b_2 \in B, x, y \in M\) and \(\alpha, \beta \in \Gamma\).

Moreover \((M, \Gamma)\) is said to act faithfully on \((A, B)\) if for any \((0 \neq x)\) in \(M\), \(x \neq 0\) and for any \((0 \neq \alpha)\) in \(\Gamma\), \(B \neq 0\).

Thus we have the following theorem.

3.2 **Theorem.** Given a pair \((A, B)\) of additive abelian groups, there exists a \(\Gamma\)-ring \(M\) such that \((M, \Gamma)\) acts faithfully on \((A, B)\).

3.3 **Theorem.** Let \(M\) be a \(\Gamma\)-ring and \(A, B\) be two additive abelian groups. If \((M, \Gamma)\) acts faithfully on \((A, B)\) then the \(\Gamma\)-ring \(M\) is isomorphic to a \(\Gamma_1\)-ring \(M_1\) such that each element
of $M_1$ is a homomorphism of $A$ into $B$ and each element of $P_j$ is a homomorphism of $B$ into $A$.

**Proof.** Let $x \in M$. Define $l_x : A \rightarrow B$ by $l_x(a) = x \cdot a$.

Let $\alpha \in \Gamma$. Define $r_\alpha : B \rightarrow A$ by $r_\alpha(b) = b \cdot \alpha$.

Suppose $a_1, a_2 \in A$. Then $l_x(a_1 + a_2) = x \cdot (a_1 + a_2) = x \cdot a_1 + x \cdot a_2$ (by (i) of definition 3.1) $= l_x(a_1) + l_x(a_2)$.

Hence $l_x$ is a homomorphism of $A$ into $B$. Similarly we can show that $r_\alpha$ is a homomorphism of $B$ into $A$. Let $S = \{l_x : x \in M\}$ and $T = \{r_\alpha : \alpha \in \Gamma\}$. Let $l_{x_1}, l_{x_2} \in S$. Define $l_{x_1} + l_{x_2}$ by $(l_{x_1} + l_{x_2})(a) = l_{x_1}(a) + l_{x_2}(a) = (x_1 + x_2) \cdot a$.

So $l_{x_1} + l_{x_2} = l_{x_1 + x_2} \in S$. Define $r_\alpha + r_\beta$ by $(r_\alpha + r_\beta)(b) = r_\alpha(b) + r_\beta(b)$. We can show that $S$ and $T$ are both additive abelian groups. Define $l_{x_1}r_\alpha l_{x_2}$ by the usual mapping product. Now $(l_{x_1}r_\alpha l_{x_2})a = x_1((x_2 \cdot a) \cdot \alpha) = (x_1 \cdot x_2) \cdot a$.

So $l_{x_1}r_\alpha l_{x_2} = l_{x_1 \cdot x_2} \in S$. We can show that $S$ is a $T$-ring. Define $f = (f_1, f_2)$ from the $\Gamma$-ring $M$ into the $T$-ring $S$ by $f_1(x) = l_x$, $f_2(\alpha) = r_\alpha$. Then $f_1 : M \rightarrow S$ and $f_2 : \Gamma \rightarrow T$ are group-homomorphisms. Also $f_1(x \cdot y) = l_{x \cdot y} = l_x r_\alpha l_y = f_1(x) f_2(\alpha) f_1(y)$. Hence $f$ is a homomorphism of the $\Gamma$-ring $M$ onto the $T$-ring $S$. Suppose $f_1(x) = 0$. Then $l_x = 0$. Hence $l_x(a) = 0$, $\forall a \in A$. So $\forall A = 0$.

Now since $(M, \Gamma)$ acts faithfully on $(A, B)$, $x A = 0$ implies $x = 0$. Consequently $f_1$ is injective. Similarly we can show that $f_2$ is also
injective. Hence the $\Gamma$-ring $M$ is isomorphic to the $T$-ring $S$.

Hence the theorem.

3.4 Theorem. Given an additive abelian group $A$ there exists a $\Gamma$-ring $R$ such that $A$ is a right $R\Gamma$-module.

Proof. Let $B$ be another additive abelian group. Also let $\Gamma = \text{Hom}(A, B)$ and $R = \text{Hom}(B, A)$. Then $R$ is a $\Gamma$-ring.

Define a mapping $\mathcal{O} : A \times \Gamma \times R \to A$ (written $\mathcal{O}(a, \gamma, x) = a \gamma x$) by $a \gamma x = (\gamma a) x$ where $\gamma a$ denotes the image of $a$ under $\gamma$ and $(\gamma a) x$ denotes the image of $\gamma a$ under $x$. It can be easily verified that the above mapping $\mathcal{O}$ satisfies the following conditions

i) $(a_1 + a_2) \gamma x = a_1 \gamma x + a_2 \gamma x,$

ii) $a \gamma (x_1 + x_2) = a \gamma x_1 + a \gamma x_2,$

iii) $a \gamma_1(x_1 \gamma_2 x_2) = (a \gamma_1 x_1) \gamma_2 x_2$ for all $a, a_1, a_2 \in A, \gamma, \gamma_1, \gamma_2 \in \Gamma$ and $x, x_1, x_2 \in R$. Consequently $A$ is a right $R\Gamma$-module.

3.5 Theorem. Let $A$ be an additive abelian group and $(f_1, f_2)$ be a homomorphism from a $\Gamma$-ring $R_1$ into a $\Gamma$-ring $R$ where $\Gamma = \text{Hom}(A, B)$ and $R = \text{Hom}(B, A)$, $B$ is an additive abelian group; then $A$ is also a right $R_1\Gamma_1$-module.
Proof. We define a mapping \( \varnothing : A \times P \times R_1 \rightarrow A \) by
\[
\varnothing (a, y_1, x_1) = a \cdot y_1 \cdot x_1 = (f_2(y_1) \cdot a) \cdot f_1(x_1).
\]
It can be easily verified that the mapping \( \varnothing \) satisfies the following conditions

1) \( (a_1 + a_2) \cdot y \cdot x = a_1 \cdot y \cdot x + a_2 \cdot y \cdot x, \)

2) \( a \cdot (y_1 + y_2) = a \cdot y_1 + a \cdot y_2, \)

3) \( a \cdot (y_1 \cdot y_2 \cdot x_3) = (a \cdot y_1 \cdot y_2) \cdot x_3 \) for all \( a, a_1, a_2 \) in \( A; \)
\( y, y_1, y_2 \) in \( P \) and \( x, x_1, x_2 \) in \( R. \) Hence \( A \) is a right \( R_1 \cdot P_1 \) - module.

3.6 Theorem. Let \( (A, B) \) be a pair of additive abelian groups
and \( R_1 \) be a \( P_1 \) - ring such that \( (R_1, P_1) \) acts on \( (B, A) \); then
\( A \) is a right \( R_1 \cdot P_1 \) - module.

Proof. Since \( (R_1, P_1) \) acts on \( (B, A) \) it follows by Theorem 3.3
that there exists a homomorphism \( (f_1, f_2) \) from the \( P_1 \) - ring \( R_1 \)
into the \( P \) - ring \( R \) where \( R = \text{Hom}(B, A) \) and \( P = \text{Hom}(A, B). \)
Consequently by Theorem 3.5 it follows that \( A \) is a right
\( R_1 \cdot P_1 \) - module.

3.7 Theorem. Let \( (A, B) \) be a pair of additive abelian groups
and \( R_1 \) be a \( P_1 \) - ring; then \( A \) will be a right \( R_1 \cdot P_1 \) - module
and \( (R_1, P_1) \) acts on \( (B, A) \) if and only if there exists a homomorphism
\( (f_1, f_2) \) from the \( P_1 \) - ring \( R_1 \) into the \( P \) - ring \( R \) where
\( P = \text{Hom}(A, B) \) and \( R = \text{Hom}(B, A). \)
Proof. If \( A \) is a right \( R_1 V_1 \) - module and \( (R_1, V_1) \) acts on \( (B, A) \) then it follows from the Theorem 3.3 that there exists a homomorphism from the \( \Gamma_1 \) - ring \( R_1 \) into the \( \Gamma' \) - ring \( R \) where \( \Gamma = \text{Hom}(A, B) \) and \( R = \text{Hom}(B, A) \). Conversely we assume that there exists a homomorphism \( (f_1, f_2) \) from the \( \Gamma_1 \) - ring \( R_1 \) into the \( \Gamma' \) - ring \( R \) where \( \Gamma = \text{Hom}(A, B) \) and \( R = \text{Hom}(B, A) \). Then it follows from the theorem 3.5 that \( A \) is a right \( R_1 V_1 \) - module. Also if we define the mappings \( R_1 \times B \to A \) and \( A \times V_1 \to B \) by \( xb = f_1(x)b \) and \( a\alpha = f_2(\alpha) \) it can be easily shown that \( (R_1, V_1) \) acts on \( (B, A) \).

Hence the theorem.
CHAPTER 2

On regular \( \Gamma \)-rings.

Introduction.

In this chapter we have introduced the notions of regularity and biregularity in \( \Gamma \)-ring and studied some properties of regular and biregular \( \Gamma \)-rings. We have shown that many of the fundamental results of classical regular and biregular rings have been held in the case of \( \Gamma \)-rings as we have expected.

Preliminaries.

We recall the following definitions and notations.

2.1 Definition. A right (left) ideal of a \( \Gamma \)-ring \( M \) is an additive subgroup \( I \) of \( M \) such that \( I \Gamma M \subseteq I \) (\( M \Gamma I \subseteq I \)) where by \( I \Gamma M \) we shall mean the subset of \( M \) consisting of all elements of the form \( a \gamma m \) where \( a \in I \), \( \gamma \in \Gamma \) and \( m \in M \). If \( I \) is a right ideal as well as a left ideal of \( M \) then \( I \) is called a two-sided ideal (or simply an ideal) of \( M \) [1].

2.2 Definition. For each element \( a \) of a \( \Gamma \)-ring \( M \) the smallest right ideal containing \( a \) is called the principal right ideal generated by \( a \) and is denoted by \( (a)_r \). Similarly we can define \( (a)_l \) and \( (a)_p \), the principal left ideal and principal ideal generated by \( a \) [1].
2.3 **Definition.** Let $I$ be an ideal of a $\Gamma$-ring $M$. If for each $a + I, b + I$ in the factor group $M/I$ and for each $\gamma \in \Gamma$ we define $(a + I) \gamma (b + I) = a \gamma b + I$ then $M/I$ becomes a $\Gamma'$-ring which is called residue class $\Gamma'$-ring [1].

2.4 **Definition.** An ideal $P$ of a $\Gamma'$-ring $M$ is said to be prime if for any two ideals $A, B$ of $M$, $A \cap B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$ [1].

2.5 **Definition.** If $S$ is a subset of a $\Gamma'$-ring $M$ then $S$ is called an $m$-system if $S = \emptyset$ (empty) or $a, b \in S$ implies that

(a) $\Gamma (b) \cap S \neq \emptyset$ [1].

2.6 **Definition.** A $\Gamma'$-ring $M$ is said to be commutative if $a \cdot b = b \cdot a$ holds for all $a, b \in M$ and for all $\gamma \in \Gamma$.

2.7 **Definition.** An element $a$ of a $\Gamma'$-ring $M$ is said to be idempotent if there exists an element $\gamma \in \Gamma$ such that $a \gamma a = a$. An ideal $I$ of a $\Gamma'$-ring $M$ is said to be idempotent if $I \cap I = I$.

2.8 **Definition.** Let $R$ be a $\Gamma'$-ring. An additive abelian group $M$ is called a right $R\Gamma'$-module (to be called just $R\Gamma'$-module) if there exists a map $\phi : M \times \Gamma \times R \rightarrow M$ satisfying $\phi(m, \gamma, x)$ will be denoted by $m \gamma^R x$ in short) (1) $(m + n) \gamma x = m \gamma x + n \gamma x$

(2) $m \gamma (x + y) = m \gamma x + m \gamma y$ and (3) $m \gamma (x \gamma y) = (m \gamma x) \gamma y$ for all
The notions of submodule, proper submodule, irreducible module, quotient module in case of \( \mathbb{R}^\Gamma \)-module can be defined as in the case of classical module [26].

2.9 Definition. An \( \mathbb{R}^\Gamma \)-module \( M \) is said to be faithful if \( M \cap x = 0 \) implies \( x = 0 \) [26].

2.10 Definition. A \( \mathbb{R}^\Gamma \)-ring \( R \) is said to be primitive if \( R \) admits a faithful irreducible \( \mathbb{R}^\Gamma \)-module. An ideal \( I \) of \( R \) is said to be primitive if and only if the quotient \( \mathbb{R}^\Gamma \)-ring \( R/I \) is primitive [26].

2.11 Definition. The Jacobson radical \( J(R) \) of a \( \mathbb{R}^\Gamma \)-ring \( R \) is defined by \( J(R) = \bigcap \text{Ann}(M) \), where \( M \) runs over all irreducible \( \mathbb{R}^\Gamma \)-modules, if any. In case \( R \) does not possess any irreducible \( \mathbb{R}^\Gamma \)-module, \( J(R) \) is defined to be \( R \) itself [26].

2.12 Definition. A \( \mathbb{R}^\Gamma \)-ring \( R \) is said to be semisimple if its Jacobson radical is zero [26].

3. Regular \( \mathbb{R}^\Gamma \)-ring.

3.1 Definition. An element \( a \) of a \( \mathbb{R}^\Gamma \)-ring \( M \) is said to be regular if there exist elements \( b \in M \) and \( \gamma_1, \gamma_2 \in \Gamma \) such that \( a \gamma_1 b \gamma_2 a = a \). A \( \mathbb{R}^\Gamma \)-ring \( M \) is said to be regular if every element of \( M \) is regular.
3.2 Example 1) Let \( M \) and \( \Gamma' \) be the additive abelian groups of all \( 1 \times 2 \) and \( 2 \times 1 \) matrices respectively over the ring of integers. Then \( M \) is a \( \Gamma' \)-ring where \( a \gamma b \) denotes the product of matrices \( a, \gamma \) and \( b, a, b \in M \) and \( \gamma \in \Gamma' \). Let \((a_1, b_1) \in M\).

If \((a_1, b_1) = (0, 0)\) then obviously \((a_1, b_1)\) is regular. If \(a_1 \neq 0\) then \((a_1 b_1) \left( 1 \right) \left( 1/a_1 \right) = (a_1 b_1)\) and if \(b_1 \neq 0\) then

\[
\begin{pmatrix}
  a_1 & b_1 \\
  1/b_1 & 1/b_1
\end{pmatrix}
\begin{pmatrix}
  a_1 & b_1 \\
  1 & 1
\end{pmatrix}
= (a_1 b_1).
\]

So \((a_1 b_1)\) is regular. This is true for all elements of \( M \). So \( M \) is a regular \( \Gamma' \)-ring.

3.3 Lemma. In a regular \( \Gamma' \)-ring \( M \) the principal right (resp. left) ideal \((a)r\) and (resp. \((a)\ell\)) generated by \( a \) is given by \((a)r = a \gamma_1 M = a \Gamma' M\) (resp. \((a)\ell = M \gamma_2 a = M \Gamma' a\)) where \( a, b \in M \) and \( \gamma_1, \gamma_2 \in \Gamma' \).

Proof. \( a \gamma_1 M \) is a right ideal containing \( a \). For if \( c = a \gamma_1 m_1 \) and \( d = a \gamma_1 m_2 \) lie in \( a \gamma_1 M \) then \( c - d = a \gamma_1 m_1 - a \gamma_1 m_2 = a \gamma_1 (m_1 - m_2) \in a \gamma_1 M \) and \( a \gamma_1 m_1 \gamma m = a \gamma_1 m_3 \in a \gamma_1 M \)

where \( \gamma \in \Gamma' \), \( m \in M \) and \( m_1 \gamma m = m_3 \). Since \( a = a \gamma_1 b \gamma_2 a \), \( a \in a \gamma_1 M \). Let \( A \) be another right ideal of \( M \) containing \( a \), then \( a \gamma_1 M \subseteq A \). So \( a \gamma_1 M \) is the smallest right ideal containing \( a \).

Hence \( a \gamma_1 M = (a)r \). Now \( a \gamma_1 M \subseteq a \Gamma' M \). Also \( a \Gamma' M = a \gamma_1 b \gamma_2 a \Gamma' M \subseteq a \gamma_1 M \Gamma' M \subseteq a \gamma_1 M \). Consequently \( a \gamma_1 M = a \Gamma' M \). Thus \((a)r = a \gamma_1 M = a \Gamma' M \) where \( a = a \gamma_1 b \gamma_2 a \).
3.4 Theorem. A $\mathcal{P}$-ring $M$ is regular if and only if every principal right ideal of $M$ is generated by an idempotent and $(a)_r = a \mathcal{P} M$ for all $a \in M$.

Proof. Suppose $M$ is regular and $a \in M$. Then by lemma 3.3

$(a)_r = a \mathcal{P} M$ for all $a \in M$. Also there exist elements

$\gamma_1, \gamma_2 \in \mathcal{P}$ and $b \in M$ such that $a = a \gamma_1 b \gamma_2 a$. Let $e = a \gamma_1 b$.

Then $e \gamma_2 e = (a \gamma_1 b) \gamma_2 (a \gamma_1 b) = (a \gamma_1 b \gamma_2 a) \gamma_1 b = a \gamma_1 b = e$. So $e$ is an idempotent element of $M$. From $a = e \gamma_2 a$ we deduce that $(a)_r \subseteq (e)_r$. However $e = a \gamma_1 b \in (a)_r$ implies $(e)_r \subseteq (a)_r$.

Hence $(a)_r = (e)_r$. So every principal right ideal of $M$ is generated by an idempotent. Conversely we assume that every principal right ideal of $M$ is generated by an idempotent and $(a)_r = a \mathcal{P} M$ for all $a \in M$. Given an element $a \in M$, let $e = e \gamma e \in M$ be an idempotent of $M$ such that $a \mathcal{P} M = (a)_r = (e)_r = e \mathcal{P} M$. Then for suitable $\gamma_1, \gamma_2 \in \mathcal{P}$ and $c, d \in M$ the equations $a = e \gamma_1 c$ and $e = a \gamma_2 d$ are satisfied; but these imply that $a = e \gamma_1 c = e \gamma_2 e \gamma_1 c = e \gamma a$. Now $a \gamma_2 d \gamma a = e \gamma a = a$ whence $M$ forms a regular $\mathcal{P}$-ring.

3.5 Theorem. The ideals of a regular $\mathcal{P}$-ring $M$ form a distributive lattice.

Proof. If $A$ and $B$ are ideals of a $\mathcal{P}$-ring $M$ then $A + B$ and $A \cap B$ are also ideals of $M$. It is easy to verify that the set of all ideals of a $\mathcal{P}$-ring $M$ forms a lattice with respect
to '†' and 'n'. Now if A, B, C are three ideals of a $\Gamma$-ring M then $A + (B \cap C) \subseteq (A + B) \cap (A + C)$. Conversely let $x \in (A + B) \cap (A + C)$. Since M is regular there exists an idempotent $e = e \gamma e$ such that $(x)_e = (e)_r$. Now $x \in (A + B) \cap (A + C)$ implies $e \in (A + B) \cap (A + C)$. So $e = a + b = a' + c$ for some $a, a' \in A$, $b \in B$ and $c \in C$. Then $e = e \gamma e = (a + b) \gamma (a' + c)$ $(a \gamma a' + a \gamma c + b \gamma a') + b \gamma c \in A + (B \cap C)$. Thus e and so $x \in A + (B \cap C)$. Consequently $A + (B \cap C) = (A + B) \cap (A + C)$.

Hence the theorem.

3.6 Theorem. A $\Gamma$-ring M is regular if and only if $(a)_r = a \Gamma M$, $(a)_l = M \Gamma a$ for all $a \in M$ and $I \cap J = I \Gamma J$ holds for every right ideal I and left ideal J of M.

Proof. Let I be a right ideal and J be a left ideal of a regular $\Gamma$-ring M. Since the inclusion $I \Gamma J \subseteq I \cap J$ always holds we have only to show that $I \cap J = I \Gamma J$. Let $a \in I \cap J$. By the regularity of M there exist elements $\gamma_1, \gamma_2 \in \Gamma$ and $b \in M$ such that $a = a \gamma_1 b \gamma_2 a$. Now $b \gamma_2 a \in J$, so $a = a \gamma_1 b \gamma_2 a \in I \Gamma J$. Hence $I \cap J \subseteq I \Gamma J$ proving thereby $I \cap J = I \Gamma J$. Also, by lemma 3.3 $(a)_r = a \Gamma M$, $(a)_l = M \Gamma a$ for all $a \in M$. Conversely we assume that the given conditions hold in M. Let $a \in M$. Now $a \in (a)_r$ and also $a \in (a)_l$. So $a \in (a)_r \cap (a)_l = (a)_r \cap (a)_l$ but $(a)_r = a \Gamma M$ and $(a)_l = M \Gamma a$. Hence $a \in a \Gamma M \Gamma M \Gamma a \subseteq a \Gamma M \Gamma a$. So $a$ is regular. Hence M is also regular.
3.7 Corollary. A commutative \( \Gamma \)-ring \( M \) is regular if and only if every ideal of \( M \) is idempotent and \( \mathfrak{a} (a) = a \Gamma M \) for all \( a \in M \).

**Proof.** If \( M \) is regular we may take \( I = J \) in the above theorem to conclude that the ideals of \( M \) are idempotent. Also \( \mathfrak{a} (a) = a \Gamma M \) for all \( a \in M \) follows from lemma 3.3. Conversely if \( I \) and \( J \) are two ideals of \( M \) then the idempotency of \( I \cap J \) yields \( (I \cap J)^2 = (I \cap J)^2 \cap (I \cap J) \leq I \cap J \cap I \cap J \) and so \( I \cap J = I \cap J \). Also given that \( \mathfrak{a} (a) = a \Gamma M \) for all \( a \in M \). Appealing to the theorem once more it follows that \( M \) is a regular \( \Gamma \)-ring.

3.8 Corollary. In a commutative \( \Gamma \)-ring \( M \) the condition
\[ (a \cap b) = (a) \cap (b) \quad \text{and} \quad (a \cap a) = a \Gamma a \Gamma M \] are equivalent to the regularity of \( M \).

**Proof.** If the indicated conditions hold then, in particular, \( a \in M \) implies \( a \Gamma a \Gamma M = (a) \Gamma (a) = (a) \Gamma \cap (a) = (a) \) (2). Now \( a \in (a) \) implies \( a \in a \Gamma a \Gamma M = a \Gamma M \cap a \). So \( a \) is regular and hence \( M \) is regular. Conversely if \( M \) is regular then \( (a) \cap (b) = (a) \cap (b) = (a \cap b) \) and \( (a \cap a) = a \Gamma a \Gamma M \). (The proof of \( (a \cap a) = a \Gamma a \Gamma M \) is same as that of lemma 3.3).

Hence the corollary.

3.9 Theorem. Let \( I \) be an ideal of a regular \( \Gamma \)-ring \( M \); then any ideal \( J \) of \( I \) is an ideal of \( M \).
Proof. I itself may be regarded as a regular $\Gamma$-ring. Indeed, if $a \in I \subseteq M$ then $a = a \gamma_1 b \gamma_2 a$ for some $b \in M$ and $\gamma_1, \gamma_2 \in \Gamma$.

Setting $c = b \gamma_2 a \gamma_1 b$ the element $c \in I$ and has the property that $a \gamma_1 c \gamma_2 a = a \gamma_1 (b \gamma_2 a \gamma_1 b) \gamma_2 a \gamma_1 a$. Our aim is to show that when $a \in J \subseteq I$ and $m \in M$ then both $a \sqrt{m}$ and $m \gamma a$ lie in $J$ where $\gamma \in \Gamma$. Now $a \sqrt{m} \in I$; hence there exist elements $d \in I$ and $\gamma_3, \gamma_4 \in \Gamma$ such that $(a \sqrt{m}) \gamma_3 \gamma_4 (a \sqrt{m}) = a \sqrt{m}$. Since $m \gamma_3 \gamma_4 a \sqrt{m} \in I$ and $\sqrt{J}$ is assumed to be an ideal of $I$ it follows that the product $(a \sqrt{m}) \gamma_3 \gamma_4 (a \sqrt{m})$ i.e. $a \sqrt{m} \in J$. A symmetric argument confirms that $m \gamma a \in J$. Hence the theorem.

3.10 Theorem. The sum of two principal right (left) ideals of a regular $\Gamma$-ring $M$ in which idempotents commute with one another is itself a principal right (left) ideal.

Proof. Consider the principal right ideals $(a)_r$ and $(b)_r$ of $M$. Since $M$ is regular and $a, b \in M$, $a = a \gamma_1 a' \gamma_2 a$ and $b = b \gamma_3 b' \gamma_4 b$ where $a', b' \in M$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$. By lemma 3.3 $(a)_r = a \gamma_1 M = a \gamma_1 M$ and $(b)_r = b \gamma_3 M = b \gamma_3 M$. By Theorem 3.4 $a \gamma_1 M = e \gamma_2 M$ where $e = a \gamma_1 a'$ is an idempotent such that $e \gamma_2 e = e$. 

We also have
\[ a \gamma_1 M + b \gamma_3 M \]
\[ = e \gamma_2 M + b \gamma_3 M \]
\[ = \{ e \gamma_2 x + b \gamma_3 y \mid x, y \in M \} \]
\[ = \{ e \gamma_2 x + e \gamma_2 b \gamma_3 y + b \gamma_3 y - e \gamma_2 b \gamma_3 y \mid x, y \in M \} \]
\[ = \{ e \gamma_2 (x + b \gamma_3 y) + (b - e \gamma_2 b) \gamma_3 y \mid x, y \in M \} \]
\[ = \{ e \gamma_2 z + (b - e \gamma_2 b) \gamma_3 y \mid x, y \in M \} \]
\[ = e \gamma_2 M + (b - e \gamma_2 b) \gamma_3 M. \]

Putting \( c = b - e \gamma_2 b \) we get
\[ a \gamma_1 M + b \gamma_3 M = e \gamma_2 M + c \gamma_3 M \]
where the element \( c \) has the property \( e \gamma_2 c = e \gamma_2 (b - e \gamma_2 b) \)
\[ = e \gamma_2 b - e \gamma_2 b \gamma_2 b = e \gamma_2 b - e \gamma_2 b = 0, \]
Also \( c \gamma_2 b^t \gamma_1 c = (b - e \gamma_2 b) \gamma_2 b^t \gamma_4 (b - e \gamma_2 b) = b \gamma_3 b^t \gamma_4 b - b \gamma_3 b^t \gamma_4 e \gamma_2 b \)
\[ = e \gamma_2 b \gamma_3 b^t \gamma_4 e \gamma_2 b = b - (b \gamma_3 b^t) \gamma_4 e \gamma_2 b \]
\[ = e \gamma_2 (b \gamma_3 b^t) \gamma_4 e \gamma_2 b = b - e \gamma_2 b - (b \gamma_3 b^t) \gamma_4 e \gamma_2 b \]
\[ = b - e \gamma_2 b - (b \gamma_3 b^t) \gamma_4 e \gamma_2 b + (b \gamma_3 b^t) \gamma_4 e \gamma_2 b = b - e \gamma_2 b = c \]
(using the condition that the idempotents commute with one another
and the fact that \( b \gamma_3 b^t \) and \( e \) are idempotents of \( M \)). Again
\[ c \gamma_3 M = f \gamma_4 M \]
where \( c \gamma_3 b^t = f \) and \( f \gamma_4 f = f \). Also \( e \gamma_2 f \)
\[ = e \gamma_2 c \gamma_3 b^t = 0. \]
So \( a \gamma_1 M + b \gamma_3 M = e \gamma_2 M + c \gamma_3 M = e \gamma_2 M + f \gamma_4 M \)
Our remaining object is to demonstrate $e \in M + f \cap M$. On the other hand,

$$e \in M + f \cap M = (e + f) \cap M \subseteq e \in M + f \cap M.$$  

At the same time,

$$f = f \gamma_2 f = e \gamma_2 f + f \gamma_2 f \quad \text{ ( since } e \gamma_2 f = e \gamma_2 e \gamma_4 f = e \gamma_4 \gamma_4 e = 0)$$

$$= (e + f) \gamma_4 f \in (e + f) \cap M.\text{ Thus } e \in M + f \cap M \subseteq (e + f) \cap M.$$

Hence the theorem.

3.11 Theorem. In a regular $\Gamma$-ring $M$ each ideal is the intersection of all prime ideals containing it.

Proof. Let $A$ be an ideal of $M$. Since $M$ is a regular $\Gamma$-ring, each element $a$ of $M$ is an $m$-system. Let $P_i, i \in I$ be the family of all prime ideals containing the ideal $A$. Then $A \subseteq \cap_{i \in I} P_i$.

We claim that $A = \cap_{i \in I} P_i$. If $A \neq \cap_{i \in I} P_i$, there exists an element $a \in \cap_{i \in I} P_i$ such that $a \notin A$. Now by lemma 2 of [1] there exists a prime ideal $P$ in $M$ containing $A$ such that $a \notin P$.

So $a \notin \cap_{i \in I} P_i$, which is a contradiction. Hence $A = \cap_{i \in I} P_i$.

3.12 Proposition. A regular $\Gamma$-ring $M$ is semisimple.

Proof. Let $J(M)$ denote the Jacobson radical of $M$ and $a \in J(M)$. Since $M$ is regular, $a = a \gamma_1 b \gamma_2 a$ for some $b \in M$ and some $\gamma_1, \gamma_2 \in \Gamma$. Since $J(M)$ is an ideal of $M$ (P 542,[26]), $a \in J(M)$ implies $-b \gamma_2 a \in J(M)$. Again elements of $J(M)$ are $\alpha$-right.
quasi regular for all \( \alpha \in \Gamma \) (P 541, [26]). Hence \(- b \gamma_2 a\)
has a right quasi inverse \(c\) such that \(- b \gamma_2 a + c - b \gamma_2 a \alpha c = 0\)
for all \( \alpha \in \Gamma \). Now \( 0 = a \gamma_1 0 = a \gamma_1 ( - b \gamma_2 a + c - b \gamma_2 a \gamma_1 c) \)
\(= a \gamma_1 b \gamma_2 a + a \gamma_1 c - a \gamma_1 b \gamma_2 a \gamma_1 c = a + a \gamma_1 c - a \gamma_1 c = a \).
So \( a = 0 \). Hence \( \Gamma(M) = 0 \). Consequently \( M \) is semisimple.

3.13 Corollary. A nonzero regular \( \Gamma \)-ring \( M \) is a subdirect sum of primitive \( \Gamma \)-rings.

Proof. The theorem follows from the above proposition and the proposition 2.13 of [26].

3.14 Definition. The centre \( B \) of a \( \Gamma \)-ring \( M \) is a subset of
\( M \) consisting of all elements \( a \) of \( M \) such that \( a \gamma x = x \gamma a \)
for all \( x \in M \) and for all \( \gamma \in \Gamma \).

3.15 Proposition. The centre \( B \) of a regular \( \Gamma \)-ring \( M \) is
regular.

Proof. Let \( a, b \in B \). Then \( a \gamma x = x \gamma a \) and \( b \gamma x = x \gamma b \) for
all \( x \in M \) and for all \( \gamma \in \Gamma \). Now \( (a - b) \gamma x = a \gamma x - b \gamma x \)
\(= x \gamma a - x \gamma b = x \gamma (a - b) \). So \( a - b \in B \). Again, let \( \gamma_1 \in \Gamma \); then
\( (a \gamma_1 b) \gamma x = a \gamma_1 (b \gamma x) = (b \gamma x) \gamma_1 a = (x \gamma b) \gamma_1 a \)
\(= x \gamma (b \gamma_1 a) = x \gamma (a \gamma_1 b) \). So \( a \gamma_1 b \in B \). Consequently \( B \) is a
\( \Gamma \)-subring of \( M \).
To show $B$ is regular, let $a \in B$. Since $M$ is regular there exist elements $x \in M$ and $\gamma_1, \gamma_2 \in \gamma$ such that $a \gamma_1 x \gamma_2 a = a$. As $a$ commutes with all the elements of $M$, we have $a \gamma_1 (a \gamma_2 a)$

$$\gamma_1 (x \gamma_2 x \gamma_1 x) \gamma_2 a = (a \gamma_1 x \gamma_2 a) \gamma_1 (a \gamma_2 x \gamma_1 x) \gamma_2 a = a \gamma_1 x \gamma_2 a = a.$$

Again $a \gamma_2 (a \gamma_1 x) = a \gamma_1 x \gamma_2 a = a$ and $x \gamma_1 (a \gamma_2 a) = a \gamma_1 x \gamma_2 a = a$.

So $a = a \gamma_2 (a \gamma_1 x) = x \gamma_1 (a \gamma_2 a)$ commutes with all the elements of $M$. Hence for every $z \in M$ and for every $\gamma \in \gamma$,

$$x \gamma ( (a \gamma_2 a) \gamma_1 z) = (a \gamma_2 a) \gamma x \gamma_1 z \quad \text{[since $a \gamma_2 a \in B$ as $B$ is a subring of $M$ and $a \in B$]}.$$

Now $(a \gamma_2 a) \gamma_1 x \gamma_1 z = x \gamma_2 x \gamma_1 (a \gamma_2 a) \gamma_1 z \gamma x$.

$$x \gamma_2 ( (a \gamma_2 a) \gamma_1 z) \gamma x = (a \gamma_2 a) \gamma_1 x \gamma_2 a \gamma x = a \gamma_2 z \gamma_1 x \gamma_2 a \gamma x = a \gamma_2 z \gamma_1 x \gamma_2 a \gamma_1 (x \gamma x) $$

$$= a \gamma_2 z \gamma_1 x \gamma_2 (x \gamma x) \gamma_1 a \gamma_2 x \gamma x \gamma_1 x \gamma x \gamma_1 x \gamma x \quad \text{[since $a \in B$]}$$

$$= a \gamma_2 z \gamma_1 (x \gamma x) \gamma_1 a \gamma_2 x \gamma x \gamma_1 x \gamma x (a \gamma_2 a) \gamma_1 z \gamma x.$$

Now $(a \gamma_2 a) \gamma_1 (x \gamma_2 x \gamma_1 x) \gamma z = (x \gamma_2 x \gamma_1 x) \gamma_1 (a \gamma_2 a) \gamma z$.

$$= (x \gamma_2 x \gamma_1 x) \gamma_1 (a \gamma_2 a) \gamma z = (x \gamma_2 x \gamma_1 x) \gamma_1 (z \gamma a) \gamma z$$

$$= (x \gamma_2 x \gamma_1 a) \gamma_1 x \gamma z \gamma a \gamma_2 z = x \gamma_2 x \gamma_1 a \gamma_1 (x \gamma a) \gamma_2 z$$

$$= x \gamma_2 x \gamma_1 x \gamma a \gamma_2 z = (x \gamma_2 x \gamma_1 x) \gamma a \gamma_1 z \gamma_2 a$$

$$= (x \gamma_2 x \gamma_1 x) \gamma z \gamma_2 a \gamma_1 a \gamma_2 z = (x \gamma_2 x \gamma_1 x) \gamma (a \gamma_2 z) \gamma_1 a$$

$$= (x \gamma_2 x \gamma_1 x) \gamma (a \gamma_2 a) \gamma_1 z = (a \gamma_2 a) \gamma_1 z \gamma (x \gamma_2 x \gamma_1 x)$$
4. Biregular \( \Gamma \)-ring.

4.1 Definition. A \( \Gamma \)-ring \( M \) is said to be biregular if each principal ideal (two-sided) is generated by an idempotent lies in the centre of \( M \).

4.2 Proposition. Every homomorphic image of a biregular \( \Gamma \)-ring \( M \) is biregular.

4.3 Proposition. A biregular \( \Gamma \)-ring \( M \) is semigimple.

Proof. Let \( a \in J(M) \). Since \( M \) is biregular there exists an idempotent \( e = e \gamma e \) in the centre of \( M \) such that \( (a) = (e) \).

Now \( J(M) \) is a two-sided ideal of \( M \) so \( e \in J(M) \) implies \( -e \in (e) \subseteq J(M) \). The elements of \( J(M) \) are \( \alpha \)-right regular quasi ideals for every \( \alpha \in \Gamma \) [28]. So there exists an element
z \in M \text{ such that } -e + z - e \sqrt{z} = 0. \text{ This implies } -e \sqrt{z} + e \sqrt{z} - e \sqrt{z} = 0 \text{ i.e. } -e + e \sqrt{z} - e \sqrt{z} = 0 \text{ or } e = 0. \text{ So } a = 0. \text{ Hence the proposition.}

4.4 Corollary. A nonzero biregular \( R' \)-ring \( M \) is a subdirect sum of primitive \( R' \)-rings.

Proof. The proof follows from the above proposition and proposition 2.13 of [26].

4.5 Proposition. Let \( R \) be a biregular primitive \( R' \)-ring; then \( R \) is simple.

Proof. Let \( R \) be a biregular primitive \( R' \)-ring and \( M \) be a faithful irreducible \( R' \)-module. Let \( e = (e \gamma e) \) be a central idempotent in \( R \). Now the subgroup \( M \gamma e \) is a \( R' \)-submodule of \( M \) since \( M \gamma e \subseteq M \gamma e \subseteq M \gamma e \subseteq M \gamma e \). Also \( M \gamma e \neq 0 \). So \( M \gamma e = M \). Again \( M \gamma e = M \gamma e = M \). Now \( e \beta \gamma x = m_1 \gamma_1 x - m_2 \gamma_1 x = m_2 \gamma_1 x - m_2 \gamma_1 x = 0 \) for all \( m_1 \in M \) and for all \( \gamma_1 \in \Gamma \). So \( x = e \gamma x = x \gamma e \).

Since each principal ideal of \( R \) is generated by a central idempotent, it follows that \( R \) is simple.
4.6 Corollary. In a biregular $\Gamma$-ring each primitive ideal is a maximal ideal.

**Proof.** Let $R$ be a biregular $\Gamma$-ring and $I$ be a primitive ideal of $R$. Then the quotient $\Gamma$-ring $R/I$ is primitive. Also from the proposition 4.2 it follows that $R/I$ is biregular. So $R/I$ is a biregular primitive $\Gamma$-ring. Consequently by the above proposition it follows that $R/I$ is simple. Now since there is a one-to-one correspondence between the ideals of $R$ and the ideals of $R/I$ containing $I$, it follows that $I$ is a maximal ideal of $R$.

We can prove the following propositions as in the case of classical ring theory with slight modifications.

4.7 Proposition. Let $R$ be a $\Gamma$-ring and $I$ an ideal in $R$. If $M$ is an $(R/I)\Gamma$-module then $M$ can be considered as an $R\Gamma$-module. Conversely if $M$ is an $R\Gamma$-module and $I$ is contained in $\mathsf{A}(M)$, annihilator ideal of $M$ [26] then $M$ can be considered as an $(R/I)\Gamma$-module.

4.8 Proposition. The Jacobson radical $J(R)$ of a $\Gamma$-ring $R$ is the intersection of all primitive ideals of $R$.

**Proof.** The Jacobson radical $J(R)$ of a $\Gamma$-ring $R$ is defined by $J(R) = \bigcap \mathsf{A}(M)$ where $M$ runs over all irreducible $R\Gamma$-modules,
if any. In case $R$ does not contain any irreducible $R^\Gamma$-module, $J(R)$ is defined to be $R$. Now from the proposition 1.4 of [26] it follows that if $M$ is an irreducible $R^\Gamma$-module then $R/A(M)$ is a primitive $C^\Gamma$-ring. Consequently it follows that $A(M)$ is a primitive ideal of $R$. Conversely if $I$ is a primitive ideal of the $C^\Gamma$-ring $R$ then $R/I$ is a primitive $C^\Gamma$-ring. So $R/I$ admits a faithful irreducible $R^\Gamma$-module say, $M_\xi$ such that $A(M) = I$. Hence it follows that $J(R)$ is the intersection of all the primitive ideals of $R$.

4.9 Corollary. In a biregular $C^\Gamma$-ring $R$ the intersection of all maximal ideals is zero.

Proof. The proof follows from the propositions 4.3, 4.8 and the Corollary 4.6.

4.10 Corollary. In a biregular $C^\Gamma$-ring $R$ each ideal is the containing intersection of all the maximal ideals containing it.

Proof. The proof follows from proposition 4.2 and above corollary.

4.11 Definition. The right annihilator $A^R$ of an ideal $A$ in a $C^\Gamma$-ring $M$ consists of all $x$ with $y \\forall x = 0$ for all $y \in A$ and for all $y \in C$. The left annihilator $A^L$ of $A$ is defined analogously.
4.12 Proposition. The right annihilator \( A^R \) of an ideal \( A \) in a \( \Gamma \)-ring \( M \) is an ideal of \( M \) and if \( M \) is biregular then \( A^R = A \).

Proof. Let \( a, b \in A^R \); \( y \gamma a = c = y \gamma b \) for all \( y \in A \) and for all \( \gamma \in \Gamma \). Now \( y \gamma (a - b) = y \gamma a - y \gamma b = 0 \). \( \text{So } a - b \in A^R \).

Also \( a \in A^R \) implies \( y \gamma a = 0 \). \( \text{So } y \gamma x_1 x = 0 \) for all \( y \in A \) and for all \( \gamma \in \Gamma \) where \( x_1 \in \Gamma \) and \( x \in M \). Consequently \( a \gamma_1 x \in A^R \). Again \( y \in A \) implies \( y \gamma x \in A \). \( \text{So } y \gamma x \gamma_1 a = 0 \) (since \( a \in A^R \)) which implies that \( x \gamma_1 a \in A^R \) where \( x \in M \) and \( \gamma_1 \in \Gamma^* \). \( \text{So } A^R \) is an ideal of the \( \Gamma \)-ring \( M \).

Next let \( M \) be biregular. If \( x \in A \) then \( z \gamma x = 0 \) for all \( z \in A \) and all \( \gamma \in \Gamma \). Now for an arbitrary \( y \in A \) biregularity assures the existence of an idempotent \( e \) in the centre of \( M \) with \( y = e \gamma_3 y \) and \( e = e \gamma_3 e \in A \). Now \( x \gamma_1 y = x \gamma_1 e \gamma_3 y = e \gamma_1 x \gamma_3 y = 0 \) for all \( \gamma_1 \in \Gamma \) (since \( e \in A \), and \( x \in A^R \), \( e \gamma_1 x = 0 \)). \( \text{So } x \in A^L \). \( \text{So } A^R \subseteq A^L \). Similarly we can show that \( A^L \subseteq A^R \). Consequently \( A^R = A^L \).

Remark. In a biregular \( \Gamma \)-ring \( M \) the two annihilators of an ideal \( A \) coincide, we shall denote this annihilator by \( A^* \).

4.13 Lemma. If \( A \) is a maximal ideal in a biregular \( \Gamma \)-ring \( M \) then either \( A^* = 0 \) or \( A^* \) is minimal.
Proof. If $A^* \neq 0$ and is not minimal, there exists a principal ideal $B \subseteq A^*$. Now biregularity of $M$ assures the existence of an idempotent $f(\neq 0)$ in the centre of $M$ such that $B = (f)$ where $(f)$ denotes the ideal generated by $f$. Now $(f) \not\subseteq A^*$ and so there exists an idempotent $e \in A^*$ such that $e \not\in (f)$. Since $(f) \not\subseteq A$ we have $A \subset A + (f)$ properly. If possible, let $A + (f) = M$. Now $e \in M$ so $e = a + x$ for some $a \in A$ and $x \in (f)$. Since $e \in A^*$ and $a \in A$, $e \not= a + x$ for all $x \in M$. But since $e$ is an idempotent, $e = e \gamma_3 e$ for some $\gamma_3 \in M$ and so $e = e \gamma_3 e = e \gamma_3(a + x) = e \gamma_3 a + e \gamma_3 x = e \gamma_3 x 
subseteq (f)$ since $x \in (f)$. This contradicts our assumption that $e \not\in (f)$. Thus $A + (f) \neq M$. So $A \subset A + (f) \subset M$ properly. So $A$ is not maximal which contradicts our hypothesis. Hence the lemma.

4.14 Lemma. If $A$ is a minimal ideal of a biregular $F$-ring $M$ then $A^*$ is maximal.

Proof. Let $A$ be a minimal ideal of $M$ then $A = (e)$ where $e$ is an idempotent lies in the centre of $M$ and $e \gamma_1 e = e$ for some $\gamma_1 \in M$. Let $f(\neq 0)$ be another idempotent in the centre of $M$ but not in $(e)^*$ and $f \gamma_2 f = f$ where $\gamma_2 \in M$. Now $(e \gamma_1 f) \gamma_2 (e \gamma_1 f) = e \gamma_1 (f \gamma_2 f) \gamma_1 e = e \gamma_1 f \gamma_1 e = e \gamma_1 e \gamma_1 f = e \gamma_1 f$. So $e \gamma_1 f$ is an idempotent. Also $e \in A$ implies $e \gamma_1 f \in A$. So $(e \gamma_1 f) \subseteq A$ and $e \gamma_1 f \neq 0$. For, if $e \gamma_1 f = 0$ then $f \in (e)^*$ which contradicts
our assumption that $f \notin (e)^*$. Now minimality of $A$ implies that
$A = (e \gamma_1 f)$. So $(e) = (e \gamma_1 f) \subseteq (f)$. Since $e \notin (e)^*$, $(e)^* \neq M$.

If $B$ is an ideal of $M$ properly containing $(e)^*$ then there exists an idempotent $f \in B$ such that $f \notin (e)^*$; but then $(e) \subseteq (f)$ and so $(e) \subseteq B$. Also $(e)^* \subseteq B$. So $(e)^* + (e)^* \subseteq B$; but $(e) + (e)^* = M$ for every element $a \in M$ can be written as $a = e \gamma_1 a + (a-e\gamma_1 a)$

where $e \gamma_1 a = e$. Obviously $e \gamma_1 a \in (e)$. We shall now show that
$(a - e \gamma_1 a) \in (e)^*$. Let $a_1 \in (e)$ and $r \in \gamma$ then $(a_2 \gamma_1 e \gamma_1 a - a_2 \gamma_1 e \gamma_1 a) = a_2 \gamma_1 e \gamma_1 a = a_2 \gamma_1 e \gamma_1 a - a_2 \gamma_1 e \gamma_1 a = a_2 \gamma_1 e \gamma_1 a = a_2 \gamma_1 e \gamma_1 a = 0$ where $a_1 = a_2 \gamma_1 e$.

Hence $(e) + (e)^* = M$. So $B = M$. Consequently $(e)^* = A^*$ is maximal.

4.15 Lemma. If $M$ is a biregular $\gamma$-ring in which each maximal ideal has a nonzero annihilator then $A = A^{**}$ for each ideal $A$
in $M$.

Proof. If $M_1$ is a maximal ideal then by lemma 4.13 $M_1^{**}$ is
minimal. Since $M$ is biregular, $M_1^{**} = (e)$ where $e (e \neq 0)$ is an
idempotent in the centre of $M$; but then $e \notin M_1^{**} \supseteq M_1$ and
since $M_1$ is maximal $M_1^{**} = M_1$.

If $A \neq M$ is an ideal in $M$ and $M_1$ is a maximal ideal
containing $A$ then $M_1^{**} \subseteq A^{*}$ and $A^{**} \subseteq M_1^{**} = M_1$. So
$A^{**} \subseteq \cap M_1$ where $M_1$ runs over all the maximal ideals of $M$.
containing \( A \). Hence by corollary 4.10 \( A^{**} \subseteq A \). Also \( A \subseteq A^{**} \).
Consequently it follows that \( A^{**} = A \). Clearly \( M^{**} = M \). Hence the lemma.

4.16 Theorem. If \( M \) is a biregular \( \Gamma \)-ring \( M \) in which every maximal ideal has a nonzero annihilator then

i) the ideals of \( M \) form a Boolean Algebra,

ii) each nonzero ideal \( A \) contains a minimal ideal.

iii) \( M \) is the discrete direct sum of its minimal ideals.

Proof. The proof is the same as in the case of classical ring theory [28].
CHAPTER 3

An introduction to $\Gamma$-field.

1. Introduction.

In [15] S. Kyuno introduced the notion of commutativity to a $\Gamma$-ring in the sense of Nobusawa and defined $\Gamma$-field using operator ring. He also characterised some commutative $\Gamma$-rings.

In this chapter we have introduced the notion of $\Gamma$-field to a $\Gamma$-ring in the sense of Barnes and studied some properties of it. Most of the properties obtained are similar to those of field in the classical ring theory. For example we have obtained the following results:

1) Let $M$ be a commutative $\Gamma$-ring and $I$ be an ideal of $M$; then $I$ is a maximal ideal if and only if the quotient $\Gamma$-ring $M/I$ is a $\Gamma$-field.

2) Let $M$ be a finite commutative $\Gamma$-ring without say divisor of zero; then $M$ is a $\Gamma$-field.

3) A commutative primitive $\Gamma$-ring is a $\Gamma$-field etc.

2. Preliminaries.

We recall the following definitions and abbreviations.
2.1 Definition. A \( \Gamma \)-ring \( M \) is said to be simple if \( M \cap M \neq 0 \) and \( M \) has no ideals other than \( 0 \) and \( M \) where \( M \cap M \) is the ideal set of all finite sums of the terms \( m_1 \gamma m_2 \) with \( m_1, m_2 \in M \) and \( \gamma \in \Gamma \) [16].

2.2 Definition. An \( \Gamma \)-module \( M \) is called irreducible if \( M \cap R \neq 0 \) and if it has no proper submodule [26].

2.3 Definition. An \( \Gamma \)-module \( M \) is said to be faithful if \( M \cap x = 0 \) implies \( x = 0 \) or equivalently if the annihilator ideal \( A(M) = \{ x \in R \mid M \cap x = 0 \} \) is zero [26].

A \( \Gamma \)-ring \( M \) with the descending chain condition of right ideals is abbreviated to \( M \) has min - \( r \) condition. The terms min - \( l \) condition, min - \( m \) condition, max - \( l \) condition and max - \( m \) condition are likewise defined.

3. \( \Gamma \)-field.

3.1 Definition. A commutative \( \Gamma \)-ring \( M \) is called a \( \Gamma \)-field if for every nonzero element \( a \) of \( M \) and for every pair of nonzero elements \( \gamma_1, \gamma_2 \) of \( \Gamma \) there exists an element \( a' \) in \( M \) such that \( a \gamma_1 a' \gamma_2 b = b \) for all \( b \in M \).

3.2 Definition. A \( \Gamma \)-ring \( M \) is said to be without divisors of zero if \( a \gamma b = 0 \) implies \( a = 0 \) or \( b = 0 \) or \( \gamma = 0 \) where \( a, b \in M \) and \( \gamma \in \Gamma \).
3.3 **Proposition.** A \( \Gamma \)-field \( M \) does not contain any divisor of zero.

**Proof.** Let \( a \neq 0 \) and \( \gamma \neq 0 \). Then there exists an element \( a' \) in \( M \) such that \( a \gamma a' \gamma c = c \) for all \( c \in M \). So, since \( M \) is commutative, \( b = a \gamma a' \gamma b = a \gamma b \gamma a' = 0 \gamma a' = 0 \). Consequently \( a \gamma b = 0 \), \( a \neq 0 \), \( \gamma \neq 0 \) imply \( b = 0 \). Hence the proposition.

3.4 **Proposition.** In a \( \Gamma \)-field \( M \) corresponding to a nonzero element \( a \) and to a pair of nonzero elements \( \gamma_1, \gamma_2 \) of \( \Gamma \) the element \( a' \) for which the condition \( a \gamma_1 a' \gamma_2 b = b \) holds for all \( b \in M \) is unique.

**Proof.** Let \( a' \) and \( a'' \) be two elements of \( M \) such that \( a \gamma_1 a' \gamma_2 b = b \) and also \( a \gamma_1 a'' \gamma_2 b = b \) for all \( b \in M \). Then \( a \gamma_1 a' \gamma_2 b = a \gamma_1 a'' \gamma_2 b \) implies \( a \gamma_1 (a' - a'') \gamma_2 b = 0 \) from which it follows that \( a' = a'' \) since a \( \Gamma \)-field does not contain any divisor of zero. Hence the proposition.

3.5 **Proposition.** A \( \Gamma \)-field \( M \) does not contain any proper ideal.

**Proof.** Let \( I \) be an ideal of \( M \) and \( a( \neq 0 ) \in I \). Let \( \gamma_1( \neq 0 ) \)
and \( \gamma_2 (\neq 0) \) be two elements of \( \Gamma \). Then there exists an element \( a' \in M \) such that \( a \gamma_1 a' \gamma_2 b = b \) for all \( b \in M \). Since \( I \) is an ideal of \( M \), \( a \in I \) implies \( b = a \gamma_1 a' \gamma_2 b \in I \) for all \( b \in M \). So \( I = M \). Hence the proposition.

3.6 Proposition. A \( \Gamma \)-field \( M \) is simple.

**Proof.** Let \( a (\neq 0) \in M \) and \( \gamma_1 (\neq 0), \gamma_2 (\neq 0) \in \Gamma \). Then there exists an element \( a' \) such that \( a = a \gamma_1 a' \gamma_2 a \). Since \( a = a \gamma_1 a' \gamma_2 a \in M \Gamma M \), \( M \Gamma M \neq 0 \). Also by proposition 3.5 \( M \) does not contain any proper ideal. Hence \( M \) is simple.

The converse of the above proposition is also true. In fact, we have

3.7 Proposition. Let \( M \) be a commutative \( \Gamma \)-ring without any divisor of zero; if \( M \) does not contain any proper ideal then \( M \) is a \( \Gamma \)-field.

**Proof.** Let \( a (\neq 0) \in M \) and \( \gamma_1 (\neq 0), \gamma_2 (\neq 0) \in \Gamma \). Let also \( b (\neq 0) \in M \). Then \( a \gamma_1 M \gamma_2 b = \{ a \gamma_1 m \gamma_2 b \mid m \in M \} \) is an ideal of \( M \). Since \( a \gamma_1 a \gamma_2 b (\neq 0) \in a \gamma_1 M \gamma_2 b \), so \( a \gamma_1 M \gamma_2 b \) is not a zero ideal. Consequently, by our hypothesis \( a \gamma_1 M \gamma_2 b = M \) So corresponding to the element \( b \) of \( M \) there exists an element \( a' \) in \( M \) such that \( a \gamma_1 a' \gamma_2 b = b \). We shall show that \( a \gamma_1 a' \gamma_2 c = c \)
for all \( c \in M \). Let \( a \gamma_1 a^n \gamma_2 b = c \). Then \( a \gamma_1 a^n \gamma_2 b = a \gamma_1 a^n \gamma_2 b = a \gamma_1 a^n \gamma_2 b = c \). Hence \( M \) is a \( \Gamma' \)-field.

3.8 Proposition. Let \( M \) be a commutative \( \Gamma' \)-ring and \( I \) be an ideal of \( M \); then \( I \) is a maximal ideal if and only if the quotient \( \Gamma' \)-ring \( M/I \) is a \( \Gamma' \)-field.

Proof. To begin, let \( I \) be a maximal ideal of \( M \). Since \( M \) is a commutative \( \Gamma' \)-ring, the quotient \( \Gamma' \)-ring \( M/I \) is also commutative. To prove \( M/I \) is a \( \Gamma' \)-field, let \( b + I \) (\( b \neq 0 \)) \( \in M/I \) and \( \gamma_1(\neq o), \gamma_2(\neq o) \in \Gamma' \). Since \( b + I \neq o, b \neq I \). Now we consider the set \( J = \{ a \gamma_1 m \gamma_2 b + t \mid m \in M, t \in I \} \). Then \( J \) is an ideal of \( M \) containing \( I \). By virtue of the fact that \( I \) is a maximal ideal, we have \( J = M \). So for \( b \in M \) there exist elements \( a' \) in \( M \) and \( t_1 \) in \( I \) such that \( a \gamma_1 a' \gamma_2 b + t_1 = b \). So \( (a + I) \gamma_1 (a' + I) \gamma_2 (b + I) = b + I \). We shall now show that \( (a + I) \gamma_1 (a' + I) \gamma_2 (c + I) = c + I \) for all \( c + I \in M/I \). If \( c + I = 0 \) then obviously \( (a + I) \gamma_1 (a' + I) \gamma_2 (0 + I) = c + I \).

So we assume \( c + I \neq 0 \). Now \( c \in M \), so there exist elements \( a'' \) in \( M \) and \( t_2 \) in \( I \) such that \( a \gamma_1 a'' \gamma_2 b + t_2 = c \). This implies \( (a + I) \gamma_1 (a'' + I) \gamma_2 (b + I) = (c + I) \). Now \( (a + I) \gamma_1 (a'' + I) \gamma_2 (c + I) = (a + I) \gamma_1 (a' + I) \gamma_2 (a + I) \gamma_1 (a'' + I) \gamma_2 (b + I) = c + I \). Hence \( M/I \) is a \( \Gamma' \)-field.
For the opposite direction we suppose \( M/I \) is a \( \Gamma' - \) field and \( J \) is any ideal of \( M \) for which \( I \subseteq J \subseteq M \). The argument consists of showing that \( J = M \) for then \( I \) will be a maximal ideal. Since \( I \) is a proper subset of \( J \), there exists an element \( a \in J \) with \( a \notin I \). Consequently \( a + I \neq 0 \). Then for a pair of nonzero elements \( \gamma_1, \gamma_2 \) of \( \Gamma' \) there exists an element \( a + I \) in \( M/I \) such that

\[
( a + I) \gamma_1 (a + I) \gamma_2 (b + I) = b + I \quad \text{i.e.,} \quad a \gamma_1 a' \gamma_2 b + I = b + I.
\]

So \( a \gamma_1 a' \gamma_2 b - b \in I \) for all \( b \in M \). Since \( a \in J \), \( a \gamma_1 a' \gamma_2 b \in J \). So \( a \gamma_1 a' \gamma_2 b - b \in I \subseteq J \) and \( a \gamma_1 a' \gamma_2 b \in J \), imply \( b \in J \). So \( J = M \) as desired. Hence the proposition.

3.9 Proposition. Let \( M \) be a \( \Gamma' - \) ring without any divisor of zero; then if \( M \) is finite, \( \Gamma' \) is also finite.

Proof. Let \( a \neq 0 \), \( b \neq 0 \) \( \in M \). If \( \gamma_1 \) and \( \gamma_2 \) are two distinct elements of \( \Gamma' \) then \( a \gamma_1 b \) and \( a \gamma_2 b \) are also distinct for \( a \gamma_1 \ b = a \gamma_2 \ b \) implies \( a ( \gamma_1 - \gamma_2 ) b = 0 \) which implies \( \gamma_1 = \gamma_2 \), a contradiction. Now if \( \Gamma' \) contains infinite number of elements \( \gamma_i \), \( i \in I \) where \( I \) is an index set then \( a \gamma_i b \neq a \gamma_j b \) for \( i \neq j \) and also each \( a \gamma_i b \in M \) which implies that \( M \) also contains infinite number of elements, a contradiction. This contradiction forces us to assume \( \Gamma' \) is finite.

3.10 Proposition. Let \( M \) be a finite commutative \( \Gamma' - \) ring without any divisor of zero; then \( M \) is a \( \Gamma' - \) field.
Proof. Let \( a(\neq 0), b(\neq 0) \in M \) and \( \gamma_1(\neq 0), \gamma_2(\neq 0) \in F \).

Then if \( m_1, m_2 \) are distinct elements of \( M \) then \( a\gamma_1 m_1 \gamma_2 b \) and \( a\gamma_1 m_2 \gamma_2 b \) are also distinct for \( a\gamma_1 m_1 \gamma_2 b = a\gamma_1 m_2 \gamma_2 b \)
implies \( a\gamma_1 (m_1 - m_2) \gamma_2 b = 0 \) which implies \( m_1 = m_2 \), a con-
tradiction to our assumption. So if \( M \) contains a distinct elements, say \( m_1, m_2, \ldots, m_n \) then \( a\gamma_1 m_1 \gamma_2 b, a\gamma_1 m_2 \gamma_2 b, \ldots, a\gamma_1 m_n \gamma_2 b \)
are also distinct elements of \( M \). Hence \( a\gamma_1 M \gamma_2 b = M \).

Now we can show that as in proposition 3.7 that there exists an element \( a' \) in \( M \) such that \( a\gamma_1 a' \gamma_2 c = c \) for all \( c \in M \).

Hence \( M \) is a \( F \)-field.

3.11 Proposition. A commutative \( F \)-ring is primitive if and only if it is a \( F \)-field.

Proof. Let \( R \) be a commutative primitive \( F \)-ring. Then \( R \) admits
a faithful irreducible \( R' \)-module \( M \). Since \( M \) is an irreducible
\( R' \)-module, \( M \cong R/I \) as an \( R' \)-module, (lemma 1.3,[26])for
some maximal regular right ideal \( I \) of \( R \). Moreover there exist:
an \( \alpha \) in \( \Gamma \) and an \( e \) in \( R \) such that \( x - e \alpha x \in I \) for all
\( x \in R \). Since \( R \) is commutative \( I \) is an ideal of \( R \). Now
\( I \subseteq A(R/I) = A(M) = 0 \), since \( M \) is faithful, So for every element
\( x \) in \( R \) there exist \( \alpha \) in \( \Gamma \) and an \( e \) in \( R \) such that
\( x - e \alpha x = 0 \). So every ideal of \( R \) is regular [26]. Consequently
\( I \) is a maximal ideal of \( R \). So \( R/I \) i.e. \( R \) is a \( F \)-field. Con-
versely, let \( R \) be a \( F \)-field. Then \( R \) is a commutative \( F \)-ring
which may be considered as an $R^\gamma$module. Since $R$ is simple
(proposition 3.6) $R$ is irreducible. Again, let $R^\gamma x = 0$ where
$x \in R$. Then $x R x \in R^\gamma x = 0$ where $x(\neq 0) \in R$ and $y(\neq 0)
\in R$. Since $R$ does not contain any divisor of zero, so $x R x = 0$
implies $x = 0$. Hence $R$ is faithful, so $R$ admits a faithful
irreducible $R^\gamma$-module. In other words, $R$ is a commutative
primitive $R^\gamma$-ring.

3.12 Proposition. Any commutative $R^\gamma$-ring with min-condition
and without any divisor of zero is a $\Gamma$-field.

Proof. Let $a (\neq 0) \in M$ and $\gamma_1 (\neq 0), \gamma_2 (\neq 0) \in \Gamma$. Let $b (\neq 0)$
be an arbitrary but fixed element of $M$. Now $a \gamma_1 b \gamma_2 M$ is an ideal
of $M$. Also $\gamma_1 b \gamma_2 M \neq 0$, since $0 \neq a \gamma_1 b \gamma_2 a \in a \gamma_1 b \gamma_2 M$. If
$a \gamma_1 b \gamma_2 M = M$ then $M$ is a $\Gamma$-field. If $a \gamma_1 b \gamma_2 M \neq M$, we consider the
set $(a \gamma_1 b) \gamma_2 (a \gamma_1 b) \gamma_2 M$ which is also a nonzero ideal of $M$ and is
contained in $a \gamma_1 b \gamma_2 M$. Continuing this process we get a descending
chain of ideals $a \gamma_1 b \gamma_2 M \supset (a \gamma_1 b) \gamma_2 (a \gamma_1 b) \gamma_2 M \supset \cdots$. Then
by min-condition on $M$ the above chain must be of finite length.
Hence there exists a positive integer $n$ such that $(a \gamma_1 b)^{n-1} \gamma_2 M
= (a \gamma_1 b)^n \gamma_2 M$. So for the element $b$ there exists an element $a'$
in $M$ such that $(a \gamma_1 b)^{n-1} \gamma_2 b = (a \gamma_1 b)^n \gamma_2 a'$ i.e.
$(a \gamma_1 b)^{n-1} \gamma_2 \{ b - a \gamma_1 b \gamma_2 a' \} = 0$. This implies $a \gamma_1 b \gamma_2 a' = b$,
since $M$ does not contain any divisor of zero. Now, as in proposition
3.7 we can prove that $M$ is a $\Gamma$-field.
3.13 **Corollary.** A commutative \( \Gamma \)-ring without any divisor of zero and with a finite number of ideals is a \( \Gamma \)-field.

We can introduce the notions of subdirect sum and direct sum of commutative \( \Gamma \)-rings (\( \Gamma \) being the same for all of them) exactly in the same manner as in classical ring theory \([2]\).

Now the following results can be deduced in the same manner as in the case of classical ring theory.

3.14 **Proposition.** A commutative \( \Gamma \)-ring \( M \) is isomorphic to a subdirect sum of \( \Gamma \)-fields if and only if the intersection of all maximal ideals of \( M \) is zero.

3.15 **Proposition.** A commutative \( \Gamma \)-ring \( M \) is isomorphic to a subdirect sum of \( \Gamma \)-fields if and only if for each nonzero ideal \( I \) of \( M \) there exists an ideal \( J \not= \emptyset \) such that \( I + J = M \).

3.16 **Proposition.** Any commutative \( \Gamma \)-ring with min-condition in which intersection of all maximal ideals is zero, is the direct sum of a finite number of \( \Gamma \)-fields.

3.17 **Corollary.** Any commutative \( \Gamma \)-ring satisfying min-condition in which the intersection of all maximal ideals is zero, also satisfies max-condition for ideals.

A \( \Gamma \)-ring \( M \) is said to be semisimple if its Jacobson radical is zero \([26]\).

Now we have the following result.

3.18 **Theorem.** A commutative semisimple \( \Gamma \)-ring is a subdirect sum of \( \Gamma \)-fields.

**Proof.** The proof follows from proposition 2.13 of \([26]\) and our proposition 3.11.
Bibliography.


4. Dutta, T. K : A characterisation of \( \Gamma \)-ring. (communicated)

5. - Do - : On regular \( \Gamma \)-rings. (communicated)


12. - Do - : Coincidence of right Jacobson radical and left Jacobson radical in a $\Gamma$-ring.
Tsukuba J. Math. 3 (1979), P 31 - 35.

13. - Do - : A $\Gamma$-ring with right and left unities.

14. - Do - : Nobusawa's $\Gamma$-ring with right and left unities.

15. - Do - : On a Nobusawa's Gamma-ring \( \begin{array}{c} R \\ M \\ L \end{array} \).

16. - Do - : A $\Gamma$-ring with minimum condition.

17. - Do - : Notes on Jacobson radicals of Gamma rings.

18. - Do - : Prime ideals in Gamma rings.


20. Luh, J : On primitive $\Gamma$-rings with minimum one-sided ideals.
Osaka J. Math. 5 (1968), P 165 - 173.
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