Chapter 4

Variable fractional delay FIR filter design using Gaussian derivatives and cosine Gabor

4.1 Introduction

Fractional delay interpolators are well studied in the field of vocal track modelling, digital keyboard, audio frequency conversion, digital beamforming for RADAR and ultrasound imaging. Especially in digital beamforming, it is desirable that filter coefficients should be updated when the focus delay gets changed and thus design of variable fractional delay filter is necessary [46]. There exists several methods to design Variable Fractional delay FIR filter e.g., windowing [45, 47], matrix transform [50], Discrete Fourier transform [45, 52], weighted least square (WLS) [51], maximally flat approximation [45] etc. Each method has its unique features. Out of which, truncation of ideal interpolation kernel ('Sinc') with a shifted tapered window is easier and faster one to obtain a fractional delay filter. Real time update of the filter coefficients is also possible using interpolation but it is difficult to control the magnitude of Gibbs oscillation when filter length is short [45, 47]. On the other hand, multiplying the signal to interpolate with a window of proper truncation properties and then applying Sampling Theorem can reduce passband magnitude error as well as computational complexity with less arithmetic operation [48] compared to other adjustable FD filters like Farrow structure (FS) [49].

The Farrow structure or modified FS [123] provides a comprehensive design tool to control delay on the fly, where each filter coefficient has been approximated by a polynomial of fixed order. Thus, FS operates with a bank of parallel sub-filters with or without having coefficient symmetry. Coefficients of each sub-filter are generated by minimizing the root mean square error in the frequency and or in group delay domain [49, 51, 124]. However, approximation error in the passband depends on the order of the polynomial used in this approximation. Lagrange interpolator, followed maximally flat FIR approximation, is the most attractive one for coefficient update...
using small number of multiplication, addition and can be realized using Farrow structure [45]. Oetken's method [50] which provides the equiripple FD FIR design, can also be used for real time coefficient update. It only requires calculating a few sin and cosine terms to update filter coefficients on the fly according to the desired delay. However, Oetken's method only designs even length FIR FD.

In a different approach, variable fractional delay can be achieved when ideal interpolator is approximated by the Taylor series expansion and the design uses 2nd order FIR differentiator for reasonable approximation [53]. Similarly, the ideal frequency response of a fractional delay filter is approximated via Taylor series and the design utilizes only the first order FIR differentiator successively [54]. The later design method though generates smaller approximation error, produces longer integer delay compared to the Farrow structure. A tradeoff among the storage requirement of the filter coefficients, computational complexity and delay of the filter is further proposed [56]. In order to lower arithmetic complexity, optimized differentiators with different filter orders of same symmetry [54] or hybrid symmetry [125] have also been employed to design VFD using the Taylor series expansion. However, in many biomedical applications, smooth differentiators are frequently used when the signal is corrupted by noise [57]. Gaussian derivatives that stems from the computational model of Human as well as primate's visual system [16-18] are preferred as smooth differentiators in edge detection, image coding, image enhancement and scale space representation [4-9] because of its compactness both in the signal and Fourier space. Compactness property of Gaussian and its derivatives is further explored to construct near ideal interpolation kernel using their linear combination [22]. Linear sum of multi-scale Gaussians is also studied to generate near ideal interpolation kernel [58]. These near ideal interpolation kernels though offering good mathematical manipulation in 2D or higher dimension [22, 58], lose their bandwidth heavily when they are delayed by half of the sample (see Fig. 1.3a for Appledron's kernel). Even the maximally flat approximation in the passband of Appledron's continuous interpolation kernel does not remain good when it is discretized (see Fig. 1.3b). On the other hand, the FIR approximation of lowpass filter with equiripple passband is possible using linear combination of multi-scale Gaussian derivatives (see chapter 2) [31].

Like Gaussian derivatives, cosine Gabor, also produced bandpass frequency response, serves as a computational basis to model primate's receptive field [36, 37]. Linear combination of the cosine Gabor functions has been used to design lowpass FIR filter with a closed form solution (see chapter 3) [38]. Unlike other window-based FIR design, the mentioned design is able to keep the band edges at the user specified magnitude. Cosine basis is further studied for FD design using discrete
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Fourier Transform-based interpolation (DFT) [52]. However, the RMS error as well as phase delay error will grow up while the truncation length of cosine will be short.

Gaussian derivative and cosine Gabor functions possess one attractive mathematical property that they are continuously differentiable. Therefore a filter kernel constructed by linear combination of these smooth functions, can be interpolated by Taylor series expansion. Smoothness property has been utilized in this study to generate continuously variable fractional delay lowpass filter and interpolation filter using linear combination of Gaussian derivatives and cosine Gabor respectively. Using the Taylor series expansion, it is also possible to obtain Farrow-structure based implementation.

Furthermore, computational complexity (storage memory for coefficients and multiplication) will be almost halved in the proposed design because the linear phase prototype is constructed by linear combination of even symmetric continuous functions resulting in the sub-filters in the Farrow structure, to be of either even or odd symmetry depending upon the derivative orders. Additionally, the proposed continuous domain formulation can design FIR VFD of odd and even length.

4.2 Filter design

4.2.1 Lowpass filter with Gaussian derivative family

Similar to [31] and chapter 2, ideal lowpass filter kernel can be approximated as a linear combination of family of Gaussian and its even order derivatives. Approximated kernel is expressed as:

\[ h(x) = \sum_{k=0}^{N} a_k g^k(x, \sigma) \quad (4.1) \]

Where,

\[ g^0(x, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{x^2}{2\sigma^2} \right) \quad (4.2) \]

represents a normalized Gaussian with scale \( \sigma \) and
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\[ g^k(x) = \frac{d^k}{dx^k} g^0(x, \sigma) \]  

(4.3)

are the Gaussian derivatives. Filter function can be expressed as:

\[ H(\omega) = \sum_{k=0, \text{even}}^{N} \alpha_k G_k(\omega, \sigma) \]  

(4.4)

\[ G_k(\omega, \sigma) = \left( \frac{\omega}{2} \right)^k \exp\left( -\frac{\omega^2 \sigma^2}{2} \right) \]  

(4.5)

are the spectral modes (Fourier transform of Gaussian derivatives) of \( g^k(x) \) at \( \omega_k = \sqrt{k} / \sigma \) where \( N = 2M \).

Scale \( (\sigma) \) of the Gaussian can be obtained from the following relation between spectral mode \( (P) \) corresponding to highest order derivative and passband edge frequency \( \omega_p \) and expressed below.

\[ \sigma = \frac{P}{\omega_p} \]  

(4.6)

Fig. 4.1. Magnitude response (red line) of a fractional delay [-0.5 0.5] FIR filter constructed of family of Gaussian derivatives. Linear phase prototype (green line) has been designed in (a) equiripple sense (b) passband of (a). Dotted magenta lines represent Gaussian derivatives in frequency domain. Black line indicates approximation using continuous functions.

\( H(\omega) \) is an even symmetric polynomial of order \( N \ (> M + 1) \) multiplied by an even symmetric Gaussian window. According to the alternation theorem, the best mini-max approximation of the ideal frequency response \( D(\omega) \) with these \( M+1 \) number of Gaussian derivatives is possible if and only if the peak absolute value \( \varepsilon \) of the passband approximation error \( E(\omega) \)

\[ E(\omega_i) = H(\omega_i) - D(\omega_i) = (-1)^i \varepsilon, \text{ for } 0 \leq i \leq M+1, \]  

(4.7)
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exhibits \( M+2 \) number of extrema (\( \omega_i \)) in the range \([0, \omega_p]\) \([29, 126]\). Where,

\( D(\omega) = 1 \) within \([0, \omega_1]\). Alternation theorem holds good if the set of \( M+1 \)
continuous, real valued function \( G(\omega) \) satisfies Haar condition \([126]\).

Since, \( G_0(\omega) \sigma > 0 \) \( \forall \omega \), \( H(\omega) \) can have at most \( N/2 \) number of real zeros in the
interval \([0, \omega_p]\). Therefore, the linear combination of \( M+1 \) number of Gaussian
derivatives (or spectral modes) can vanish in at most \( M \) times in the said interval. Thus family of Gaussian derivatives satisfies Haar condition in \([0, \omega_p]\).

![Magnitude response of a fractional delay FIR filter constructed by (a) Cosine Gabors (magenta) (b) Cosine basis (red) \([52]\). (c)-(d) Passband of (a) and (b). Blue line represents the Cosine Gabor in frequency domain.](image)

So, lowpass filter with equiripple magnitude response can be produced by iterative
Remez exchange algorithm \([29, 126]\).

Gaussian derivative can be computed using the following recurrence relation.

\[
g^k(x) = \frac{x}{v} g^{k-1}(x) - \frac{k-1}{v} g^{k-2}(x) \tag{4.8}
\]
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With \( v = \sigma^2 \) and \( n > 1 \)

### 4.2.2 Interpolation filter with cosine Gabor

Lowpass FIR approximation \((\omega_p < \pi)\) can be obtained by linear sum of mean shifted Gaussian in the frequency domain (see chapter 3) \[38\]. If there are \( 2M+1 \) number of mean shifted Gaussians in the interval \([-\omega_p, \omega_p]\).

Filter function becomes

\[
H(\omega) = \frac{1}{S} \sum_{k=-M}^{M} \exp \left( -\frac{(\omega - k\Delta/\sigma)^2}{2} \right)
\]  

(4.9)

corresponding filter kernel \( h_{LP} \) can be expressed in terms of cosine Gabor basis \[38\] (see Eq. 3.4).

\[
h_{LP}(x) = \frac{1}{S} \left( \sum_{k=0}^{M} \beta_k \cos \left( \frac{k\pi x}{\sigma} \right) \right) g^0(x,\sigma)
\]  

(4.10)

where, \( k\Delta/\sigma = \omega \) is the mean of the \( k^{th} \) shifted Gaussian in the frequency domain and \( S \) is the normalization factor. When, \( \omega_p = \pi \), \( M^{th} \) Gaussian in the frequency domain will be centered on \( \pi \) and so \( \Delta/\sigma = \frac{\pi}{M} \). Like zero-th Gaussian, coefficient of \( M^{th} \) Gaussian is reduced by 50% because discrete Fourier transform of \( \cos(\pi x) \) is unimodal around \( \pi \) for all \( x = 0, \pm 1, \ldots, \pm M \). The filter kernel then becomes,

\[
h(x) = \frac{1}{S} \left( \sum_{k=0}^{M} \beta_k \cos \left( \frac{k\pi x}{M} \right) \right) g^0(x,\sigma)
\]  

(4.11)

Where,

\[
\beta_k = 1 \quad ; k = 0, M
\]

\[
= 2 \quad ; \text{otherwise}
\]

The above expression except the Gaussian window also arrives from the discrete Fourier Transform (DFT) based interpolation \[52\] of an even length sequence.

Filter kernel is a linear sum of cosine modulated Gaussian which in the frequency domain behaves as the mean-shifted Gaussian. Gibb’s oscillation produced due to truncation of the infinite kernel can be reduced by decreasing scale of this Gaussian window. On the other hand, multiplying the cosine basis with a Gaussian window will be convolution of their Fourier transform in the frequency domain. So, scale of
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the Gaussian window should be large enough to make transition band sharper or effective passband wider. A tradeoff is necessary and the optimal scale is selected from the following empirical relation (see Fig 4.3). For a given filter length \( L \), the scale that produces minimum approximation error Eq. (4.21) is numerically searched. The scale of the Gaussian may be expressed as

\[
\sigma = a_0 + a_1 L + a_2 L^2
\]  

(4.12)

![Fig. 4.3. RMS error as a function of scale of the Gaussian window for a fractional delay FIR filter constructed by (a) cosine Gabors (b) Optimal scale for a given filter order. Filter order increases from top to bottom in (a). Dotted Magenta line represents optimal scale values.](image)

**4.2.3 Variable fractional delay FIR construction**

Since Gaussian is differentiable up to any order, kernel of the fractional delay filter can be obtained by interpolating actual kernel at any non integer sample index using the Taylor’s series expansion retaining terms up to \( q^{th} \) order.

\[
h(x+\delta) = h(x) + \delta h'(x) + \frac{\delta^2}{2!} h''(x) + \ldots + \frac{\delta^q}{q!} h^{(q)}(x)
\]

and \( \delta \leq 0.5 \)

\[
h(x+\delta) = \sum_{q=0}^{Q} \delta^q C_q(x)
\]

(4.14)

In general notation, the interpolated kernel can be expressed as a linear combination of ‘\( Q+1 \)’ sub-filters \( C_q \). For Gaussian derivative based design, \( C_q \)s are computed using Eq. (4.14) and recurrence relation Eq. (4.8) and for cosine Gabor based design they are calculated by the Leibnitz’s rule as:
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Prototype linear phase FIR filter of length \( L \) is expressed as:

\[
N \sum_{k=0}^{L} \alpha_k g^{k}[n-D]
\]

for Gaussian derivatives

\[
N \sum_{k=0}^{L} \beta_k \cos\left(\frac{2\pi k}{N}[n-D]\right) g^0[n-D]
\]

for \( 0 < n < (L-1) \) and \( D = \frac{L-1}{2} \)

and its corresponding variable fractional delay FIR will be

\[
h(n, \delta) = \sum_{q=0}^{Q} \delta^q C_q[n]
\]

4.3 Design example

4.3.1 Specification

Gaussian derivatives based designs (see Fig. 4.1) have been made with passband edge frequency \( \omega_p = 0.8 \pi \). Number of tap or filter length \( L \) for all designs is 31. Scale \( \sigma \) of the Gaussian for a family of Gaussian derivatives was considered to be 1.9292 and that of cosine Gabor was 5.707. Highest order of Gaussian derivatives, used to construct linear phase filter, is 18. Highest order derivative for Taylor series expansion is considered to be 10. \( \alpha_0 = 3.0282 \), \( \alpha_1 = 0.1020 \) and \( \alpha_2 = -0.0004 \).

4.3.2 Magnitude response error

Ideal frequency response of a fractional delay filter is defined as;

\[
H_d(e^{j\omega}) = e^{-j\omega(D+\delta)}
\]

(4.18a)
where,
\[
\text{Phase } \theta_d(\omega) = \arg\{H_d(e^{j\omega})\} = -\omega(D + \delta) \quad (4.18b)
\]
\[
\text{Phase delay, } \tau_{p,d} = -\frac{\theta_d(\omega)}{\omega} = D + \delta \quad (4.18c)
\]

Frequency response of the proposed filter is,
\[
H(e^{j\omega}, \delta) = \sum_{q=0}^{Q} \delta^n C_q(e^{j\omega}) \quad (4.19)
\]

To evaluate the performance of our proposed design, RMS error ($E_{\text{RMS}}$) and maximum absolute error ($E_{\text{max}}$) of magnitude response are defined as
\[
E_{\text{RMS}} = \left( \int_0^{\alpha} \int_{-0.5}^{0.5} |H(e^{j\omega}, \delta) - H_d(e^{j\omega}, \delta)|^2 \, d\delta \, d\omega \right)^{1/2} 

E_{\text{max}} = \max \left\{ \frac{1}{\alpha} \int_{\omega_0}^{\omega_1} |H(e^{j\omega}, \delta) - H_d(e^{j\omega}, \delta)| \, d\omega \right\} \quad \delta \in [-0.5, 0.5] \quad \text{and} \quad 0 < \alpha < 1 

(4.20)
(4.21)

Table 4.1 Passband magnitude and phase delay error

<table>
<thead>
<tr>
<th>Method</th>
<th>$E_{\text{RMS}} \times 10^3$</th>
<th>$E_{\text{max}} \times 10^3$</th>
<th>$\text{RE}_{\text{RMS}}$</th>
<th>$\text{RE}_{\text{max}}$</th>
<th>$E_{\tau_p}$ (RMS) $\times 10^{-3}$</th>
<th>$E_{\tau_p}$ (max) $\times 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lagrange</td>
<td>12.6</td>
<td>81.3</td>
<td>-</td>
<td>-</td>
<td>2.3</td>
<td>20.1</td>
</tr>
<tr>
<td>Oetken</td>
<td>0.015</td>
<td>0.027</td>
<td>-</td>
<td>-</td>
<td>0.00353</td>
<td>0.00468</td>
</tr>
<tr>
<td>Gdev-ER</td>
<td>11.1</td>
<td>13.8</td>
<td>0.0007</td>
<td>0.0055</td>
<td>0.136</td>
<td>1</td>
</tr>
<tr>
<td>MSG</td>
<td>0.18</td>
<td>1.1</td>
<td>-</td>
<td>-</td>
<td>0.18</td>
<td>0.686</td>
</tr>
<tr>
<td>Cosine (DFT)</td>
<td>5</td>
<td>14.1</td>
<td>-</td>
<td>-</td>
<td>6.6</td>
<td>27.0</td>
</tr>
<tr>
<td>Farrow (G. LS)</td>
<td>0.25</td>
<td>0.83</td>
<td>-</td>
<td>-</td>
<td>0.0353</td>
<td>0.125</td>
</tr>
</tbody>
</table>

Peak approximation error ($E_{\text{max}}$) for Gaussian derivative-based design (Gdev-ER) is much greater than Oetken's method and Farrow structure design based on generalized least square method (see Table-4.1) but less than Lagrange's interpolation. However, the peak ($\text{RE}_{\text{max}}$) and RMS ($\text{RE}_{\text{RMS}}$) error when measured with respect to the prototype linear phase filter becomes fairly low (39.86% less than the prototype) in the Gaussian derivative-based design (see Table-4.1). Use of Gaussian window with the cosine basis reduces the Gibb's oscillation near the passband edge and thus RMS and peak error, compared to the FD design based on cosine basis and Lagrange interpolator (see Table 1). However, peak error of the MSG is little higher but the RMS error is less (see Table 1) relative to the
4.3.3 Phase delay error

Phase delay \( \tau_{p,d} \) of the ideal FD filter is constant throughout the band of interest, but it \( \tau_p = -\theta'(\omega)\omega \) differs significantly when FIR FD approximation is made.

Phase delay error \( \Et_p \) measures the phase delay deviation of FIR FD from that of ideal FD filter. We have measured RMS value and peak absolute (max) value within the frequency band of interest and for the entire fractional delay range.

RMS and peak absolute phase delay error are defined as the following,

\[
\Et_{p,\text{RMS}} = \left( \frac{1}{\Delta \omega} \int_{0}^{\Delta \omega} \left( \tau_p(\omega, \delta) - \tau_{p,d}(\omega, \delta) \right)^2 d\delta d\omega \right)^{\frac{1}{2}}
\]  
(4.21a)

\[
\Et_{p,\text{max}} = \max_{\delta \in [-0.5, 0.5]} \max_{0 < a < 1} \left( \tau_p(\omega, \delta) - \tau_{p,d}(\omega, \delta) \right)
\]  
(4.21b)

Typical phase delay response for FIR FD, designed using family of Gaussian derivatives and cosine Gabor functions, are depicted in Fig. 4.4a and 4.4b. In both the designs, phase delay response of the FIR FD follows that of the ideal FD filter till the bandedge (here 0.87\( \pi \)) is reached. It is observed by our study that Gaussian derivative-based design (see blue line in Fig. 4.5a) produces lowest phase delay error in the middle of the passband and around \( \omega = 0 \), compared to the cosine and cosine Gabor interpolation kernels (see red and magenta lines in Fig. 4.5a and Fig. 4.5b). Moreover, it is observed that cosine interpolation basis produces large phase delay error around \( \omega = 0 \) (see magenta lines in Fig. 4.5b) compared to cosine Gabor basis for lower filter order \( (L=1=30) \) (see red lines in Fig. 4.5a). Our proposed Gaussian derivative based design produces maximum phase delay error around the passband edge frequency (see blue line in Fig. 4.4a). A comparative study with other benchmark methods demonstrates that maximum and RMS phase delay error produced by Gaussian derivative based FIR FD filters are relatively small compared to those produced by FIR FD filters designed using Lagrange interpolator (see Table-4.1). But, FIR FD filters designed by Oetken’s method and generalized least square based Farrow structure produce much smaller phase delay error compared to the FIR FD designed using Gaussian derivatives (see Table-4.1). The other FD design method based on cosine Gabor functions produces smaller peak and RMS phase delay error when compared to those in DFT based interpolator, Lagrange interpolator
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and Gaussian derivative based FIR FD filters (see Table-4.1). Phase delay errors in our proposed design method are relatively higher than those produced by FIR FD filters designed by Oetken’s method and generalized least square based Farrow structure (see Table-4.1).

Fig. 4.4. Typical phase delay response for FIR FD designed using (a) family of Gaussian derivatives (b) cosine Gabor functions according to the specifications given in sub section 4.4.1.

4.3.4 DC constancy error

Fig. 4.5. Phase delay error of fractional delay FIR filter constructed by (a) family of Gaussian derivatives in equiripple sense (blue line), cosine Gabor (dotted red line) and (b) only with cosine interpolation basis (dotted magenta line).

Sum of the kernel must be unity for an ideal fractional delay interpolator. For the FIR FD kernel, the sum does not remain constant or equal to unity with the change in fractional delay. It grows up with increasing fractional delay. It has also been observed by our study that DC error with respect to the prototype linear phase filter
is almost negligible for the Gaussian derivative based designs (see Fig. 4.6a) whereas, the same for cosine Gabor kernel is relatively higher.

4.3.5 Taylor series truncation error

Instead of computing error due to the remainder term in Taylor series, we have evaluated the RMS error between the actual filter kernel, sampled at the non integer sample index and the kernel obtained from Taylor series expansion. In this example, truncation error reduces as the number of derivatives used for the Taylor series expansion increases. It is also evident (see Fig. 4.6b) that the RMS error \( T_{\text{RMS}} \) also increases with the increase in fractional delay and it reaches a maximum at the maximum fractional delay.

We define,

\[
Gd(n, \delta) = \sum_{k=0, \text{even}}^{N} a_k g^n[n-D-\delta] \tag{4.22a}
\]

and

\[
cG(n, \delta) = \left( \sum_{k=0}^{N} \beta_k \cos \left( \frac{2\pi k}{N} [n-D-\delta] \right) \right) g^n[n-D-\delta] \tag{4.22b}
\]

for Gaussian derivatives and cosine Gabors. The corresponding \( T_{\text{RMS}} \) is defined as following:

![Fig. 4.6. (a) DC error with respect to DC magnitude of the prototype linear phase filter. Blue and red line represent Gdev-ER and cosine Gabor kernels respectively, (b) Taylor series truncation error for variable fractional delay FIR filter constructed by family of Gaussian derivatives (Blue line) and cosine Gabor (Red star dotted line).](image-url)
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\[
T_{\text{RMS}} = \left( \frac{1}{L} \sum_{\tau=0}^{L-1} \left( \sum_{q=\tau}^{Q} \delta^q C_q[n] - Gd(n, \delta) \right) \right)^{1/2}
\]

\[
T_{\text{RMS}} = \left( \frac{1}{L} \sum_{\tau=0}^{L-1} \left( \sum_{q=\tau}^{Q} \delta^q C_q[n] - cG(n, \delta) \right) \right)^{1/2}
\]

(4.23)

4.3.6 Structure for implementation

Coefficients of the derivative filters \((C_q)\) are stored in a memory. There are \(Q+1\) sub filters \((C_q)\) in the proposed structure to update the coefficients of a fractional delay filter (see Fig. 4.7). Similar to Farrow structure, Horner’s rule is applied to compute delay \((\delta)\) multiplication (see Fig. 4.7). Gaussian derivative based design would require \(Q+1\) number of filtering operation whereas cosine Gabor requires \(Q\) number to produce fractionally delayed output since the sub-filter \(C_0\) is a single impulse at \(n=D\) or simple delay, for the cosine Gabor based interpolator. In our proposed design, computational complexity (storage memory for coefficients and multiplication) is almost halved compared to conventional Farrow structure because the linear phase prototype is constructed by linear combination of even symmetric continuous functions resulting in the sub-filters in the Farrow structure to be of either even or odd symmetry depending upon the derivative orders.

![Fig. 4.7. Structure for implementation of VFD FIR. ‘X’ circles denote multiplication.](image)

4.4 Discussion

Family of Gaussian derivatives can produce nearly equiripple FIR FD filter (see Fig 4.1). The term ‘nearly’ has been used since the ripple magnitude gets enhanced little bit near the passband edge from its prototype equiripple magnitude. Like
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Oetken’s method, our proposed design does not require to preserve the zeros of passband approximation error function of linear phase prototype for the equiripple FD transformation. Oetken’s method can design almost equiripple FIR FD only for even length whereas the proposed methodology produces both odd and even length (see Fig. 4.8a) filters. When the passband becomes closer to half of the sampling frequency, unlike the Oetken’s method, ripple magnitude is large and phase delay errors becomes higher for Gaussian derivatives based design. In general, ripple magnitude in the passband decreases with the increase in the number of Gaussian derivatives (see Fig. 4.8b). However, highest spectral mode is assigned in the proposed design to a fixed spectral location (0.7π) which is less than the passband edge (0.8π). As a result, ripple magnitude starts increasing once the passband edge frequency is above that spectral location. Reallocating the highest spectral mode close to the passband edge, ripple magnitude could be further reduced in the linear phase prototype. But some of the discretized derivatives cannot retain their spectral shape when they are fractionally delayed. Resulting error in FD FIR could not produce equiripple behavior throughout the passband and effective passband gets reduced. On the other hand, in the proposed design, there is no matrix multiplication or calculation of sin or cosine terms required for online delay control. Gaussian derivative based design has one more advantage in order to generate 2-D fractional delay filter since the Gaussian derivatives can be made isotropic.

![Graphs showing magnitude response and normalized frequency](image)

**Fig. 4.8.** Magnitude response (red line) of a fractional delay [-0.5 0.5] FIR filter constructed of family of Gaussian derivatives. (a) Linear phase prototype (green line) has been designed in equiripple sense and even (L=30) length (b) passband magnitude of another FIR FD with reduced passband [0 0.7π].

Linear combination of Mean shifted Gaussian in the frequency domain can also be used for FIR FD design (see Fig 4.2). Initially cosine function as interpolation basis was used to design FIR FD but introduction of Gaussian window reduces the Gibbs
oscillation, phase delay error sufficiently (see Fig. 4.2c and 4.4c) and making it useful for shorter length filter. Windowing each sub-filter made up of cosine and or sin basis can reduce the RMS error from the non-windowing FD filter [52] but the error is still higher in order of magnitude 10 from our proposed design.

4.5 Conclusion

We propose an alternative design of variable fraction delay (VFD) FIR filter using linear combination of continuous functions like Gaussian derivatives and cosine Gabor. As Gaussian and cosine both functions are differentiable up to any order, it is possible to obtain VFD FIR filter from prototype linear phase filter using the Taylor series expansion and consequently Farrow-structure (FS) based implementation. The proposed VFD design based on Gaussian derivative only optimizes passband approximation error of the linear phase prototype in a mini-max sense. Since the magnitude response of the passband of FD filters does not vary significantly (see Table 4.1) than that of the linear phase prototype and phase delay error remains well within the acceptable range (< 0.0015 samples see Table 4.2), no more optimization is required in the frequency and fractional delay domain. In terms of magnitude and phase delay error, Gaussian derivative and cosine Gabor based FIR FD perform better than Lagrange interpolator based FIR FD. Cosine Gabor also out performs Gaussian derivatives and only cosine basis in the FIR FD filter design. Even, with respect to Farrow structure (coefficients are 3rd order polynomial), it produces less RMS error of the magnitude response. Moreover, computational complexity of the proposed VFD is almost halved due to the coefficient symmetry in the sub-filters of FS. Design of odd and even length FIR VFD also becomes straight forward because of continuous domain formulation.