§ 5.0 INTRODUCTION

In this chapter, we have found some more properties of function spaces with the help of the new spaces introduced in previous chapter, when the function space is endowed with almost compact open topology.

R.H. Fox [27], Arens and J. Dugundji [6] have established some results related to the subbasis, composition maps and associates, when the function space is endowed with compact open topology. They also studied function spaces of equi-continuous family when endowed with compact-open topology. In a similar fashion, we have studied those topics on function spaces endowed with almost compact open topology.

In Section 1, some results related to subbases of almost compact open topology have been established.

In Section 2, some results related to continuity of the composition maps have been established with the help of new spaces introduced in the previous chapter.
In the penultimate section, results related to associates have been established when the function space is endowed with almost compact open topology.

In the final section, some results concerning function spaces of equi-continuous family endowed with almost compact open topology have been established. Here the co-domain space is taken as metric space.

§ 5.1 SUBBASIS OF ALMOST COMPACT OPEN TOPOLOGY

**THEOREM 5.1.1** (a) Let \( B = \{ W_\alpha : \alpha \in A \} \) be a subbasis for \( Z \). Then the family \( \{ T(A, W) : A \subset Y, W \in B \} \) is also a subbasis for the topology on \( C_{st\theta}(Y, Z) \) where \( C_{st\theta}(Y, Z) \) denote the collection of all strong \( \theta \)-continuous function on \( X \) to \( Y \).

(b) Let \( \mathcal{K} = \{ A_\alpha : \alpha \in A \} \) be a family of almost compact sets in \( Y \) with the following property : For each almost compact \( A \) and \( \theta \)-open \( U \supseteq A \) there are finitely many \( A_i \in \mathcal{K} \) with \( A \subset \bigcup_{i=1}^n A_i \subset U \). Then the family \( \{ T(A, W) : A \in \mathcal{K}, W \in B \} \) is also a subbasis for the topology on \( C_{st\theta}(Y, Z) \).

Here the function space is endowed with almost compact open topology for both (a) and (b).
PROOF (a) It is enough to show that, given \( f \in \mathcal{T}(A, V) \), \( A \) almost compact in \( Y \) and \( V \) open in \( Z \), \( \exists \) finitely many \( \mathcal{T}(A_i, W_i), W_i \in \mathcal{B} \) with \( f \in \bigcap_{i=1}^{n} \mathcal{T}(A_i, W_i) \subset \mathcal{T}(A, V) \). Here \( V \) is open in \( Z \) and \( \mathcal{B} \) is a subbasis for \( Z \). So, \( V = \bigcup_{\beta} V_{\beta} \) where each \( V_{\beta} \) is a finite intersection \( \bigcap_{j=1}^{K(\beta)} W_{\beta_j} \) with \( W_{\beta_j} \in \mathcal{B} \). Now as \( f \) is strong \( \Theta \)-continuous, \( f^{-1}(V_{\beta}) \) is \( \Theta \)-open in \( Y \); \( f^{-1}(V_{\beta}) \cap A \) is \( \mathcal{T}_A - \Theta \)-open [53] (where \( \mathcal{T}_A \) denote the subspace topology on \( A \)). Also \( \{f^{-1}(V_{\beta}) \cap A\} \) covers \( A \). Let \( x \in A \), then \( \exists f^{-1}(V_{\beta_x}) \cap A \) such that \( x \in f^{-1}(V_{\beta_x}) \cap A \). As \( f^{-1}(V_{\beta_x}) \cap A \) is \( \mathcal{T}_A - \Theta \)-open, \( \exists U_x \ni x \) open in \( A \) such that \( \text{cl}_{\mathcal{T}_A} U_x \subseteq f^{-1}(V_{\beta_x}) \cap A \) [\( \text{cl}_{\mathcal{T}_A} U_x \) denote \( \mathcal{T}_A \)-closure of \( U_x \)].

Now, \( \{U_x : x \in A\} \) forms a cover of \( A \) by open sets in \( A \).

As \( A \) is almost compact, \( \exists x_1, x_2, \ldots, x_n \) in \( A \) such that
\[
A = \text{cl}_{\mathcal{T}_A} U_{x_1} \bigcup \text{cl}_{\mathcal{T}_A} U_{x_2} \bigcup \ldots \bigcup \text{cl}_{\mathcal{T}_A} U_{x_n}.
\]

Now \( \text{cl}_{\mathcal{T}_A} U_{x_i} \), \( i=1,2,\ldots,n \), are regular closed subsets of \( A \) in \( \mathcal{T}_A \) topology. As \( A \) is almost compact, \( \text{cl}_{\mathcal{T}_A} U_{x_i} \) is almost compact as a subspace of \( A \) [62]. Hence \( \text{cl}_{\mathcal{T}_A} U_{x_i} \) is almost compact as a subspace of \( Y \). So if we put \( A_i = \text{cl}_{\mathcal{T}_A} U_{x_i} \), \( i=1,2,\ldots,n \) then \( A = \bigcup A_i \).
Also \( A_i \subseteq f^{-1}(V_{\beta_x}) \), \( i = 1, 2, \ldots, n \). Since
\[
K(\beta_x) \subseteq K(\beta_{x_j})
\]
for each \( i \). So we find
\[
f \in \bigcap_{i=1}^{n} \bigcap_{j=1}^{\infty} T(A_i, W_{\beta_{x_j}}) = \bigcap_{i=1}^{n} T(A_i, V_{\beta_{x_i}}) \subseteq T(A, \bigcup_{i=1}^{n} V_{\beta_{x_i}})
\]
\( \subseteq T(A, V) \) as required.

(b) Given \( f \in T(A, W) \). As \( f \) is strong \( \Theta \)-continuous, \( f^{-1}(W) \) is \( \Theta \)-open and \( f^{-1}(W) \subseteq A \), we choose finitely many \( A_i \in \mathcal{A} \) with
\[
A \subseteq \bigcup_{i=1}^{n} A_i \subseteq f^{-1}(W).
\]
Then \( f \in T(A_i, W) \) for \( i = 1, 2, \ldots, n \), so that
\[
f \in \bigcap_{i=1}^{n} T(A_i, W) = T(\bigcup_{i=1}^{n} A_i, W) \subseteq T(A, W)
\]
and as in (a), the proof is complete.

**Corollary 5.1.2** In \( C_{\text{st}_{\Theta}}(X \times Y, Z) \), the sets \( T(A_1 \times A_2, V) \), \( A_1 \subseteq X \), \( A_2 \subseteq Y \) are almost compact and \( V \subseteq Z \) is open, form a subbasis for the almost compact open topology.

**Proof** Let \( A \subseteq X \times Y \) be almost compact and let \( W \) be a \( \Theta \)-open set containing \( A \). If \( A_1 \) and \( A_2 \) are projections of \( A \) on \( X \) and \( Y \) respectively, then \( A_1 \times A_2 \) is almost compact and contains \( A \). So
\[
W \cap (A_1 \times A_2) \subseteq A \quad \text{and as } W \text{ is } \Theta \text{-open in } X \times Y, \ W \cap (A_1 \times A_2) \text{ is}
\]
Let $a \in A$, then there exist $U_a$ open in $A_1$ and $V_a$ open in $A_2$ such that $a \in U_a \times V_a$ and $\text{cl}_{T_{A_1} \times A_2} (U_a \times V_a) \subseteq (A_1 \times A_2) \cap W$. So $\{U_a \times V_a : a \in A\}$ forms an open cover of $A$ in $A_1 \times A_2$. As $A$ is almost compact, there exist $U_{a_i} \times V_{a_i}, i = 1, 2, \ldots, n$ such that

$$A \subseteq \bigcup_{i=1}^{n} \text{cl}_{T_{A_1} \times A_2} (U_{a_i} \times V_{a_i}).$$

Also $\bigcup_{i=1}^{n} \text{cl}_{T_{A_1} \times A_2} (U_{a_i} \times V_{a_i}) \subseteq W$.

As $\text{cl}_{T_{A_1} \times A_2} (U_{a_i} \times V_{a_i})$ is regularly closed in $A_1 \times A_2$, so $\text{cl}_{T_{A_1} \times A_2} (U_{a_i} \times V_{a_i}), i = 1, 2, \ldots, n$, are almost compact. Hence condition of Theorem 5.1.1(b) is satisfied and the result follows.

§ 5.2 CONTINUITY OF THE COMPOSITION MAP

Let $X, Y, Z$ be three spaces; for $f \in C_{st\theta}(X, Y)$ and $g \in C(Y, Z)$, the composition $gof \in C_{st\theta}(X, Z)$ [38]; so that $F(f, g) = gof$ defines a map $C_{st\theta}(X, Y) \times C(Y, Z) \rightarrow C_{st\theta}(X, Z)$. For $f \in C_{st\theta}(X, Y)$, $g \in C(Y, Z)$, we investigate the continuity of $F$. For $f \in C(X, Y)$, $g \in C(Y, Z)$, $F(f, g) = gof$ also defines a map $C(X, Y) \times C(Y, Z) \rightarrow C(X, Z)$. Then also we investigate the continuity of $F$. In fact,

**Theorem 5.2.1** $F$ is continuous in each argument separately:
(a) \( f^+ : C(Y,Z) \to C(X,Z) \) defined by
\[ f^+(g) = gof \text{ for each fixed } f \in C(X,Y), \text{ is continuous.} \]

(b) \( f^+ : C(Y,Z) \to C_{\text{st} \theta}(X,Z) \), defined by
\[ f^+(g) = gof \text{ for each fixed } f \in C_{\text{st} \theta}(X,Y), \text{ is continuous.} \]

(c) \( g^+ : C(X,Y) \to C(X,Z) \) defined by
\[ g^+(f) = gof \text{ for each fixed } g \in C(Y,Z), \text{ is continuous.} \]

(d) \( g^+ : C_{\text{st} \theta}(X,Y) \to C_{\text{st} \theta}(X,Z) \) defined by
\[ g^+(f) = gof \text{ for each fixed } g \in C(Y,Z), \text{ is continuous.} \]

**Proof** (a) Let \( T(A,V) \) be a subbasic nbd. of \( gof \) for a fixed \( f \in C(X,Y) \) where \( A \) is almost compact in \( X \) and \( V \) open in \( Z \). Then \( g(f(A)) \subseteq V \) and \( A \) is almost compact. As \( f \) is continuous, \( f(A) \) is almost compact subset. So \( g \in T(f(A),V) \) and \( f^+(T(f(A),V)) \subseteq T(A,V) \). So \( f^+ \) is continuous.

(b) Let \( T(A,V) \) be a subbasic nbd. of \( gof \), for a fixed \( f \in C_{\text{st} \theta}(X,Y) \), when \( A \) is almost compact in \( X \) and \( V \) open in \( Z \). Then \( g(f(A)) \subseteq V \) and as \( A \) is almost compact and \( f \) is strong \( \Theta \) continuous, \( f(A) \) is a compact set. So \( g \in T(f(A),V) \) and \( f^+(T(f(A),V)) \subseteq T(A,V) \). So \( f^+ \) is continuous.

(c) Let \( gof \in T(A,V) \) for fixed \( g \in C(Y,Z) \). So \( g(f(A)) \subseteq V \) or, \( f(A) \subseteq g^{-1}(V) \). As \( g \) is continuous, \( g^{-1}(V) \) is open. So \( f \in T(A,g^{-1}(V)) \) and \( g^+(T(A),g^{-1}(V)) \subseteq T(A,V) \). Hence \( g^+ \) is continuous.

(d) The proof is similar to (c).
**THEOREM 5.2.2** Let \( Y \) be \( \Theta \)-CR and locally \( \Theta \)-H-closed. Then

(a) the map \( F : C(X,Y) \times C(Y,Z) \to C(X,Z) \) is continuous where \( F(f,g) = gof \).

(b) the map \( F : C(X,Y) \times C(Y,Z) \to C_{st}^\Theta(X,Z) \) is strong \( \Theta \)-continuous where \( F(f,g) = gof \).

**PROOF** (a) Let \( T(A,W) \) be a subbasic nbd. of \( gof \). Then \( g(f(A)) \subseteq W \). Now as \( f \) is continuous and \( A \) is almost compact, \( f(A) \) is almost compact and hence an \( H \)-set in \( Y \). As \( Y \) is \( \Theta \)-CR and locally \( \Theta \)-H-closed, \( \exists \) a \( \Theta \)-open set \( V \) such that \( \overline{V} \) is an \( H \)-set and \( f(A) \subseteq V \subseteq \overline{V} \subseteq g^{-1}(W) \) [53]. Now as \( \overline{V} \) is regularly closed, \( \overline{V} \) is almost compact [59]. So \( f \in T(A,V) \) and \( g \in T(\overline{V},W) \) and clearly \( F[T(A,V) \times T(\overline{V},W)] \subseteq T(A,W) \). Hence \( F \) is continuous.

(b) Let \( T(A,W) \) be a subbasic nbd. of \( gof \). Then \( g(f(A)) \subseteq W \). Now as \( f \) is continuous and \( A \) is almost compact, \( f(A) \) is almost compact and hence an \( H \)-set in \( Y \). As \( g \) is strong \( \Theta \)-continuous, \( g^{-1}(W) \) is \( \Theta \)-open in \( Y \). As \( Y \) is \( \Theta \)-CR and locally \( \Theta \)-H-closed, \( \exists \) a \( \Theta \)-open set \( V \) such that \( \overline{V} \) is an \( H \)-set and \( f(A) \subseteq V \subseteq \overline{V} \subseteq g^{-1}(W) \); again \( \overline{V} \) is an \( H \)-set and \( g^{-1}(W) \) is \( \Theta \)-open and hence \( \exists \) a \( \Theta \)-open \( V_1 \) such that \( \overline{V} \subseteq V_1 \subseteq \overline{V}_1 \subseteq f^{-1}(W) \). Now as \( \overline{V} \) is regularly closed, \( \overline{V} \) is almost compact and clearly

\[
F[T(A,V) \times T(\overline{V},V_1)] \subseteq F[T(A,\overline{V}) \times T(\overline{V},\overline{V}_1)] \subseteq T(A,W).
\]

Hence \( F \) is strong \( \Theta \)-continuous.

**COROLLARY 5.2.3** (a) For each fixed \( y_0 \in Y \), the map
$w_Y : C(Y,Z) \to Z$ given by $f \mapsto w(f,y_o) = f(y_o)$ is continuous.

(b) If $Y$ be $\theta$-CR and locally $\theta$-H-closed, then the map $w : C_{st\theta}(Y,Z) \times Y \to Z$ defined by $w(f,y) = f(y)$ is strong $\theta$-continuous.

**PROOF** If we take $X$ to be a space consisting of single point, then (a) and (b) follow from Theorem 5.2.1(a) and Theorem 5.2.2.

**THEOREM 5.2.4** If $Y$ be locally compact Hausdorff space, then

(a) $F : C(X,Y) \times C(Y,Z) \to C(X,Z)$ defined by $F(f,g) = gof$ is continuous.

(b) $F : C(X,Y) \times C(Y,Z) \to C_{st\theta}(X,Z)$ defined by $F(f,g) = gof$ is continuous.

**PROOF** Here $C(X,Y) = C_{st\theta}(X,Y)$ as sets as $Y$ is regular and the space $C(Y,Z)$ is the same as the space of all continuous function on $Y$ to $Z$ with compact open topology as $Y$ is regular.

Also here the proofs of (a) and (b) are similar to Theorem 5.2.2(a) and Theorem 5.2.2(b) respectively.
§ 5.3 ASSOCIATES

Given three spaces $X, Y, Z$, a function $\alpha(x, y) = z$ can be regarded as a map $X \times Y \rightarrow Z$ or as a family of maps $Y \rightarrow Z$ with $X$ as a parameter space. In this section, we consider the effect that shifting from one point of view to the other has on the continuity or strong $\Theta$-continuity of maps; for notation, let $\alpha : X \times Y \rightarrow Z$ be strong $\Theta$-continuous in $y$ for each fixed $x$; the formula $[\hat{\alpha}(x)](y) = \alpha(x, y)$ ... (1), defines for each fixed $x$ an $\hat{\alpha}(x) : Y \rightarrow Z$ which is strong $\Theta$-continuous.

Thus $\hat{\alpha} : X \rightarrow C_{\Theta}(Y, Z)$

$\left\{ \begin{align*}
x & \rightarrow \hat{\alpha}(x) \\
\end{align*} \right\}$

Conversely, given an $\hat{\alpha} : X \rightarrow C_{\Theta}(Y, Z)$, the formula $[\hat{\alpha}(x)](y) = \alpha(x, y)$, defines an $\alpha : X \times Y \rightarrow Z$ strong $\Theta$-continuous in $Y$ for each fixed $x$.

DEFINITION 5.3.1 [25] Two maps $\alpha : X \times Y \rightarrow Z$ and $\hat{\alpha} : X \rightarrow C_{\Theta}(Y, Z)$ related by the formula (1) are called associates.

LEMMA 5.3.2 Let $X, Y$ be spaces and $0 \Theta$-open in $X \times Y$. If $x_0 \times A \subset 0$ where $x_0 \in X$ and $A$ almost compact set in $Y$, then $\exists$ open $W \ni x_0$ such that $\overline{W} \times A \subset 0$. 
PROOF Let $y \in A$. As $(x_0, y) \in 0$ and $0$ is $\theta$-open in $X \times Y$, $\exists \, U_y$ and $V_y$ open in $X$ and $Y$ respectively such that $x_0 \in U_y$ and $y \in V_y$ and $U_y \times V_y \subseteq 0$. Now $\{V_y : y \in A\}$ forms an open cover of $A$ in $X$ and as $A$ is almost compact (hence an $H$-set), $\exists \, y_1, y_2, \ldots, y_n$ in $A$ such that

$$A \subseteq \bigcup_{i=1}^{n} V_i.$$ 

Now $x_0 \in \bigcap_{i=1}^{n} U_i = W$ (say). Then

$$W \times A \subseteq \bigcup_{i=1}^{n} U_i \times \bigcup_{i=1}^{n} V_i \subseteq \bigcap_{i=1}^{n} U_i \times \bigcup_{i=1}^{n} V_i = \bigcup_{i=1}^{n} \left( (\bigcap_{i=1}^{n} U_i) \times V_i \right) \subseteq 0.$$

THEOREM 5.3.3 Let $\alpha$ and $\hat{\alpha}$ be the associates then

(a) If $\alpha : X \times Y \to Z$ be strong $\theta$-continuous, then $\hat{\alpha} : X \to C_{st\theta}(Y, Z)$ is strong $\theta$-continuous.

(b) If $\alpha : X \to C_{st\theta}(Y, Z)$ is continuous and $Y$ be $\theta$-CR and locally $\theta$-H-closed, then $\alpha : X \times Y \to Z$ is continuous.

PROOF (a) Let $x \in X$ and $\alpha(x) \in T(A, V)$ where as usual $A$ is almost compact in $Y$ and $V$ open in $Z$. So $\alpha(x_0 \times A) \subseteq V$ and hence $x_0 \times A \subseteq \alpha^{-1}(V)$. As $\alpha$ is strong $\theta$-continuous, $\alpha^{-1}(V)$ is $\theta$-open. So by Lemma 5.3.2, $\exists$ open $U, x_0$ such that $U \times A \subseteq \alpha^{-1}(V)$ or, $\alpha(U \times A) \subseteq V$. So $\hat{\alpha}(U) \subseteq T(A, V)$. So $\hat{\alpha}$ is strong $\theta$-continuous.
(b) Here the mapping \( \hat{\alpha} \times 1 : X \times Y \to C_{st\Theta}(Y,Z) \times Y \) defined by
\[
\hat{\alpha} \times 1(x,y) = (\hat{\alpha}(x), y)
\]
is continuous as \( \hat{\alpha} \) and 1 (identity function) are continuous. Also as \( Y \) is \( \Theta \)-CR and locally \( \Theta \)-H-closed, the map \( w : C_{st\Theta}(Y,Z) \times Y \to Z \) is continuous by Corollary 5.2.3(b). Now the composed map is \( \alpha : X \times Y \to Z \), since \( (x,y) \to w(\hat{\alpha}(x), y) = \hat{\alpha}(x)(y) = \alpha(x,y) \). So \( \alpha \) is continuous.

§ 5.4 FUNCTION SPACES OF EQUICONTINUOUS FAMILY WHEN ENDOWED WITH ALMOST COMPACT OPEN TOPOLOGY

**DEFINITION 5.4.1** Let \((Z,d)\) be a metric space and \( Y \) an arbitrary space. Let \( \mathcal{K} \subset C(Y,Z) \) be equicontinuous on \( Y \). As \( \mathcal{K} \) is equicontinuous on \( Y \), so for each \( \varepsilon > 0 \), we can find a nbd. \( V(y_0) \) containing \( y_0 \) which is a nbd. of \( \varepsilon/2 \)-equicontinuity, such that \( \forall f \in \mathcal{K}, \ f(V(y_0)) \subset B(f(y_0), \varepsilon/2) \); then \( \forall f \in \mathcal{K}, \ f(V(y_0)) \subset f(V(y_0)) \subset B(f(y_0), \varepsilon/2) \subset B(f(y_0), \varepsilon) \).

We call \( V(y_0) \) a second nbd. of \( \varepsilon \)-equicontinuity. As \((Z,d)\) is a metric space, when \( C(Y,Z) \) is endowed with almost compact open topology, it is regular as \((Z,d)\) is regular [29].

**THEOREM 5.4.2** Let \( \mathcal{K} \) be equicontinuous on \( Y \) and for each \( y \in Y \), let \( w_y : C(Y,Z) \to Z \) be the evaluation map \( f \to f(y) \). Let \( \mathcal{U} \) be a filterbase on \( \mathcal{K} \) such that \( w_y(\mathcal{U}) \) converges to \( \delta(y) \in Z \) for each \( y \). Then
(a) \( \emptyset \) is continuous.

(b) If \( U(y_0) \) be a nbd. of \( \varepsilon \)-equicontinuity for \( \mathcal{K} \) at \( y_0 \in Y \), then \( \emptyset(U(y_0)) \subseteq B(\emptyset(y_0), 3\varepsilon) \).

(c) \( \mathcal{U} \rightarrow \emptyset. \)

**Proof** (a) and (b) follows from [25].

To prove (c), let \( y_0 \in Y \) be fixed. Given \( \varepsilon > 0 \), let \( V(y_0) \) be a second nbd. of \( \varepsilon \)-equicontinuity, so that for each \( f \in \mathcal{K} \),

\[
 f(\overline{V(y_0)}) \subseteq B(f(y_0), \varepsilon). \quad \text{Since} \quad w_{y_0}(\mathcal{U}) \rightarrow \emptyset(y_0), \quad \exists \text{ some } A_{\varepsilon}
\]

with \( w_{y_0}(A_{\varepsilon}) \subseteq B(\emptyset(y_0), \varepsilon) \). So for each \( f \in A_{\varepsilon} \),

\[
 f(y_0) \in B(\emptyset(y_0), \varepsilon). \quad \text{It follows that} \quad \forall y \in \overline{V(y_0)}, \quad \forall f \in A_{\varepsilon}
\]

\[
 d(f(y), \emptyset(y_0)) \leq d(f(y), f(y_0)) + d(f(y_0), \emptyset(y_0)) < 2\varepsilon. \quad \text{So we have the following:}
\]

\[
 \forall y \in \overline{V(y_0)}: \quad w_y A_{\varepsilon} \subseteq B(\emptyset(y_0), 2\varepsilon) \quad \ldots \ldots \quad (1)
\]

Now to prove \( \mathcal{U} \rightarrow \emptyset \), it suffices to show that if \( \emptyset \in T(A, V) \), \( A \) almost compact in \( Y \) and \( V \) open in \( Z \), then \( \exists A_{\varepsilon} \in \mathcal{U} \) with \( A_{\varepsilon} \subseteq T(A, V) \). Now as \( \emptyset \) is continuous and \( Z \) is a metric space, \( \emptyset(A) \) is compact and \( d(\emptyset(A), Y-V) = 3\varepsilon \) for some \( \varepsilon > 0 \). For each \( a \in A \), let \( U(a) \) be a second nbd. of \( \varepsilon \)-equicontinuity for \( \mathcal{K} \). As \( A \) is almost compact, \( \exists U(a_1), \ldots, U(a_n) \), finite in numbers such that \( A \subseteq U(a_1) \cup U(a_2) \ldots \cup U(a_n) \). According to (1) there are \( A_i \in \mathcal{U} \) such that \( w_y A_i \subseteq B(\emptyset(a_i), 2\varepsilon) \) for each \( y \in U(a_1) \). Choosing an \( A_{\varepsilon} \in \mathcal{U} \) such that \( A_{\varepsilon} \subseteq \bigcap_{i=1}^{n} A_i \), we find
that whenever \( f \in \mathcal{A}_\alpha \), then \( d(f(a), \phi(a)) < 3\varepsilon \) for each \( a \in A \) and consequently, \( A_\alpha \subseteq T(A, V) \) as required.

**Theorem 5.4.3** Let \( C(Y, Z) \) be endowed with almost compact open topology and \( \mathcal{K} \subseteq C(Y, Z) \) be equicontinuous on \( Y \). Then its closure \( \overline{\mathcal{K}} \) is also equicontinuous.

**Proof** For each \( \phi \in \mathcal{K} \), \( \exists \) a filterbase \( \mathcal{U} \) on \( \mathcal{K} \) with \( \mathcal{U} \rightarrow \phi \); because \( \omega_Y : C(Y, Z) \rightarrow Z \) is continuous by Corollary 5.2.3(a), we have \( \omega_Y(\mathcal{U}) \rightarrow \phi(y) \) for each \( y \in Y \). Using Theorem 5.4.2 we get the required result.

This leads to:

**Theorem 5.4.4** Let \((Z, d)\) be a metric space and \( Y \) an arbitrary space. Assume \( \mathcal{K} \subseteq C(Y, Z) \) (endowed with almost compact open topology) satisfies:

(i) \( \mathcal{K} \) is equicontinuous on \( Y \).

(ii) \( \overline{\mathcal{W}_Y \mathcal{K}} \) is compact for each \( y \in Y \) (i.e., \( \{f(y) : f \in \mathcal{K}\} \) has compact closure for each \( y \)).

Then \( \overline{\mathcal{K}} \) is compact.

**Proof** Similar proof as in [25].

**Definition 5.4.5** Let \((Z, d)\) be a metric space and \( Y \) be an arbitrary space. A sequence \( \{f_n\} \) in \( C(Y, Z) \) is said to converge
uniformly on every almost compact subset if for each almost compact \( A \subseteq Y \) and each \( \varepsilon > 0 \), there exists a positive integer \( N = N(A, \varepsilon) \) such that \( d(f(a), f_n(a)) < \varepsilon \) for every \( n \geq N \) and every \( a \in A \).

**Theorem 5.4.6** Let \((Z,d)\) be a metric space and \( Y \) an arbitrary space. A sequence \( \{f_n\} \) in \( C(Y,Z) \) converges to \( f \in C(Y,Z) \) uniformly on every almost compact subset if and only if \( f_n \to f \) in the almost compact open topology.

**Proof.** Assume that \( f_n \to f \) according to Definition 5.4.5 and let \( f \in T(A,V) \), \( A \) almost compact in \( Y \) and \( V \) open in \( Z \). As \( A \) is almost compact and \( f \) is continuous, \( f(A) \) is compact, as \( Z \) is a metric space. So we have \( d(f(A), Z - V) = \varepsilon > 0 \). Let now \( N = N(A, \varepsilon) \). Then for \( n \geq N \), we have \( d(f_n(a), f(a)) < \varepsilon \ \forall a \in A \) and therefore \( f_n \in T(A,V) \ \forall n \geq N \). Then \( f_n \to f \) in the almost compact open topology.

Conversely, let \( f_n \to f \) in the almost compact open topology. Given any almost compact set \( A \) and \( \varepsilon > 0 \), each \( a \in A \) has a nbd. \( U(a) \) with \( f(U(a)) \subseteq B(f(a), \varepsilon) \) [as \( Z \) is a metric space, so it is regular, so continuity coincides with strong \( \theta \)-continuity]. Now \( \{U(a) \cap A : a \in A\} \) forms an open cover of \( A \) in \( A \); so \( \exists a_1, a_2, \ldots, a_n \) in \( A \) such that

\[
A = \operatorname{cl}_{T_A} (U(a_1) \cap A) \cup \ldots \cup \operatorname{cl}_{T_A} (U(a_n) \cap A).
\]

Now as \( \operatorname{cl}_{T_A} (U(a_i) \cap A) \) is regularly closed in \( A \) for each \( i \), it is...
almost compact as a subspace of \( A \) and hence as a subspace of \( Y \). Put \( A_i = \text{cl}_{T_A}(U(a_i) \cap A) \), \( i = 1, 2, \ldots, n \), then \( A = \bigcup_{i=1}^{n} A_i \) where \( A_i \), \( i = 1, 2, \ldots, n \), are almost compact and also \( A_i \subseteq U(a_i) \).

Letting \( W \) be the nbd. \( \bigcap_{i=1}^{n} T(A_i, B(f(a_i), \varepsilon)) \) of \( f \), we find that if \( f_n \in W \), then \( d(f_n(a), f(a)) < 2\varepsilon \) for \( a \in A \). Hence the proof.

**THEOREM 5.4.7** Let \((Z, d)\) be a metric space and \( Y \) an arbitrary space. Let \( \{f_n\} \) be a sequence in \( C(Y, Z) \) (endowed with almost compact open topology) converging uniformly on every almost compact subset to some function \( f \). Then \( f/A \) is continuous for each almost compact \( A \subseteq Y \).

**PROOF** Similar proof as in [25].