CHAPTER THREE

RUSSELL'S THEORY OF ENTAILMENT AS FORMAL IMPLICATION

3.1 Foreword

In the previous chapter, Russell's attempt at defining entailment in terms of Material implication has been considered at length. It has been seen that this attempt leads to certain paradoxical consequences. Russell himself was aware of the shortcomings in his theory of material implication. In order to overcome this difficulty, Russell, in his later writings, formulates a new kind of implication which he calls formal implication. He upholds this notion to be the most appropriate one in defining the concept of entailment and thereby in defining the relation that holds between the premises and the conclusion in any valid argument.

Before entering into Russell's definition of entailment in terms of formal implication, let us first see what Russell means by formal implication.

3.2 What Is Formal Implication?

Russell points out that it is because of confusion between propositional functions and 'other things' i.e. things other than propositional functions, that several logical and philosophical problems and mistakes crop up. It will, however, not be out of context to state the meanings of 'proposition' and 'propositional function'. The expression 'proposition' means the primary bearers of truth-value and the meanings of sentences. 'Propositional function' on the other hand, means any expression which contains an undetermined constituent, and becomes a proposition by determining the undetermined constituent either by instantiation or by generalization or by both. E.g. 'x is mortal' is a propositional function, which becomes a proposition when 'x' is substituted by, say, 'John' to form the proposition 'John is mortal' or by way of generalization (x) (x is mortal). As Russell says:

> We may explain (but not define) this notion as follows: ϕx is a propositional function if, for every value of x, ϕx is a proposition, determinate when x is given. Thus "x is a man" is a propositional function. In any proposition, however complicated, which contains no real variables, we may imagine one of the terms, not a verb or adjective, to be replaced by other terms: instead of "Socrates is a man" we may put "Plato is a man", "the number 2 is a man", and so on.*Thus we get successive propositions all agreeing except as to the one variable term.

> Putting *x* for the variable term, "*x* is a man" expresses the type of all such propositions. A propositional function in general will be true for some values of the variable and false for others. The instances where it is true for *all* values of the variable, so far as they are known to me, all express implications, such as "*x* is a man implies *x* is a mortal…".[1]

Russell holds the root of all types of philosophical problems to be the confusion between propositional functions and 'other things'. Existence, according to him, is one such notions, which, though being a property of propositional function, is nonetheless supposed to be an attribute of an individual. Thus 'Socrates exists' is supposed to be a significant assertion according to the latter view. However, 'Socrates exists'- is a non-sensical assertion in Russell's view, because 'existence', as Russell points out, can never be predicated of an individual but only of a propositional function. Thus, the proposition 'Philosophers exist' is significant in the sense that it asserts that the propositional function 'x is a philosopher and x exists' is sometimes true.

Russell points out that we tend to ascribe certain predicates to propositions which can actually be predicated of propositional functions alone. Necessity is another such term, as also is possibility, impossibility etc.

It is propositional functions which can be necessary, possible and impossible; propositions, on the other hand, can be true or false but never necessary, possible or impossible. Now let us see what is meant by a propositional function's being necessary, possible and impossible. A propositional function is said to be necessary when all its substitution instances are true; again a propositional function is possible when some of its substitution instances are true, while it is impossible when its instances are *never* true i.e. none of the propositions which are substitution instances of that propositional function are true.

In the light of Russell's above interpretation of necessity, possibility, and impossibility, material implication cannot be predicated of necessity i.e. it cannot be said to be necessary (as material implication is a proposition). This can be exemplified with the help of the following example:

All men are mortal

All Philosophers are men

∴ All Philosophers are mortal

In the above argument although the premises entail the conclusion, however, it cannot be said that the material implication [('All men are mortal' and 'all Philosophers are men') ⊃ 'All Philosophers are mortal'] is necessary. Even though "There is some necessity involved here, but it is not true necessity of the above material implication…"[2] but rather of the implicative formula i.e. of the propositional function $[(x)(Hx\rightarrow Mx)\cdot(x)(Px\rightarrow Hx)]\rightarrow(x)\,(Px\rightarrow Mx)$, (where Hx = x is a man, Mx = x is mortal' and Px = x is a Philosopher).

In other words, the necessity involved here is not that of material implication but of the propositional function, and a propositional function is necessary means that all its substitution instances are true, i.e. each and every material implication which is a substitution instance of the above implicative propositional function is true.

By formal implication is meant an implicative propositional function, all of whose substitution instances are true. Now, if all the substitution instances of an implicative formula or propositional function are true then the universal quantification of that propositional function is true as well. And, according to Russell, formal implication is nothing but this i.e. the universal quantification of the implicative propositional function. To explain:

Socrates was a Philosopher

∴ Either Socrates was a Philosopher or Socrates was a king

The above inference is valid, meaning thereby that 'Socrates was a Philosopher' entails 'Either Socrates was a Philosopher or Socrates was a king', which in turn means that the material implication 'Socrates was a Philosopher ⊃ (Either Socrates was a Philosopher or Socrates was a king)' is necessary. This again is equivalent to saying that the propositional function 'p ⊃ (Either p or q)' (all of whose substitution instances are true) is necessary. This yet again means that the universal quantification of this propositional function i.e. the formal implication

'(p) (q) [p⊃ (Either p or q)]'

is true.

To write it in symbols: (p) (q) {p⊃ (p v q)} is a tautology.

Thus a new definition of entailment can be given as follows:

'p entails q' if and only if 'p materially implies q' is a specification of some formal implication that is true.

According to Russell, as a rule material implication may be considered to be a particular instance of some formal implication,[3] which can be obtained by giving some constant value to the variable or variables in the said formal implication.[4]

Russell goes on to remark:

> "For the technical study of Symbolic Logic, it is convenient to take as a single indefinable the notion of a formal implication, *i.e.* of such propositions as "*x* is a man implies *x* is a mortal, for all values of *x*"—propositions whose general type is: "ϕ (*x*) implies ψ (*x*) for all values of *x*", where ϕ (*x*), ψ (*x*), for all values of *x*, are propositions".[5]

It can be noted here that "x is a man implies x is a mortal" is not a relation holding between two propositional functions, it is a single propositional function that is always true. According to *Principia*, a formal implication stands basically for a true conditional universal statement. [6] Russell goes on to distinguish implication from formal implication by saying that implication 'holds between any two propositions provided the first be false or the second true', while formal implication 'is not a relation, but the assertion, for every value of the variable or variables, of a propositional function which, for every value of the variable or variables, asserts an implication'.[7]

Russell sums up his discussion of formal implication by saying that both material implication and formal implication are "essential to every kind of deduction"[8]

3.3 Evaluation of Russell's Theory of Entailment as Formal Implication

Even though Russell formulated the theory of formal implication in order to overcome the difficulties encountered in the theory of material implication, despite that his theory of formal implication has been attacked from diverse angles. In the book *Logic Induction and Ontology*, Pranab Kumar Sen[9] points out that even this attempt of defining entailment fails due to the fact that like other theories of entailment, this also involves some paradoxical consequences which may be stated as follows:

(i) A proposition that is a substitution instance of an impossible formula entails any proposition. (By an impossible formula is meant that of which every substitution instance is false).

(ii) A proposition, which is a substitution instance of a necessary formula, is entailed by any proposition. (By a necessary formula is meant that of which every substitution instance is true).

The question arises: how do these paradoxes arise?

Suppose P and Q are any two formulae which are chosen arbitrarily and P formally implies Q. Now P can be said to formally imply Q when (P⊃Q) is necessary. Again, to say that (P⊃Q) is necessary is to say that its equivalent (~PvQ) is necessary. Again, the disjunctive formula (~PvQ) is necessary means that either '~P' is necessary or 'Q' is necessary. '~P' turns out to be necessary when 'P' is impossible (the concepts of necessity, possibility, impossibility being interdefinable). Thus (~PvQ) as also (P⊃Q) turns out to be necessary when either P is impossible or Q is necessary. Or, to put it simply: P formally implies Q either if P is impossible or if Q is necessary.

Nonetheless, according to Russell, neither of these consequences is paradoxical in nature when taken as a truth about formal implication. To explain: Any formula P is impossible, means that all the propositions, which are substitution instances of the formula P, are false. Now if all the substitution instances of P are false then the implicative formula (p→q) will be valid irrespective of whatever Q, or, for that matter, any other formula, may be (i.e. be it necessary or possible or impossible). If the implicative formula is valid then its universal quantification is true also. Thus, P being an impossible formula formally implies any formula Q whatsoever. Thus P (an impossible formula) formally implies Q (any formula) holds.

Analogously, if any formula Q is necessary then all the substitution instances of this formula has to be true also. If this be the case, then whatever P (another formula) might be (whether necessary, possible or impossible), the implicative formula (p→q) must be valid. The implicative formula being valid, its universal quantification is true also. So whether P be impossible or Q be necessary, –in both the cases '(p)(q)(P→Q) is a tautology'.

It may be pointed out here that, even though they are true of formal implication yet they are absolutely false of entailment. Considering the above consequences as truths of entailment leads to the acceptance of the following types of argument:

1. All objects are both coloured and not coloured.

   ∴ 7+5=12

2. Plato is a great philosopher.

   ∴ Either it is snowing or not snowing.

In each of these examples formal implication holds, for the premise of the first argument is a substitution instance of an impossible propositional function, while the conclusion of the latter is a substitution instance of a necessary propositional function. So if entailment is identified with formal implication then it has to be said that in case of each of the above inferences, the premises entail the conclusion, which is clearly not the case as these types of arguments are generally not considered as valid.

Moreover, it may be objected that this theory cannot successfully analyse the indispensable notion of necessity in the sense in which the term is generally used. According to Russell, necessity and other modal concepts are predicable of a propositional function alone, they are not predicable of a proposition. A proposition can be either true or false but never necessary or impossible. If this be the case, then it follows that whatever is necessary cannot be true and vice-versa. However, that necessity entails truth is commonly accepted and it is also accepted in classical modal systems, 'for in each of them that 'p is necessary' implies 'p is true' is recognized as a truth'[10] Again, C.I. Lewis maintains that Russell's exposition of 'entailment' in terms of 'formal implication' doesn't stand the test of reason. As he puts it:

> In fact, a formal implication holds whenever a class inclusion, a$\subset$b, holds – e.g., the class of featherless bipeds is contained in the class of animals that laugh. But the essence of 'featherless biped' does not include or imply the attribute of laughter; and "x is a featherless biped" does not imply "x laughs" in the strict sense of implication.[11]

He doesn't think formal implication to be "'formal' in the strict logical sense". According to him, "all logically formal relations are intensional; while formal implication shares the extensional character of material implication".[12]

Finally, Ronald Jager in his book *The Development of Bertrand Russell's Philosophy* [13] charges Russell's notion of formal implication of being unclear. He points out that Russell sometimes takes it in the sense of (i) a general rule, sometimes as (ii) a universal quantification and yet again in the sense of (iii) a connection among meanings. Jager goes on to deride Russell by saying that formal implication in the sense of (i) is just a general rule i.e. that of modus ponens; in the sense of (ii) i.e. as a universal quantification, it belongs to the next level of logic i.e. to the predicate calculus, which he presumes has nothing to do with the notion of implication; while the third sense, he believes, has no place in Russell's logic as it belongs to modal logic. Jager therefore, insists that the notion of formal implication as a special kind of implication is to be rejected.

Thus it may be concluded from the above discussion that the attempt to define entailment in terms of formal implication is not satisfactory.

NOTES AND REFERENCES

1. Bertrand Russell, *The Principles of Mathematics*, George Allen & Unwin, London, tenth impression, second edition, 1979. (first published in 1903, Routledge.), p. 19- 20.
2. Pranab Kumar Sen, ‘Paradoxes of Formal Implication’, in *Journal of Indian Academy of Philosophy*, vol-iv, nos. 1 and 2, 1965, p. 73.
3. Bertrand Russell, *The Principles of Mathematics, op. cit.*, p. 34.
4. “formal implication asserts a class of [material] implications”. Bertrand Russell, *The Principles of Mathematics*, vol. 1, *op. cit.*, p. 38.
5. *Ibid.*, p. 11.
6. For brevity we say “Φ x always implies ψx” when we mean that “Φ x implies ψx” is always true. Propositions of the form “Φ x always implies ψx” are called formal implication”... Bertrand Russell, *Introduction to Mathematical Philosophy*, Allen and Unwin, London, 1919, p.163.
7. Bertrand Russell, *The Principles of Mathematics*, vol. 1, *op. cit.*, p. 1066.
8. *Ibid.*, p. 33.
9. Pranab Kumar Sen, “The Problem of Entailment”, in *Logic Induction and Ontology: Essays in Philosophical Analysis*, Macmillan India Press, Madras, Calcutta, 1980, p. 29.
10. Sen, Pranab Kumar, ‘Paradoxes of Formal Implication’, *op. cit.*, p. 74.

11. Clarence Irving Lewis, and Cooper Harold Langford, *Symbolic Logic*, Dover Publication, INC., New York, 1959. p. 100.
12. *Ibidem*.
13. Ronald Jager, *The Development of Bertrand Russell's Philosophy*, London, George Allen and Unwin, 1972, p. 520.