INTRODUCTION

As the title of the thesis indicates, the present work touches on a variety of topics which fall within the purview of special functions from classical and group-theoretic points of view.

Mathematicians including Professor Harry Bateman developed the interplay between mathematical analysis and physical understanding and made an extensive study of special functions which are solutions of a wide class of mathematically and physically relevant functional equations. Valuable contributions in this field by some other well-known mathematicians throughout the world, such as Professor A. Erdelyi, Professor L. Carlitz, Professor G. Szegö, Professor F. G. Tricomi, Professor L. Toscano, late Professor E. D. Rainville, Professor F. Brafman, Professor L. Weisner, Professor W. Miller Jr., Professor W. A. Al-Salam, Professor J. L. Burchnall, Professor N. Ya Vilenkin, Professor J. D. Talman, Professor B. Kaufman, Professor T. S. Chihara, Professor O. Frink, Professor A. E. Danese, Professor H. W. Gould and others, which are relevant to the present work, will be referred in proper places of our investigations embodied in the thesis.

The Legendre polynomials $P_n(x)$ introduced by A. M. Legendre in 1785, are defined by means of the generating relation

$$\left(1 - 2xt + t^2\right)^\frac{1}{2} = \sum_{n=0}^{\infty} P_n(x) t^n$$

The Gegenbauer polynomials $G_n^\nu(x)$ introduced by L. Gegenbauer in 1874, are defined by means of the generating relation

$$\left(1 - 2xt + t^2\right)^\nu = \sum_{n=0}^{\infty} G_n^\nu(x) t^n$$
The Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \) introduced by C. G. J. Jacobi (the original memoir was published after Jacobi's demise by E. Heine in 1869) are defined by means of the generating relation
\[
2 \, \frac{1}{\rho}(1 + t + \rho)(1 - t + \rho) = \sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(x)t^n
\]
where \( \rho = (1 - 2xt + t^2)^{\frac{1}{2}} \).

It may be noted that the Legendre polynomials are Jacobi polynomials with \( \alpha = \beta = 0 \), and also Gegenbauer polynomials with \( \nu = \frac{1}{2} \). Furthermore, Gegenbauer polynomials are constant multiples of Jacobi polynomials with \( \alpha = \beta = \nu - \frac{1}{2} \).

The Hermite polynomials \( H_n(x) \) introduced by C. Hermite in 1864, are defined by means of the generating relation
\[
\exp \left( 2xt - t^2 \right) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.
\]

The Laguerre polynomials \( L_n^{(\alpha)}(x) \), which is a generalization of the simple Laguerre polynomials \( L_n^{(0)}(x) = L_n(x) \) introduced by E. N. Laguerre in 1898, are defined by means of the generating relation
\[
(1 - t)^{-\alpha-1} \exp \left( \frac{-xt}{1 - t} \right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n.
\]

Sometimes (especially in the French literature) orthogonal polynomials in general are called Tchebichef polynomials. There are also several special systems of orthogonal polynomials called Tchebichef polynomials (A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi: Higher Transcendental Functions, Vol.2, P.185).
Besides the above well-known polynomials, there are other polynomials recently studied by various investigators. For examples, the generalized Bessel polynomials $Y_n(x,a,b)$ were introduced by H.L. Krall and O. Frink [Trans. Amer. Math. Soc. 65 (1949), 100-115]; the generalized Laguerre polynomials were introduced by S.K. Chatterjea [Rend. Sem. Mat. Univ. Padova 34 (1964), 180-190]; two generalizations of the classical Hermite polynomials were given by H.W. Gould and A.T. Hopper [Duke Math. J. 29 (1962), 51-64]; the generalized ultraspherical or Gegenbauer polynomials were introduced by P. Barrucand [C.R. Acad. Sc. Paris 264 Ser. A (1967) 792-794, also 265 (1967), 807] and also by H.W. Gould [Duke Math. J. 32 (1965), 697-711].

In Chapter I we have found some new properties of the generalized ultraspherical polynomials of H.W. Gould as well as of P. Barrucand, whereby the corresponding properties of the well-known ultraspherical polynomials follow as fascinating results.

The study of classical polynomials by operational methods was strongly influenced by the works of J.L. Burchnall [Quart. J. Math. Oxford 12 (1941), 9-11] and L. Carlitz [Michigan Math. J. 7 (1960), 219-223]. Such operational methods are frequently very useful as well as elegant. It seems particularly interesting that several properties for classical orthogonal polynomials as well as their generalizations can be readily derived by using operational methods. It may be also remarked that a class of bilateral generating relations for classical polynomials was the subject-matter of mathematicians just after the investigation of Burchnall and Carlitz. For examples, W.A. Al-Salam [Duke Math. J. 31 (1964), 127-] found a class of bilateral generating relations for Laguerre
polynomials; S.K. Chatterjea [C.R. Acad. Sc. Paris 266 (1968) Ser. A 979-981; Pacific J. Math. 29 (1969), 73-76] found a class of bilateral generating relations for the Hermite and the ultraspherical polynomials. Prior to the works of Al-Salam and Chatterjea, some particular forms of bilateral generating relations for classical polynomials are available in the field of special functions without having a unified theory for the existence of a class of bilateral generating relations.

In Chapter II we have utilized such operational methods in order to derive some new theorems for bilateral generating relations. The usefulness of these theorems is discussed by means of illustrative exam examples.

The name 'Generating Function' was introduced by Laplace in 1812. The formula involving generating function is termed as 'Generating relation' by us. Generating functions play a large role in the study of classical orthogonal polynomials. In Chapter III we have considered the properties of functions generated by functions of the forms:

$$A(t) \gamma(xt)$$ and $$A(t) \exp \left( \frac{xt}{t-1} \right)$$

whereby the corresponding properties of the Laguerre polynomials $$L_n^{(\kappa)}(x)$$, the Bernoulli polynomials $$B_n(x)$$, the Bernoulli polynomials of higher order and the $$K$$ th Cesaro mean $$\gamma_n^{(K)}(x)$$ follow easily as particular cases.
Let us now turn to the group-theoretic study of special functions. This group-theoretic study of the hypergeometric function, the Hermite function and the Bessel function was made by L. Weisner [Pacific J. Math. 5 (1955), 1033-1039; Canad. J. Math. 11 (1959), 141-147, 148-156]. Weisner's method consists in constructing a partial differential equation by giving a suitable interpretation to the index or parameter of the special function under consideration and then finding a non-trivial group of continuous transformations which is admitted by the partial differential equation. Later the study of special functions as matrix elements of representations of the simplest Lie groups was made by N. Ya Vilenkin [Uspehi Mat. Nauk II (1966), No. 3 (60), 69-112, etc., his book was published in Russian (1968) and in English (1968)]. The group-theoretic study of special functions was also made by W. Miller Jr. [Nat. Sc. Found. Tech. Report, Univ. of California (1963), his book was published in 1968], also by J. D. Talman (book published in 1968, based on unpublished lecture-notes of E. Wigner of Princeton Univ. 1955), also by B. Kaufman [Jour. Math. Phys. 7 (1966), 447-457], also by E. E. Mc. Bride [Obtaining generating functions (1971)] and also by some mathematicians in recent years.

In Chapter IV we have made a group-theoretic study of generalized Bessel functions and generalized Bessel polynomials and derived several properties, some of which are believed to be new.

Lastly in Chapter V we have also made a group-theoretic study of Tchebychev polynomials and derived a general theorem for a class of mixed trilateral generating relations for such polynomials. In course of this study we have found a novel extension of Lee's trilateral generating relation involving Tchebychev and Charlier polynomials.
Thus the present thesis touches on a variety of theorems and results in the field of special functions from the classical and group-theoretic viewpoints.

**NATURE OF WORK.**

Before we present a succinct account of our present investigations, we like to mention a few general observations on the nature of work of the subject-matter of the thesis. Broadly the contents of the thesis may be classified into the following categories:

(i) a set of new theorems and results for some special functions and their generalizations.

(ii) a set of known theorems and results derived by new methods.

(iii) a set of new theorems for a class of bilateral and mixed trilateral generating relations for some special functions.

(iv) inter-relations between some known results established by some new techniques.

(v) methods adopted are both classical and group-theoretic.

**SYNOPSIS : MAIN THEOREMS AND RESULTS.**

We now propose to give a synopsis of our investigation as a preliminary to their detailed discussions. Here the main theorems and results of our work along with those of other mathematicians, which are relevant to our work, are incorporated with a view to show clearly what new investigations are actually done in the present work.
Some generalizations of Ultraspherical polynomials.

A generalization of Ultraspherical polynomials due to H. W. Gould [Duke Math. J. 32 (1965), 697-711] is given by means of the following generating relations:

\[(0.1.1) \quad (c - mxt + yt)^m \sum_{n=0}^{\infty} t^n P_n(m, x, y, p, c)\]

where \(m \geq 1\) is an integer and the other parameters are unrestricted and \(P_n(m, x, y, p, c)\) is a polynomial in \(x\).

We have studied these polynomials of Gould in some details. The following differentiation formula for \(P_n(m, x, y, p, c)\) is worthy of notice:

\[(0.1.2) \quad P_n(m, x, y, p, c) = \frac{n(m-1)(m-2)^n}{n! \left(\frac{m}{m-1}\right)^n} \left[\begin{array}{c} -n, \frac{2n-p-m}{m-1}, \frac{2n-p-m+1}{m-1}, \ldots, \frac{2n-p-mn+m-2}{m-1} \\ \frac{1}{m}, \frac{2}{m}, \frac{m-1}{m}, \frac{(m-1)^{m-1}}{c^{m-1}} \end{array}\right] x^m \]

where \(D = d/dx\).

By means of this differentiation formula we have deduced the generating relation for \(P_n(m, x, y, p, c)\) by operational method.


\[(0.1.3) \quad (1-ktx + t^k)^\lambda \sum_{n=0}^{\infty} \frac{c_n}{n!} \binom{\lambda + k, x} t^n\]
It may be noted that such type of generalization was already considered by P. Humbert [Proc. Edin. Math. Soc. 39 (1920), 21-24]. However we have considered some new properties of this kind of generalized ultraspherical polynomials with a slight change of notation as follows:

\[(1 - ktx + t^k)^{-\lambda} = \sum_{n=0}^{\infty} P_n^\lambda(x, k) t^n,\]

So that \(P_n^\lambda(x, 2) = P^\lambda(x)\) which is the usual ultraspherical polynomials.

The following are the chief properties derived in connection with \(P_n^\lambda(x, k)\):

\begin{equation}
(0.1.5) \quad P_n^\lambda(x, k) = \sum_{p=0}^{[\lambda/k]} \frac{(-1)^p \lambda^{n-(k-1)p}}{p! (n-kp)! (a+r)!} \end{equation}

where \(a_r = a(a+1) \cdots (a+r-1)\)

\begin{equation}
(0.1.6) \quad a \cdot x D P_n^\lambda(x, k) = n P_n^\lambda(x, k) + DP_{n-k+1}^\lambda(x, k), \quad D = d/dx
\end{equation}

\begin{equation}
(0.1.7) \quad (n+\lambda) P_n^\lambda(x, k) = x(1-k) DP_n^\lambda(x, k) + D P_{n+1}^\lambda(x, k)
\end{equation}

\begin{equation}
(0.1.8) \quad K(n+\lambda) P_n^\lambda(x, k) = DP_{n+1}^\lambda(x, k) + (1-k) DP_{n-k+1}^\lambda(x, k)
\end{equation}

\begin{equation}
(0.1.9) \quad DP_{n-k+2}^\lambda(x, k) + x^2(1-k)D P_n^\lambda(x, k) = n(1-k)xP_n^\lambda(x, k) +
\end{equation}

\begin{equation}
+(n+\lambda k - k + 1!) P_{n-k+1}^\lambda(x, k)
\end{equation}
where \( W = \left[ x^{\lambda} t^{k(1-t)} e^{k\theta} + y^{\lambda} (l-t)^k e^{-k\theta} \right]^{\lambda/k} \)

In course of the discussion of the above two generalizations of Ultraspherical polynomials \([2]\) we have found that some particular cases of the above properties are worthy of interest.

0.2 (SYNOPSIS OF CHAPTER II)

Operational derivation of generating functions for classical polynomials and their generalizations.

If

\[(0.2.1) \quad f_n(x) = \mu(n) G(x) D^n g(x), \quad D = \frac{d}{dx}\]

where \(g(x)\) and \(G(x)\) are independent of \(n\), and

\[(0.2.2) \quad F(x,t) = \sum_{m=0}^{\infty} a_m t^m f_m(x)\]

then

\[(0.2.3) \quad \frac{G(x) F(x-t,tv)}{G(x-t)} = \sum_{r=0}^{\infty} \frac{(-t)^r}{\mu(r)} b_r(y) f_r(x)\]

where \(b_r(y) = \sum_{m=0}^{r} (-1)^m \mu(m) a_m y^m\)

we have pointed out that the theorem of Saran fails to supply a bilateral generating function for \(f_n(x)\) where \(g_n(x)\) contains terms like \(x^n\), which is possible in the case of Laguerre polynomials. For this purpose we [3] have generalized the theorem of S. Saran in the following form:

**Theorem**

If

\[(0.2.4) \quad P_n(x) = \left\{k_n w(x)\right\}^{-1} \frac{n}{D} \left(w(x) X^n\right),\]

where \(k_n\) is a constant and \(W(x)\) is independent of \(n\) and

\[(0.2.5) \quad G(x,t) = \sum_{m=0}^{\infty} a_m t^m P_m(x)\]

then
Our theorem can be directly applied to \( L_n^{(\alpha)}(x) \), \( P_n^{(\alpha-n, \beta)}(x) \) and \( Y_n(x, a-n, b) \), where \( Y_n(x, a, b) \) is the generalized Bessel polynomial introduced by H.L. Krall and O. Frink [Trans. Amer. Math. Soc. 65, 1949, pp. 100-115].

Following particular results of our investigation are interesting:

\[
(0.2.7) \quad (1-xt)^{-\alpha} c^{-1} (1-xt-ty) \exp \left( \frac{-x^2 t}{1-xt} \right) \cdot

= \sum_{r=0}^{\infty} \frac{(-r)^m}{m!} \frac{a_m y^m}{K_m} \cdot

\text{where} \quad L_n^{(\alpha)}(x) \quad \text{is the Laguerre polynomial defined by the generating relation}
\[
\sum_{n=0}^{\infty} \frac{(c)_n}{(1+\alpha)_n} t^n \quad L_n^{(\alpha)}(x) = (1-t)^{-c} \quad _1F_1\left(c; 1+\alpha; \frac{-xt}{1-t} \right)
\]
Replacing $xt$ by $t$ and $y$ by $-y$, (0.2.7) reduces to the following formula due to L. Weisner

\[ (1 - t)^{-\alpha + c - 1} \left( 1 - t + ty \right)^{-c} \exp \left( \frac{xt}{1 - t} \right) \]

\[ \frac{\Gamma(a) \Gamma(\alpha)}{(a-1)!} \]

\[ C \left( 1, \alpha \right) \cdot \frac{xt}{(1-t)(1-t+ty)} \]  

\[ \Gamma(\alpha)(1-t)^{-\alpha} \cdot \frac{yt}{1-t+ty} \exp \left( \frac{xt}{1 - t} \right) \]

\[ \left( 1 - xt - txy \right)^{1-a} e^{bt} (y + 1) = \sum_{r=0}^{\infty} b^r B_r(y) Y_r(x; a-r, b) \]

where

\[ B_r(y) = \sum_{m=0}^{r} \frac{(-1)^{r-m} y^m}{m!} \]

and $Y_n(x; a-n, b)$ is the generalized Bessel polynomials defined by

\[ \sum_{n=0}^{\infty} \frac{(bt)^n}{n!} Y_n(x, a-n, b) = (1-xt)^{1-a} e^{bt} . \]

\[ (1 - ty)^{-\frac{1}{2}} \left( l + \alpha \right) \frac{txy}{(1-ty)(1-xt)} e^{xt} \left( \frac{1}{1-xt} \right)^{\alpha + 1} \]

\[ = \sum_{r=0}^{\infty} \frac{2^r B_r(y) \cdot \frac{x}{x-y} \cdot t^r}{r!} \]

where

\[ B_r(y) = \sum_{m=0}^{r} \frac{(2)^{m} (-r)^{m} Y^m y}{2^m} \]
and \( Y_n(\alpha)(x) = Y_n(x, \alpha + \beta, \alpha) \) in the notation of Krall and Frink \([\text{Trans. Amer. Math. Soc.} 65, 1949, \text{pp} 100-115]\)

\[
(0.2.10) \quad \left[1 - tx \left( \frac{2-y}{2-x} \right) \right]^{\alpha} (1 - xt + \frac{txy}{2})^{-\alpha - \beta - 1}
\]

\[
= \sum_{\nu=0}^{\infty} b_{\nu}(y) P_{\nu}^{(\alpha, \beta)}(x-1) \left( \frac{2xt}{x-2} \right)^{\nu}
\]

Where \( b_{\nu}(y) = \sum_{m=0}^{\nu} \frac{(\nu-m)!}{m!} \frac{y^m}{m!} \)

and \( P_{\nu}^{(\alpha, \beta)}(x-1) \) is the Jacobi polynomial defined by

\[
p_{\nu}^{(\alpha, \beta)}(x) = \frac{(-1)^n (2-x)^{\nu - n}}{2^n n!} \frac{x^{\nu}}{D^{n}} \left[ (2-x)^{\nu} n + \beta \right].
\]

\[
(0.2.11) \quad \left[1 - \frac{xt(2+y)}{2-x} \right]^{\beta} (1 - xt + \frac{txy}{2})^{-\alpha - \beta - 1}
\]

\[
= \sum_{\nu=0}^{\infty} b_{\nu}(y) P_{\nu}^{(\alpha, \beta)}(1-x) \left( \frac{2xt}{x-2} \right)^{\nu}
\]

Where \( b_{\nu}(y) = \sum_{m=0}^{\nu} \frac{(\nu-m)!}{m!} \frac{y^m}{m!} \)

and \( P_{\nu}^{(\alpha, \beta)}(x) = (-1)^n (1-x)^{\nu} (1+x)^{-\beta + n} \frac{n!}{2^n n!} \frac{x^{\nu}}{D^n \left[ (1-x)^{\nu} + (1+x)^{\beta} \right]} \)
(0.2.12) \[
\begin{align*}
&{}_{1}F_{1} \left\{ -\beta; \alpha + 1; \frac{txv}{2(1-xt)(2-2xt-x)} \right\} \\
&= \exp \left( \frac{txv}{2(1-xt)} \right) \cdot (1-\frac{2xt}{x-2})^{\beta} (1-xt)^{-\alpha-\beta-1} \\
&= \sum_{r=0}^{\infty} \left( \frac{2xt}{x-2} \right)^{r} \sum_{m=0}^{r} \frac{(-r)_{m} \cdot y^{m}}{2^{m} (\alpha+1)_{m} m!} b_{r}(y) p_{\beta-r}(1-x) \\
&\text{where } b_{r}(y) = \sum_{m=0}^{\infty} \frac{(-r)_{m} \cdot y^{m}}{2^{m} (\alpha+1)_{m} m!} \\
&\text{and } \sum_{n=0}^{\infty} \frac{t^{n}}{(\alpha+1)_{n}} p_{\beta-n}(x) = \frac{1}{\alpha} \\
&= \int_{0}^{1} \left( -\beta; \alpha + 1; (1-xt)^{\beta} \cdot (1-\frac{2xt}{x-2})^{\beta} \right) \\
&= \sum_{r=0}^{\infty} b_{r}(y) p_{\beta-r}(1-x) \left( \frac{2xt}{x-2} \right)^{\beta} \\
&\text{where } b_{r}(y) = \sum_{m=0}^{\infty} \frac{(-r)_{m} \cdot y^{m}}{2^{m} (\alpha+1)_{m} m!} \\
&\text{and } \sum_{n=0}^{\infty} \frac{(\alpha)_{n}}{(\alpha)_{n}} t^{n} p_{\beta-n}(x) \\
&= \left( \frac{1+xt}{2} \right)^{\beta} F_{2} \left[ \alpha+1; -\beta; \lambda; \alpha+1; c; \frac{1-x}{x}, \frac{1+xt}{2} \right]
\end{align*}
\]

(0.2.13) \[
\begin{align*}
&\left( 2 - \frac{x}{1-xt} \right)^{\beta} F_{2} \left[ \alpha+1; -\beta; \lambda; \alpha+1; c; \frac{x}{2(1-xt)} \right] \\
&= \sum_{r=0}^{\infty} b_{r}(y) p_{\beta-r}(1-x) \left( \frac{2xt}{x-2} \right)^{\beta} \\
&\text{where } b_{r}(y) = \sum_{m=0}^{\infty} \frac{(-r)_{m} \cdot y^{m}}{2^{m} (\alpha+1)_{m} m!} \\
&\text{and } \sum_{n=0}^{\infty} \frac{t^{n}}{(\alpha)_{n}} p_{\beta-n}(x) \\
&= \left( \frac{1+xt}{2} \right)^{\beta} F_{2} \left[ \alpha+1; -\beta; \lambda; \alpha+1; c; \frac{1-x}{x}, \frac{1+xt}{2} \right]
\end{align*}
\]
where
\[ F_2(a, b, b^1; c, c^1; x, y) = \sum_{n+k=0}^{\infty} \frac{(a)_{n+k} (b)_k (b^1)_n x^n y^n}{k! n! (c)_n (c^1)_n} \].

Next for the well known ultraspherical or Gegenbauer polynomials \( P_n^\lambda (x) \) which may be defined by means of the following explicit representation
\[ P_n^\lambda (x) = \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^p (\lambda)_{n-p}}{p! (n-2p)!} (2x)^{n-2p} \]  

P.Humbert \[ \text{Proc.Edin.Math.Soc. 39(1920), 21-24} \]  
considered independently one generalization of the ultraspherical polynomials. We \[ 4 \] shall exhibit the importance of the operational derivation of the generating functions for ultraspherical and analogous polynomials. It is interesting to observe that the well-known generating function of the ultraspherical polynomials \( P_n^\lambda (x) \) can be easily derived with the help of our operational formula
\[ P_n^\lambda (x) = \frac{(-1)^n}{2^n n!} D^n \, _2F_1 (-n, \lambda; \frac{1}{2}; x^2) \]

S.K.Chatterjea \[ \text{Bull.Cal.Math.Soc. 67,115-127(1975)} \] made an attempt to prove a theorem in connection with the unification of a class of bilateral generating relations for certain special functions. 

Here we \[ 5 \] have generalized the theorem of S.K.Chatterjea. 

Actually we have proved the following theorems:
Theorem:

For a set of functions $S_{<}(x_1, x_2, ..., x_n)$ of order $<
$ generated by

\[(0.2.16) \sum_{n=0}^{\infty} A_n S_{n+m}(x_1, x_2, ..., x_n) t^n\]

where the sequence of the coefficients, $A_n$ is selected
in such a way that the series on the left of (0.2.3) gives rise
to a generating function separated like the right member of (0.2.15)

Let

\[(0.2.16) F(x_1, x_2, ..., x_n, t) = \sum_{n=0}^{\infty} a_n S_{n+m}(x_1, x_2, ..., x_n) t^n\]

where $F(x_1, x_2, ..., x_n)$ is of arbitrary nature then

the following bilateral generating relation for $S_{m}(x_1, x_2, ..., x_n)$ holds

\[(0.2.17) \frac{f(x_1, x_2, ..., x_n, t)}{[g(x_1, x_2, ..., x_n)]^m} F[h(x_1, ..., x_n, t), ...]\]

\[= \sum_{n=0}^{\infty} S_{n+m}(x_1, ..., x_n) \frac{t^n}{g(x_1, ..., x_n)}\]
Theorem I can be modified in the case when $A_n$ and $a_n$ are functions of $\alpha$ and $m$ respectively. The modified theorem can be stated as follows:

**Theorem II.**

For a set of functions $S_\alpha(x_1, x_2, \ldots, x_n)$ of order $\alpha$ generated by

$$(0.2.19) \quad \sum_{n=0}^{\infty} A_n(\alpha) S_{n+m}(x_1, x_2, \ldots, x_n) t^n$$

where the coefficient sequence of $A_n(\alpha)$ is selected in such a way that the series on the left of (0.2.19) gives rise to a generating function separated like the right member of (0.2.19).

Let

$$(0.2.20) \quad F(x_1, x_2, \ldots, x_n, t) = \sum_{n=0}^{\infty} a_n(m) S_{n+m}(x_1, x_2, \ldots, x_n) t^n$$

where $F(x_1, x_2, \ldots, x_n)$ is of arbitrary nature then the following bilateral generating relation for $S_m(x_1, x_2, \ldots, x_n)$ holds:

$$(0.2.21) \quad \frac{f(x_1, x_2, \ldots, x_n, t)}{\left[ g(x_1, x_2, \ldots, x_n, t) \right]^m} F \left[ h_1(x_1, \ldots, x_n, t), \ldots, h_n(x_1, \ldots, x_n, t), \left[ g(x_1, x_2, \ldots, x_n, t) \right]^m t \right]$$

$$_n \sum_{n=0}^{\infty} S_{n+m}(x_1, x_2, \ldots, x_n) a_n(m)(z)^n$$
where

\[(0.2.22) \quad \sigma_{n,m}(z) = \sum_{k=0}^{n} a_k(m) A_{n-k}(m+k) z^k\]

the following particular cases of our theorem are worthy of notice:

\[(0.2.23) \quad (1-t-yt)^{-1} \exp \left( \frac{xt(1+y)}{t+yt-1} \right) L_{m}^{(\alpha)} \left( \frac{x}{l-t} \right) \]

\[= \sum_{n=0}^{\infty} L_{n+m}^{(\alpha)}(x) \Delta_{n,m}(y)t^n\]

where \(\Delta_{n,m}(y) = \sum_{k=0}^{n} \binom{k+m}{k} \binom{n+m}{n-k} y^k = (\frac{m+n}{n}) (1+y)^n\)

and \(L_{n}^{(\alpha)}(x)\) is the well-known Laguerre polynomial.

In particular, putting \(y = 0\) (0.2.23) becomes

\[(1-t)^{-1} \exp \left( \frac{-xt}{1-t} \right) L_{m}^{(\alpha)} \left( \frac{x}{1-t} \right) \]

\[= \sum_{n=0}^{\infty} \frac{(n+m)!}{n! m!} L_{n+m}^{(\alpha)}(x) t^n,\]

which is a well-known generating relation for Laguerre polynomial. Again, for \(m = 0\) (0.2.23) reduces to

\[\left[1-t(l+y)\right]^{-1} \exp\left( \frac{xt(l+y)}{t(l+y)-1} \right) = \sum_{n=0}^{\infty} L_{n}^{(\alpha)}(x) \left[ (l+y)t \right]^n,\]

which is identical with the well-known generating relation

\[(1-t)^{-1} \exp \left( \frac{-xt}{1-t} \right) = \sum_{n=0}^{\infty} L_{n}^{(\alpha)}(x) t^n\]
We observe that \( y = 0 \) in (0.2.24) gives rise to the following generating relation for the ultraspherical polynomial

\[
\sum_{k=0}^{\infty} \binom{n+k}{k} \delta_{k,n}(y) t^k \]

where \( \delta_{k,n}(y) = \binom{n+k}{k} (1+y)^k \)

and \( C_n(x) \) is the well-known Gegenbauer polynomial.

We observe that \( y = 0 \) in (0.2.24) gives rise to the following generating relation for the ultraspherical polynomial

\[
\sum_{k=0}^{\infty} \binom{n+k}{k} C^{(\nu)}_{n+k}(x) t^k = (1-2xt+t^2)^{-\nu} \frac{x-t}{(1-2xt+t^2)^{\nu}}
\]

and for \( n = 0 \) in (0.2.24) we have

\[
\left[ (1-2xt+t^2)^{-2} -2(x-t)yt +y^2 t^2 \right]^{-\nu} = \sum_{k=0}^{\infty} C^{(\nu)}_{k}(x)(1+y)t^k,
\]

which is identical with the generating relation

\[
(1-2xt+t^2)^{-\nu} = \sum_{k=0}^{\infty} C^{(\nu)}_{k}(x)t^k.
\]
(0.2.25) \((1-\alpha x) \beta (1-\gamma y)^\nu\) \\
\[ \frac{x}{1-\alpha x}, \frac{y}{1-\gamma y} \]

\[ g_m \left( \frac{x}{1-\alpha x}, \frac{y}{1-\gamma y} \right) \]

\[ = \sum_{n=0}^{\infty} g_{n+m}(x,y) \sigma_{n,m}(z)t^n \]

where \( \sigma_{n,m}(z) = \binom{m+n}{n} (1+z)^n \) and \( g_n(x,y) \)

is the Lagrange polynomial in two variables defined by

\[ (1-\alpha x)^\alpha (1-\gamma y)^\beta = \sum_{n=0}^{\infty} g_n(x,y)t^n. \]

We notice that \( z = 0 \) in (0.2.25) yields the following generating relation for Lagrange's polynomial of two variables:

\[ \sum_{n=0}^{\infty} \binom{m+n}{n} g_{n+m}(x,y)t^n = (1-\alpha x)^\alpha (1-\gamma y)^\beta g_m \left( \frac{x}{1-\alpha x}, \frac{y}{1-\gamma y} \right). \]

Again, letting \( m = 0 \) in (0.2.25) we obtain

\[ (1-\alpha x-zx)^\alpha (1-\gamma y-yz)^\beta = \sum_{n=0}^{\infty} g_n(x,y)(1+z)\gamma^n, \]

which is equivalent to the generating relation

\[ (1-\alpha x)^\alpha (1-\gamma y)^\beta = \sum_{n=0}^{\infty} g_n(x,y) t^n. \]
SYNOPSIS OF CHAPTER III

Some properties for certain sets of polynomials given by their respective generating functions.

If a function $F(x,t)$ has a formal power series expansion in $t$, viz.,

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n,$$

we say that the above expansion of $F(x,t)$ has generated the set $f_n(x)$ and $F(x,t)$ is a generating function for the $f_n(x)$. If for some set of values of $x$, usually a region in the complex plane, the function $F(x,t)$ is analytic at $t=0$ and the series converges in some region around $t=0$.

But the convergence is not necessary for the above generating relation to define $f_n(x)$ and to be useful in obtaining properties of these functions. For this reason we have considered the properties of functions generated by functions of the forms $A(t)\psi(x)$ and $A(t)\exp\left(-\frac{x}{t-1}\right)$ in this chapter. We shall prove the following properties of $Y_n(x)$ generated by $A(t)\psi(x)$

$$y_n(x) = \sum_{n=0}^{\infty} y_n(x)t^n$$

(0.3.1.) \[ \sum_{k=0}^{n} \frac{a_{n-k}}{(n-k)!} \left[ x y_k'(x) + (n-2k)y_k(x) \right] = 0, \]

where $A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$

(0.3.2) If $\psi(u) = \sum_{i=1}^{p} \left[ \alpha_i, \alpha_i', \cdots, \alpha_i^{(p)} \right] u^{(i)}$

then $y_n(x) = \sum_{k=0}^{n} \frac{a_{n-k}}{(n-k)!} \frac{(\alpha_1)_k \cdots (\alpha_p)_k x^k}{(\beta_1)_k \cdots (\beta_p)_k}$
We shall make applications of these results in the case of Laguerre polynomials \( L_n(x) \), the Bernoulli polynomials \( B_n(x) \) and the \( k \)th Cesàro mean \( s_n^k(x) \). On applying our result (0.3.2) in case of Bernoulli polynomials \( B_n(x) \) we have obtained the following theorem:

For arbitrary \( c \), it follows that

\[
(0.3.3) \quad \sum_{n=0}^{\infty} (c)_n B_n(x) t^n = \sum_{n=0}^{\infty} \frac{c_n a_n t^n}{n!} \left\{ \sum_{\pi} \frac{c+n, x_{\pi}, \ldots, x_{\pi}}{p_1, \ldots, p_1} x^t \right\}
\]

Extending our theorem in the case of generalized Bernoulli Polynomials \( B_n^m(x) \) and Bernoulli polynomials of higher order \( B_n^m(x/x_{\alpha_1}, \ldots, x_{\alpha_m}) \),

where

\[
(0.3.5) \quad \sum_{n=0}^{\infty} \frac{B_n^m(x) t^n}{n!} = \sum_{n=0}^{\infty} \frac{B_n^m(x) t^n}{n!}
\]

\[
(0.3.6) \quad \alpha_1 \ldots \alpha_m t^m \left[(e^{\alpha_1} - 1) \ldots (e^{\alpha_m} - 1)\right]^{-1} e^{xt}
\]

\[
= \sum_{n=0}^{\infty} \frac{B_n^m(x/x_{\alpha_1}, \ldots, x_{\alpha_m}) t^n}{n!}
\]
we have derived the following results

\[
\sum_{n=0}^{\infty} \frac{(c)_n B_n^{(1)}(x)t^n}{n!} = (1-xt)^{-c} \sum_{n=0}^{\infty} \frac{(c)_n B_n^{(1)}(t)}{n!} \left( \frac{t}{1-xt} \right)^n
\]

and

\[
\sum_{n=0}^{\infty} (c)_n B_n^{(m)}(x) \left( \frac{xt}{\alpha_1 \cdots \alpha_m} \right)^n = ((1-xt)^{-c} \sum_{n=0}^{\infty} \frac{(c)_n B_n^{(m)}(\alpha_1 \cdots \alpha_m)}{n!} \left( \frac{t}{1-xt} \right)^n
\]

Moreover the following particular results of our investigation is worthy of notice:

\[
\sum_{k=0}^{n} \frac{1}{(1+\alpha)_k} (n-k)! \left[ x \left( DL_k^{(\alpha)}(x) + (n-2k) L_k^{(\alpha)}(x) \right) \right] = 0
\]

where \( D = \frac{d}{dx} \) and \( L_k^{(\alpha)}(x) \) is the Laguerre polynomial.

\[
\sum_{k=0}^{n} \frac{B_{n-k}}{k!(n-k)!} \left[ x B_k^{(1)}(x) + (n-2k) B_k(x) \right] = 0
\]

where \( B_n(x) \) is the Bernoulli polynomials defined by

\[
\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!}
\]

\[
\sum_{k=0}^{n} \frac{B_{n-k}}{(k(n-k)!} \left[ x DB_k^{(1)}(x) + (n-2k) B_k^{(1)}(x) \right] = 0
\]

where \( B_n^{(1)}(x) \) is the generalized Bernoulli polynomials defined by (0.3.5).
\[ (0.3.12) \quad \sum_{k=0}^{n} \frac{B_{n-k}^{(m)} (x_{1}, \ldots, x_{m})}{k! (n-k)!} \left[ xDB_{k}^{(m)} (x/x_{1}, \ldots, x_{m}) \right] = 0 \]

where \( B_{n}^{(m)} (x_{1}, \ldots, x_{m}) \) denotes Bernoulli number of order \( m \).

\[ (0.3.13) \quad \sum_{p=0}^{n} \frac{(k+1)^{n-p}}{p! (n-p)!} \left[ xDg_{p}^{(k)} (x) + (n-2p)g_{p}^{(k)} (x) \right] = 0 \]

where \( g_{n}^{(k)} (x) \) is the \( k \) th Cesàro mean of the first \( n \) partial sums of the series.

\[ 1 + x + x^{2} + \ldots \text{ is defined by} \]
\[ (l-t)^{-k} (l-xt)^{-1} = \sum_{n=0}^{\infty} \frac{g_{n}^{k} (x) t^{n}}{n!} \]

\[ (0.3.14) \quad \sum_{n=0}^{\infty} \frac{(c)_{n} L_{n}^{(\alpha)} (x) t^{n}}{(1 + \alpha)_{n}} = (l-t)^{-c} _{1}F_{1} (c; \alpha+1; \frac{-xt}{1-xt}) \]

\[ (0.3.15) \quad (l-xt)^{-c} \left[ 1 - \frac{c}{\alpha} \left( \frac{t}{l-xt} \right) + \sum_{n=1}^{\infty} \frac{(c)_{2n} B_{2n}}{(2n)!} \left( \frac{t}{l-xt} \right)^{2n} \right] \]
\[ = \sum_{n=0}^{\infty} \frac{(c)_{n} B_{n} (x) t^{n}}{n!} \]

\[ (0.3.16) \quad \sum_{n=0}^{\infty} (c)_{n} g_{n}^{(k)} (x) t^{n} \quad \text{is defined by} \]
\[ \sum_{n=0}^{\infty} (c)_{n} (xt)^{n} \]
Next we [7] have proved the formula

\[(0.3.17) \quad p+1 F_q \left[ -n, \alpha_1, \ldots, \alpha_p \left| \begin{array}{c}
\beta_1, \ldots, \beta_q \end{array} \right| x \right] = \sum_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} \frac{p+1 F_q}{p} \left[ -k, \alpha_1, \ldots, \alpha_p \left| \begin{array}{c}
\beta_1, \ldots, \beta_q \end{array} \right| x \right] \]

As a nice application of this formula we have pointed out the following multiplication formula:

\[(0.3.18) \quad \left( \frac{\sin \beta}{\tan \alpha} \right)^n \binom{\lambda}{\tan \alpha} = \sum_{k=0}^{n} \binom{\lambda+n}{k} \left[ \frac{\sin (\beta-\alpha)}{\sin \alpha} \right]^{-k} \cos \frac{n-k}{\beta} \binom{\lambda}{\tan \beta} \]

For \( \lambda = 0 \) which reduces to a result of S.K. Chatterjea [Rend. Sem. Mat. Univ. Padova, Vol. 31(1961) pp 243-248].

Here we like to point out that above result can be generalized to hold for the generalized hypergeometric polynomials

\[ p+1 F_q \left[ -n, \alpha_1, \ldots, \alpha_p \left| \begin{array}{c}
\beta_1, \ldots, \beta_q \end{array} \right| x \right] \]

Next we (3) have drawn several conclusions about \( y_n(x) \) generated by

\[(0.3.19) \quad A(t) \exp \left( \frac{xt}{t-1} \right) = \sum_{n=0}^{\infty} y_n(x) t^n \]

after deriving the following differential recurrence relation:
where \( y_0^\prime(x) = 0 \) and \( y_l(x) = 0 \).

Furthermore we have been able to demonstrate the following theorem on multiplication formulas:

Theorem.
If \( f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \),
\[
\phi(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!},
\]
and \( \phi(t) f(xt) = \sum_{n=0}^{\infty} g_n(x) \frac{t^n}{n!} \),

then \( g_n(x) \) satisfies the multiplication formula:

\[
(0.3.21) \quad g_n(xy) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} y^k (1-y)^{n-k} g_k(x)
\]

provided that \( b_1 b_j = b_1 + j \).

The existence of such type of multiplication formula is exhibited by assuming \( \phi(t) = e^t \), so that \( b_1 = 1 \) and obviously \( b_1 b_j = b_1 + j \).

and in such case the following multiplication is well-known:

\[
(0.3.22) \quad g_n(xy) = \sum_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} g_k(x).
\]
Certain properties of generalized Bessel functions and generalized Bessel polynomials from the view point of Lie-algebra.

The basic idea of the Lie group (or Lie algebra) is that from an operator of the form

\[ U = \xi(x,y) \frac{\partial}{\partial x} + \eta(x,y) \frac{\partial}{\partial y} \]

Known as 'infinitesimal operator' (or generator) in connection with one parameter group of continuous transformations, which shifts the point \( X = (x,y) \) to a neighbouring point \( X + dX \), whereas the finite operator \( \exp(tU) \) shifts the point \( X \) to a point \( X' \) at a finite distance along the 'path curve' of one parameter group. Such operator can be constructed for some special functions (not for all special functions, for example, no group-theoretic basis is yet known for the Gamma and elliptic function) from the differential difference recursion relations. In fact, the underlying idea of the group-theoretic method is to consider a pair of first-order differential-difference relations which are equivalent to a given second order differential equation. Such pair of operators raise and lower the index of the parameter of the special function under consideration and from them we can generate the finite operators which are considered as generators of the Lie-algebra.

We have considered the generalized Bessel functions and
generalized Bessel polynomials from the viewpoint of Lie Algebra
For the generalized Bessel functions $I_{m}^{p,q}(r)$ we have noticed
the relations

\[
(q \frac{d}{dr} + \frac{m}{r}) I_{m}^{p,q}(r) = I_{m-p}^{p,q}(r)
\]

\[
(p \frac{d}{dr} - \frac{m}{r}) I_{m}^{p,q}(r) = I_{m+q}^{p,q}(r)
\]

and then by giving suitable interpretation to $m$, we have
formed the Lie elements for Lie-algebra. For the generalized
Bessel polynomials

\[
Y_{n}^{(\alpha)}(x) = \sum_{\gamma} x^{\gamma} \left( -n, n + \alpha - 1; -x/\beta \right)
\]

we have noticed the relations

\[
\left( \frac{x^2}{y} \frac{\partial}{\partial x} - \frac{nx-\beta}{y} \right) Y_{n}^{(\alpha)}(x) y^{\alpha} = \beta Y_{n}^{\alpha-1}(x) y^{\alpha-1}
\]

\[
( xy \frac{\partial}{\partial y} + y^2 \frac{\partial}{\partial y} + n(n-1)y ) Y_{n}^{(\alpha)}(x) y^{\alpha} = (n+\alpha -1) Y_{n}^{\alpha+1}(x) y^{\alpha+1}
\]

In this chapter we [10] propose to consider some properties
of the functions $I_{m}^{p,q}(r)$ which are group theoretic generalization
of Bessel functions and may be defined by the following generating
relation:

\[
(0.4.1) \exp \left[ \frac{r}{p+q} (z^{p} + z^{-q}) \right] = \sum_{l=-\infty}^{\infty} I_{l}^{p,q}(r) z^{l}
\]

Indeed, $I_{1}^{p,q}(r) = (-i)^{l} J_{l}(ir)$, $J_{0}$ is the ordinary
modified Bessel function.
The following are the results of our investigation

\[(0.4.2) \quad \left[ \frac{r^+ (p+q)}{r + \frac{(p+q)}{p+q}} \right]^{\frac{m}{p+q}} \quad I_m \left[ (r + \alpha(p+q))^{\frac{q}{r + \beta(p+q)}} \right] \]

\[= \sum_{s,t=0}^{\infty} \frac{\alpha^s \beta^t}{s! t!} \quad I_{m-sp+tq} \left( r \right) \]

\[(0.4.3) \quad \text{If} \quad \sum_{m=0}^{\infty} \frac{a_m}{m!} I_m \left( r, z \right) Z^m = F(r, z), \text{then} \]

\[\sum_{m=0}^{\infty} (yz)^m \quad I_m \left( r, z \right) b_m(yz) \]

\[= F \left[ \left( r^q \left( r + \alpha(p+q) \right) \right)^{\frac{1}{p+q}}, \quad zy \left( \frac{r}{r + \alpha(p+q)} \right)^{\frac{1}{p+q}} \right] \]

where \( b_m(x) = \sum_{l=0}^{[m/4]} \frac{a_{m-4l} \left( x \right)^l}{1!(m-4l)!} \)

\[(0.4.4) \quad \text{If} \quad G(r, z, \theta) = \sum_{m=0}^{\infty} \frac{a_m}{m!} I_m \left( r, z \right) Z^m p_m(\theta) \]

\[\text{then} \quad \sum_{m=0}^{\infty} F_m \left( r, \theta \right) (yz)^m b_m(yz, \theta) \]

where \( b_m(x, \theta) = \sum_{n=0}^{[m/4]} \frac{a_{m-nq} \left( x \right)^n}{n!(m-nq)!} \quad p_{m-nq}(\theta) \)
It may be of interest to remark that the result (0.4.2) does not seem to appear earlier in the form given here.

Again, we have derived a class of mixed trilateral generating functions for the generalized Bessel polynomials $Y_n(x)$ defined by

$$Y_n(x) = \sum_{k=0}^{\infty} A_k(n) W^k Y_n(x) p_k(u)$$

Indeed we shall prove the following theorem:

Theorem

If there exists a bilateral generating relation of the form

$$G(x,w,u) = \sum_{k=0}^{\infty} A_k(n) W^k Y_n(x) p_k(u)$$

where $A_k(n)$ contains $n$ and $p_k(u)$ is an arbitrary classical polynomial (or function) of degree $n$, then there exists a mixed trilateral generating relation of the form

$$\left(\frac{1}{1-t}\right)^{n-1} G(\frac{x}{1-t}, \frac{t}{1-t}, u) =$$

$$= \sum_{p=0}^{\infty} y_p(x) t^p b_p(z,u)$$

where

$$b_p(z,u) = \sum_{k=0}^{p} A_k(n)(n+k+1)_{p-k} z^k p_n(u)$$
Following particular case of our theorem is interesting

\[(1-t-1-z)^{l-n} \frac{y^n}{1-t(l+1+z)} = \sum_{p=0}^{\infty} y_n^p (x) \frac{t^p}{p} b_p(z)\]

where \[b_p(z) = \sum_{k=0}^{p} \left( \begin{array}{c} p \\ k \end{array} \right) (n-1)p z^k.\]

§ 06 (SYNOPSIS OF CHAPTER V)

A class of mixed trilateral generating relations for Tchebychev polynomials: a novel extension of Lee's trilateral generating relation involving Tchebychev and Charlier polynomials.

P.A. Lee [Nanta Math.8(1975), 83-87] has considered some generating functions involving the Charlier polynomial

\[C_n(x;a) = \frac{\,_2F_0 \left[ \begin{array}{c} -n, -x \\ 0 \end{array} \right]}{\left[ 1-a \right]}\]

from the viewpoint of differential operators. He has also derived a trilateral (which he calls trilinear) generating function of Charlier polynomials with the Tchebychev polynomials. Of all generating functions the trilateral generating function seems to be a novel one to us and so we are inclined to extend that trilateral generating relation. Here we [12] have considered the Lie-algebra for Tchebychev polynomials and proved the following theorems for a class of mixed trilateral generating functions for Tchebychev polynomials. Moreover we have found a quadrilateral generating relation involving Tchebychev and Charlier polynomials from Lee's result.

According to L. Weisner, we have replaced \[D = \frac{d}{dx}\] by \[\frac{\partial}{\partial x}\] and \[n\] by \[\frac{\partial}{\partial y}\] in the
differential equation satisfied by the Tchebychev polynomials of the first kind and then investigated the following raising and lowering operators $B$ and $C$ such that

$$B \left[ T_n(x) y^n \right] = n T_{n-1}(x) y^{n-1}$$

$$C \left[ T_n(x) y^n \right] = n T_{n+1}(x) y^{n+1}$$

where

$$B = (1 - x^2) y^{-1} \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

and

$$C = y (x^2 - 1) \frac{\partial}{\partial x} + xy^2 \frac{\partial}{\partial y} .$$

By means of commutator relation viz.


We have found that the operators $A$, $B$, $C$ generate a Lie group. The extended forms of the transformation group generated by the operators $B$ and $C$ are expressed as follows:

$$e^{B B} f(x, y) = f \left( \frac{-xy - b}{\sqrt{y^2 + 2xyb + b^2}}, -\sqrt{y^2 + 2xyb + b^2} \right)$$

$$e^{C C} f(x, y) = f \left( \frac{x-cy}{\sqrt{1-2xyc + c^2y^2}}, \frac{y}{\sqrt{1-2xyc + c^2y^2}} \right)$$

Using this notion of Lie group (or the corresponding Lie algebra) we have been able to deduce the following main results of our investigation:
Theorem I.

If there exists a unilateral generating relation of the form

\[ G(x,t) = \sum_{m=0}^{\infty} \frac{a_m}{m!} T_m(x) t^m \]

where \( a_m \) is an arbitrary constant, then there exists a bilateral generating relation of the form

\[ G\left(\frac{x-w}{\sqrt{1-2xw+w^2}}, \frac{tw}{\sqrt{1-2xw+w^2}}\right) = \sum_{p=0}^{\infty} T_p(x)^w b_p(t) \]

where \( b_p(t) = \sum_{m=0}^{p} a_m \binom{p}{m} t^{m} \).

Theorem II

If there exists a bilateral generating relation of the form

\[ G(x,t,u) = \sum_{m=0}^{\infty} \frac{a_m}{m!} T_m(x) p_m(u) t^m \]

where \( p_m(u) \) is an arbitrary classical polynomial or function of degree \( m \) and \( a_m \) is an arbitrary constant, then there exists a mixed trilateral generating relation of the form

\[ G\left(\frac{x-w}{\sqrt{1-2xw+w^2}}, \frac{tw}{\sqrt{1-2xw+w^2}}, u\right) = \sum_{r=0}^{\infty} \frac{T_r(x)^w}{r!} b_r(t,u) \]
where \( b_r(t,u) = \sum_{m=0}^{\infty} a_m(t^m) \left( \frac{r}{m} \right) m \cdot p_m(u) \cdot t^m \)

The following is a nice extension of Lee's trilateral generating relation:

\[
(0.5.3) \quad \sum_{p=0}^{\infty} \frac{T_p(x) w^p}{p!} \sum_{n=0}^{\infty} \binom{p}{n} (n)_{p-n} c_n(k; \alpha) c_n(\ell; \beta) Z^n
\]

\[
= z \left\{ e^{\lambda_1} (1-\frac{\lambda_1}{\alpha})^k (1-\frac{\lambda_1}{\beta})^l c_k(l, (\alpha-\lambda_1)(\beta-\lambda_1)) \right. \\
+ e^{\lambda_2} (1-\frac{\lambda_2}{\alpha})^k (1-\frac{\lambda_2}{\beta})^l c_k(l, (\alpha-\lambda_2)(\beta-\lambda_2)) \right\}
\]

where \( \lambda_1 = \frac{(x-w + \sqrt{x^2-1}) ZW}{1-2xw + w^2} \)

and \( \lambda_2 = \frac{(x-w - \sqrt{x^2-1}) ZW}{1-2xw + w^2} \)

Now we humbly state that we have finished the introduction as well as the synopsis of the five chapters as a preliminary to their full discussion in the next stage. We have shown how known generating relations involving various classical polynomials and their generalizations have been refined and generalized in order to obtain novel and interesting relations. Next we proceed formally to dispose of the five chapters in the serial order.
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