Appendix

An expansion formula and Parseval relation

The following expansion formula and the Parseval relation of which we have made extensive use in Chapter IV (The inverse problem) of the present thesis, occurs in Tiwari's Calcutta D.Phil. Thesis, 1971 (unpublished) [See Tiwari (164), p. 38-44]. For convenience we outline the proofs of the theorems as given by Tiwari.

An Expansion Theorem

Let \( f(x) = \{f_1(x), f_2(x)\} \) have continuous derivatives up to the order two in \([0, \infty)\) (or more generally suppose \( f(x) \) to be the integral of an absolutely continuous vector) and let \( \tilde{f}(x) = L f(x) \) (where \( L \) is the matrix operator treated in the present thesis) belongs to \( L^2 [0, \infty) \) and let \( f(x) \) satisfy the boundary conditions in the \( b \)-case (i.e., the boundary conditions at \( x = 0 \) and at \( x = b \), where \( b \) is arbitrary but \( > 0 \). \( b \) is ultimately made to tend to infinity so as to obtain the problem of the interval \([0, \infty)\)). Further, let

\[
\Psi_n(x, \lambda) = \{\Psi_{n1}(x, \lambda), \Psi_{n2}(x, \lambda)\} = \sum_{\lambda=1}^{\lambda} \int_{0}^{\lambda} \phi_n(0/x, u) d\phi_x(u)
\]

\((r = 1, 2, \lambda \text{ real})\) and

\[
\int_{n}(\lambda) = \langle \Psi_n(y, \lambda), f(y) \rangle > 0, \infty \quad (r = 1, 2).
\]
Then
\[ f(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \phi_{-n}(0/x, \lambda) \, d\mathcal{F}(\lambda) \]  
(A)

In particular,
\[
f(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \phi_{-n}(0/x, \lambda) \, d\mathcal{F}(\lambda) \left( \int_{0}^{\lambda} (\phi_{n}(0/y, \lambda), f(y)) \, dy \right) \]  
(B)

where \((\mathcal{F}(\lambda))\) is defined as in Chapter I.

The formula (A) follows in exactly the same way as the corresponding theorem of Titchmarsh in (151), p. 60.

Under the stated conditions \(\mathcal{F}(\lambda)\), \(r = 1, 2\) are of bounded variation over \((-\infty, \infty)\).

Also
\[
\mathcal{F}(\lambda) = \int_{0}^{\lambda} \left( \sum_{n=1}^{\infty} \phi_{n}(0/y, u) \, d\mathcal{F}(u) \right) f(y) \, dy
\]
\[
= \sum_{n=1}^{\infty} \int_{0}^{\lambda} d\mathcal{F}(u) \int_{0}^{\lambda} (\phi_{n}(0/y, u), f(y)) \, dy,
\]
the \(y\)-integral uniformly convergent.

Then
\[
d\mathcal{F}(\lambda) = \sum_{n=1}^{\infty} d\mathcal{F}(\lambda) \int_{0}^{\lambda} (\phi_{n}(0/y, \lambda), f(y)) \, dy
\]
and the expansion formula (A) reduces to (B) i.e.,
\[
f(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \phi_{-n}(0/x, \lambda) \, d\mathcal{F}(\lambda) \left( \int_{0}^{\lambda} (\phi_{n}(0/y, \lambda), f(y)) \, dy \right) \]
Parseval relation

Let \(( f(x), f(x) ) \in L[0, \infty) \). Then the sequence of vectors

\[
F_n(\lambda) = \{ F_{1n}(\lambda), F_{2n}(\lambda) \},
\]

where

\[
F_{rn}(\lambda) = \langle \phi_r(0/x, \lambda), f(x) \rangle \quad n \quad (r = 1, 2)
\]

converges in mean with respect to \( f_{\nu_\lambda}(\lambda) \), over \((-\infty, \infty)\) to a limit

\[
P(\lambda) = \{ P_1(\lambda), P_2(\lambda) \}.
\]

I.e.

\[
\lim_{n \to \infty} \| F - F_n \times d\nu(\lambda) \| \to 0 \quad \text{and}
\]

\[
\| f \|_{d, \nu} = \frac{1}{N} \| F, d\nu \|_{-\infty, \infty}
\]

where \( f_{\nu_\lambda}(\lambda) \) is defined as in Chapter I.

The theorem is proved in the same way as that by Titchmarsh in (151), theorem 3.7.