CHAPTER TWO

RUSSELL'S THEORY OF ENTAILMENT AS MATERIAL IMPLICATION

2.1 The Nature of Material Implication

There is great amount of dispute among logicians as regards the nature of entailment. Various logicians define the notion in various ways. Bertrand Russell too, among many others, has focused on entailment. In *Principia Mathematica* Russell proceeds to define entailment in terms of material implication. For an understanding of Russell's notion of entailment let us first see what is meant by implication.

An implication is a binary, truth-functional (where the truth-value of the whole proposition depends on the truth-value of it's parts), propositional connective which is formed by connecting two propositions with ‘if’, and ‘then’. It is formed by placing the word ‘if’ before the first proposition and introducing a ‘then’ between the two propositions.

An implication is also known as a conditional, hypothetical or even an implicative statement. The statement following “if” i.e. the antecedent, implicans or the protasis, is the part in which some condition is mentioned and on the basis of which something is asserted, while the statement following “then”, known as the consequent, the implicate or the apodosis, is the part that follows from and is asserted on the basis of the antecedent.

This can be explained with the help of the following example: “If you study hard, then you can get good results”. In the implication just mentioned, ‘you study hard’ is the antecedent, while ‘you can get good results’ is the consequent. Here, studying hard is the condition for getting good results. In an implication, what is asserted is that in case its antecedent is true, its consequent is true also. Neither does it assert that its antecedent is true, nor does it assert that its consequent is true; what it simply asserts is just that if its antecedent is true, its consequent is true also, or that it’s consequent is true in case its antecedent is true. Different phrases such as *if, only if, given that, provided that, supposing that, implies, even if*, and *in case* etc. are used to identify the conditional in ordinary language.

The meaning of the implicative statement can be made clear by understanding the relationship that holds between its antecedent and consequent. Ordinarily there can be more than one kind of implication. Let us consider the following types of implicative statements. Each of these statements projects a different sense of "if-then", and states a different kind of implication.

1. **Logical** Implication: where the consequent follows from its antecedent logically, e.*g.*, "If all logicians are mathematicians and Joe is a logician, then Joe is a mathematician."

2. **Definitional** Implication: The consequent follows from its antecedent by the very definition of the term in the antecedent. *E. g.* (i), "If Kate is a spinster, then Kate is unmarried"*; E.*g. (ii), "If the contents of this glass is water, then the contents of this glass is H_2O".

3. **Causal** Implication: where the relation between the antecedent and the consequent is a causal one. *E. g.*, "If it rains then the ground will get wet." [In this example raining is the cause of the ground being wet.]

4. **Decisional** Implication: where the consequent is the decision of an individual to act in a particular manner in a particular situation. *E. g.*, "If India wins the semifinals of the world cup, then I will watch the finals."

It is clear from the above instances that these conditionals differ from each other in so far as they express a different type of implication relation between the antecedent and consequent; they, however, have a common partial meaning. This common meaning can be seen by considering the instances when the implicative statement turns out to be false. For example, the implicative statement: "If it rains then the ground will get wet" turns out to be false in case it actually rains but the ground does not get wet; or yet again the conditional "If all logicians are mathematicians and Joe is a logician, then Joe is a mathematician" turns out to be false, in case all logicians are mathematicians and Joe is a logician but Joe is not a mathematician. Thus it is clear that in case of a conditional statement, be it truth-functional or non truth- functional, the conditional turns out to be false in case its antecedent is true but the consequent is false. And this is the common partial meaning of all the cases of implication. Thus, for a conditional to be true what is needed is that the antecedent being true the consequent does not turn out to be false. And this is exactly the meaning of material implication.

Thus material implication is the fifth kind of implication. For example "If Neil Armstrong was the first man to step on the moon, then 2+2=5" – this proposition does not demonstrate any logical, definitional or causal implication, nor does it signify any decisional implication. What is expressed here is just that it is not the case that the antecedent is true and the consequent is false.

To put it in symbols:

$\sim (P \cdot \sim Q)$

Where P= 'Neil Armstrong was the first man to step on the moon' and Q= '2+2=5'

The entire conditional statement $P \supset Q$ will be false in case the antecedent P is true and the consequent Q is false.

Thus, to assert "If p then q" means to deny the conjunction of p with the negation of q; in other words, asserting "If p then q" is denying (p· ~q), and thereby asserting ~(p·~q). Thus, ~(p·~q) can be considered to be a part of the meaning of "If p then q".

The symbol "⊃", (called a *horse-shoe*) is used to stand for this common partial meaning of the "if-then" phrase. Thus, the "if-then" phrase is defined by considering p⊃q to be an abbreviation of ~(p·~q), which is again equivalent to ~p v q (by the application of De Morgan rule). That is, it is another way of saying 'either p is false or q is true'. As a result, the truth table for any sentence of the form p ⊃ q should be the same as that of an analogous sentence of the form ~(p· ~q) , which in turn should be the same as the truth table of an analogous sentence having the form ~p v q. Thus in logic the following equivalences are accepted.

p ⊃ q ≡ ~p v q ≡ ~ (p· ~q)

This can be made clear with the help of the following truth table:

p	q	~p	~q	p ⊃ q	~p v q	(p . ~q)	~ (p . ~q)
T	T	F	F	T	T	F	T
T	F	F	T	F	F	T	F
F	T	T	F	T	T	F	T
F	F	T	T	T	T	F	T

It is clear from the above truth table that p ⊃ q, ~p v q and ~(p· ~q) have the same truth value, they are false when p is true and q is false, as is evident from the second row. The other three rows, namely, the first, third and fourth rows indicate where they come out true; i.e. they turn out to be true if either p is false or q is true.

This common partial meaning of the "if – then" phrase called by Russell as material implication is the one in terms of which Russell chooses to define entailment. The following section will deal with the same.

2.2 Russell's Definition of Entailment in Terms of Material Implication

Russell coined the expression "material implication" to refer to the relation of entailment. Russell in *The Principles of Mathematics*, *Introduction to Mathematical Philosophy* and *Principia Mathematica.*, defines "p entails q" as "p materially implies q" which he symbolizes as p⊃q, where '⊃' stands for horse-shoe. According to Russell, if the relation between the two propositions 'p' and 'q' is such that 'q' follows from 'p', i.e. 'p' being true, 'q' must also be true, then it can be said that 'p' implies 'q'. To put it in Russell's words: "When a proposition q follows from a proposition p, so that if p is true, q must also be true, we say that p implies q…"[1, 2]

Deducibility is dependent on the relation of implication, and according to Russell, 'The relation in virtue of which it is possible for us validly to infer is what I call material implication.[3] It is with material implication that Russell equates entailment.

He proceeds to define this truth-functional notion of material implication in terms of truth-functional operators such as negation, disjunction, and conjunction, which are represented by tilde-'~', vel-'v' and dot -'.', respectively. According to him, 'p materially implies q' can be defined as 'either p is false or q true', (where 'either … or …' is taken in the inclusive sense). This may be symbolically represented as: $p \supset q$ = df ~ pvq[4]

According to Russell, this definition of material implication is equivalent to saying that 'it is not the case that p is true and q is false'. In symbols: $p \supset q$ = df ~ (p· ~q). Russell maintains:

> …if p implies q, then it cannot be the case that p is true and q is false, i.e., it must be the case that either p is false or q is true. The most convenient interpretation of implication is to say, conversely, that if either p is false or q is true, then "p implies q" is to be true. Hence "p implies q" is to be defined to mean: "Either p is false or q is true". Hence we put:
>
> * 1.01 $p \supset q$.=. ~pvq Df[5]

Let us exemplify it with the help of the following example: "If the Sun is cold" materially implies "Antarctica is sunny'" means it is not the case that "'the Sun is cold' is true and 'Antarctica is sunny' is false", or in the equivalent form, Either "'it is not the case that 'the Sun is cold' or 'Antarctica is sunny'". Here it is clear that there is no 'real' connection between the antecedent (p) 'the Sun is cold' and the consequent (q) 'Antarctica is sunny' apart from the fact that it is not the case that p is true and q is false.

So in case of material implication no 'real connection' is indicated between the antecedent and the consequent. What it indicates is just that when its consequent is false, then it is not the case that its antecedent is true.

Russell defines his notion of 'material implication' as per the requirement of extensional logic, where 'necessity' will have no role to play. Being an empiricist, Russell was interested in doing extensional logic, where he would be able to evade notions such as 'necessity', 'a-priori', etc., which are irksome to the empiricists. So, Russell has defined his notion of 'material implication' in keeping with his empiricism. As a result, if we go through the *Principia Mathematica*, we come across some peculiar theorems which are a part of all those systems that are based on material implication. Let us see what these peculiar theorems are.

1. $\sim p \supset (p \supset q)$

The above theorem could be read as the following:

p is false implies that p implies q i.e., a false proposition implies any proposition.

2. $q \supset (p \supset q)$

A possible reading of this could be the following:

‘p implies q’ is implied by q is true, i.e., a true proposition (q) is implied by any proposition (p). Russell names this proposition as the “Principle of Simplification”, because “…it enables us to pass from the joint assertion of q and p to the assertion of q simply”.[6]

3. $\sim (p \supset q) \supset p$

The above theorem could be read as the following.

If it is not the case that p implies q, then p is true.

4. $\sim (p \supset q) \supset \sim q$

A possible reading of the above mentioned theorem could be the following:

If it is not the case that p implies q, then q is false.

5. $\sim (p \supset q) \supset (p \supset \sim q)$

If it is not the case that p implies q, then p implies that ‘q is false’ – this could be a possible reading of the above theorem.

6. $\sim (p \supset q) \supset (\sim p \supset q)$

A possible reading of this theorem could be the following:

If it is not the case that p implies q, then ‘p is false’ implies q.

7. $(p.q) \supset [(p \supset q) \cdot (q \supset p)]$

The above theorem could be read as:

If p is true and q is true, then p implies q, and q implies p.

8. $(\sim p \cdot \sim q) \supset [(p \supset q) \cdot (q \supset p)]$

Again, possible reading of the above theorem is as follows:

If p is false and q is false, then p implies q and q implies p.

9. $(p\supset q) \vee (q\supset p)$

This theorem could be read as follows:

Of any two propositions either the first implies the second or the second implies the first.

These theorems are characteristic of the relation of material implication. It is clear that material implication is a relation that holds not between any content or logical import of propositions but between the truth-values of propositions. These theorems, along with many others, follow necessarily from the very definition of material implication.

In fact, Russell contends about "Principle of Simplification", (i.e., a true elementary proposition is implied by any proposition, 2.02. $\vdash$: q. $\supset$. p$\supset$q),[7] as also about others, that: "…When the special meaning which we have given to implication is remembered, it will be seen that this proposition is obvious".[8] Hence propositions like $\sim p\supset(p\supset q)$, i.e. a false proposition materially implies any proposition (for if *p* is false it is not the case that *p* is true and *q* false), and $q \supset (p\supset q)$, i.e. any proposition materially implies a true proposition, are called the 'paradoxes' of material implication. The above theorems along with some others are considered by many to be paradoxical, in the sense of sounding odd and being counter-intuitive. Any calculus of propositions based on material implication is packed with such ‘peculiar’ theorems.

Even though Russell holds material implication to be the one relation by virtue of which deduction is possible, however, it will be evident that such a notion of implication does not go without demur. Several theorems that follow from Russell’s notion of material implication, are contested to be counter intuitive and paradoxical. There is also the doubt whether at all material implication can be identified with entailment. The next section will deal with such objections and the paradoxes of material implication.

2.3.1 Examination of Russell’s Definition of Entailment as Material Implication

As stated in the previous section, *Principia* is vitiated with a number of peculiar theorems which are paradoxical in nature, in the sense of being counter-intuitive. However, before discussing it let us first see what is meant by paradox.

The word ‘paradox’ is derived from the Latin word paradoxum, and also from the Greek word ‘paradoxon’. The etymological meaning of ‘paradox’ is ‘contrary to opinion’, where ‘para’ means beyond and ‘doxa’ [from ‘dokein’ (to think)] means opinion. Thus, a paradox may stand for any statement that appears to be contradictory or absurd but may nonetheless be true. According to Oxford Dictionary ‘paradox’ is ‘a statement that seems self-contradictory but is in fact true’. It may also mean any statement that contradicts commonly accepted view.

According to Johnson, when a thinker proceeds in accordance with the principles of the system of logic and arrives at a formula which is in opposition to the common-sense, then he is said to have obtained a paradox. As Johnson puts it:

> …When a thinker accepts step by step the principles or formulae propounded by the logician until a formula is reached which conflicts with his common-sense, then it is that he is confronted with a paradox...It is the formulae that are *derived* –by apparently unexceptionable means from apparently unexceptionable first principles – that appear to be exceptionable… [9]

Thus, it can be stated that a paradox is a perplexing conclusion which is obtained from a set of apparently indubitable premises, but which in itself is highly counterintuitive and absurd nonetheless.

Principia Mathematica has a number of paradoxes which have perplexed even the logicians; as such it has been attacked from various corners. Firstly, it has been objected by logicians that the meaning of ‘implies’ or ‘entails’ as given in *Principia* does not correspond to the ordinary sense of implication. They further refute the fact that ‘p⊃q’ (i.e., “It is not the case that p and not – q’) is a correct analysis of ‘p’ entails ‘q’, in so far as it leads to certain unacceptable paradoxical consequences.

Russell, in *Principia Mathematica,* we have seen, has defined material implication as, 'it is not the case that p is true and q is false' or, plainly as, 'either p is false or q is true'. It straight away follows from this that any false proposition (no matter how irrelevant it is to its consequent) implies any proposition, i.e. in symbols, '$\sim p \supset (p \supset q)$' and any true proposition (no matter how irrelevant it is to its antecedent) is implied by any proposition, which again can be symbolized as, '$q \supset (p \supset q)$'.

Thus, we obtain two paradoxes which are generally known as the paradoxes of material implication. To restate:

1) A false proposition materially implies any proposition.
2) A true proposition is materially implied by any proposition.

The above paradoxes can be shown to be true with the help of the following proofs:

I Proof to show that any proposition (say q) follows from a false proposition (say ~p)

Proof:

1. $\sim p$ (p is false)
2. $\sim p v q$ 1, Add
3. $p \supset q$ 2, Def. '$\supset$'

II Proof to show that any proposition (say ~q) follows from a false proposition (say ~p)

Proof:

1. ~p (p is false)
2. ~pv~q 1, Add
3. p⊃~q 2, Def. '⊃'

III Proof to show that a true proposition (say q) follows from any proposition (say p)

Proof:

1. q (q is true)
2. qv~p 1, Add
3. ~pvq 2, Com
4. p⊃q 2, Def. '⊃'

IV Proof to show that a true proposition (say q) follows from any proposition (say ~p)

Proof:

1. q (q is true)
2. qvp 1, Add
3. pvq 2, Com
4. ~ ~pvq 3, DN.
5. ~p⊃q 2, Def. '⊃'

Proofs I and II show that if p is false then it implies any proposition q or ~q, i.e. if the antecedent is false then the material implication turns out to be true, no matter what the consequent be. Proofs III and IV show that if q is true then it is implied by any proposition p or ~p, i.e. if the consequent is true then the material implication turns out to be true irrespective of (the truth value of) the antecedent.

Applying these theorems to non-symbolic propositions leads to startling results thereby exposing their paradoxical nature. Thus "The earth is flat" being false, implies anything, from "the earth is round" to "Socrates was a Philosopher" to "Fishes swim in water" or, even to fishes fly in the sky. Similarly "The sun rises in the east" being true is implied by anything, from "Socrates drank Hemlock" to "Socrates was a goldsmith". Let us analyse with the help of the following three propositions.

(1) If the earth is flat then Socrates was a Philosopher.

(2) The earth is flat ⊃ Socrates was a Philosopher.

(3) The earth is flat entails Socrates was a Philosopher.

Clearly (1) is false, because the earth's shape doesn't have anything to do with Socrates' being a Philosopher or for that matter not being one. Let us now represent (1) in terms of material implication thus and see the effect.

(2) The earth is flat ⊃ Socrates was a Philosopher.

Now, the antecedent being false the material implication, i.e. (2) turns out to be true, whereas (1) is false. Thus, it is obvious that (2) is not a satisfactory representation of (1). Hence material implication is not equivalent to "if…then…". Let us now proceed to check the proposition (3) i.e. 'the earth is flat entails Socrates was a Philosopher'. Since entailment is the converse of deducibility, so for the above entailment statement to be true, what is required is that 'Socrates was a Philosopher' be deducible from 'the earth is flat'. However, it is evident that no amount of effort can lead us to deduce 'Socrates was a Philosopher' from 'the earth is flat', thereby making it clear that the former cannot be entailed by the latter. Thus, even if the material implication 'The earth is flat ⊃ Socrates was a Philosopher' turns out to be true, its corresponding entailment statement, 'The earth is flat entails Socrates was a Philosopher', turns out to be false. This shows that the notion of material implication cannot be equated with that of entailment.

In addition to these paradoxes, Principia is home to a lot of other paradoxes which are not stated often; a closer look at some of the theorems of *Principia Mathematica* will reveal their paradoxical nature as well. Let us point out some of the paradoxical theorems.

Theorems, such as

(i) $(p \cdot q) \supset [(p \supset q) \cdot (q \supset p)]$

and (ii) $(\sim p \cdot \sim q) \supset [(p \supset q) \cdot (q \supset p)]$

appear to be counter-intuitive as well. How can it be said at the same time that $(p\cdot q)\supset[(p\supset q)\cdot(q\supset p)]$ meaning that if both p and q are true, then p implies q, and q implies p, and yet again that $(\sim p\cdot\sim q)\supset[(p\supset q)\cdot(q\supset p)]$ meaning that if both p and q are false, then p implies q and q implies p. This amounts to saying that of any two propositions taken at random, one will imply the other if both are true or even if both are false. This concept of implication itself appears to be weird. To elucidate, let us take the following (pair of) propositions 'Rabindranath Tagore was a Poet', 'Birds have wings'. Both of these propositions being true, it follows according to the above theorem (i) that each implies the other. Again, if we consider the following propositions 'The sun is cold', and 'Mercury revolves around the Earth' we see that both being false, it at once follows by virtue of theorem (ii) that the first implies the second and vice-versa. This seems to be quite strange.

In addition to the above theorems, even the theorem $(p\supset q) \vee (q\supset p)$ appears to be paradoxical. The above proposition could be read as: "of any two propositions either the first implies the second or the second implies the first." Theorem such as this is counter-intuitive. Say, for example p = 'fishes have fins'; q = 'Delhi is the capital of India'; neither of these propositions has anything to do with the other; p, as also q, without a doubt, is neither bound to imply nor bound to be implied by the other.

Moreover, the above theorem would violate one of the conditions of any axiomatic system, namely, independence. There are three conditions, on which an axiomatic system is based, namely completeness, soundness and independence.[10] Independence is the condition which demands that the postulates in the set of postulates of the system must not be deducible from one another. In other words, when no one of them is entailed by the other, only then can the two postulates be said to be independent of each other. Thus according to the above theorem of P.M. i.e. $(p \supset q) \vee (q \supset p)$, out of any two postulates T_1 and T_2, selected randomly either T_1 materially implies T_2, or T_2 materially implies T_1. As a result, it is clear that identifying the notion of "entailment" with that of "material implication", will lead to the violation of the condition of independence.

Furthermore, it will not be out of context to point out that the above theorem of the *Principia Mathematica* highlights another aspect in which the concept of material implication differs from that of 'entailment'. In case of material implication it can very well be said that out of any two arbitrary propositions one materially implies the other; however this does not hold in case of entailment. It cannot be said in case of entailment that out of any two arbitrary propositions one entails the other, or that if the first proposition doesn't entail the second then the second entails the first; e.g. "roses are beautiful" cannot be deduced from "Plato was a philosopher", but this doesn't mean that the latter can be deduced from the former. Thus, even though it is true of material implication that out of any two propositions taken at random, either the first implies the second or the second implies the first, but a corresponding statement about entailment does not hold.

Again, a further explanation will once again show that the concept of "p materially implies q" is not equivalent to that of "q is deducible from p", as upheld by Russell. To restate Russell's position: according to Russell, p materially implies q if either p is false or q is true i.e. $(p\supset q)=(\sim p \vee q)$ Df. Again, at the same time, he says that p materially implies q if q is the consequence of or follows from p. So, according to Russell, deduction is dependent on material implication. However, a little reflection will show the difference between the notion of deducibility and that of material implication:

Suppose p='The earth has twelve moons', q='Plato was a Greek Philosopher'; Then, the earth has twelve moons' ⊃ 'Plato was a Greek philosopher', i.e. $p\supset q$= Either it is not the case that the earth has twelve moons' or 'Plato was a Greek philosopher', i.e. $\sim p \vee q$. Then, following Russell's definition of material implication, p being false, the material implication, i.e. $p\supset q$, which is equivalent to $\sim p \vee q$, turns out to be true. Nonetheless, the proposition q-"Plato was a Greek philosopher" is in no way deducible from p – "The earth has twelve moons'. Thus, it is evident that the proposition q, "Plato was a Greek philosopher", does not follow from the proposition p, "The earth has twelve moons", nor has it anything to do with the latter. However, as already shown above, $p\supset q$ i.e. 'the earth has twelve moons ⊃ Plato was a Greek philosopher' holds true according to the definition of entailment in terms of material implication, as given by Russell. From the above example it is quite clear that where 'p materially implies q' does hold, 'q follows from p' does not hold. So it is difficult to accept the definition of material implication as ‘the converse of the relation of follows from’. Thus the notion of ‘material implication’ cannot be equated with that of the ‘converse of deducibility’ and as such with that of ‘entailment’. In other words, entailment and material implication are different relations. The relation between “This is red” and “This is coloured” is completely different from the one that holds between ‘the earth has twelve moons’ and ‘Plato was a Greek philosopher’. In such a case, then it is inappropriate that the same word “implies” is employed for both the relations.

However, Russell guards his use of “implies” in a sense different from the usual one by saying that “Provided our use of words is consistent, it matters little how we define them”.[11]

He goes on say:

> I use the word ‘implication’ in a special technical sense which does not carry with it the consequences you indicate. I say that one proposition ‘implies’ another whenever the first is false or the second true (not excluding both). I do not pretend that this is the usual meaning of the word, but it is a relation for which I need a name, and no other name occurred to me.[12]

Nevertheless, the use of a word in a sense diverse from its original familiar sense is liable to lead to confusion and to untoward consequences. In fact, Russell has not been successful in keeping the two meanings of “implies” separate. It seems that even though Russell himself stressed upon calling the horseshoe as “implies,” but it is doubtful as to whether he had simply made a verbal choice at random, or whether he himself was unaware of the difference between implication, as a relation that holds between propositions, and truth-functional conditional.

According to Moore, the truth-functional concept of material implication is unable to capture fully the notion of entailment. He points out that Russell has not been able to keep the two concepts distinct. Moore comments that in identifying "q can be deduced from p" with "p materially implies q" Russell is led into an "enormous 'howler'". Moore accordingly writes: "…the statement that "q can be deduced from p" means the same as "p) q" is simply an enormous "howler";…".[13]

It may be pointed out that 'p entails q' expresses a relation which is a *necessary* one. This means that it cannot be the case that 'p' is true and 'q' is false, i.e. the conjunction of p and not q is impossible or self contradictory; on the other hand "p materially implies q" i.e. '$p \supset q$' expresses a relation which is only contingent. What is stated in the latter is that it is not the case that p is true and q is false under the present circumstances which contingently holds. Consequently, it seems that to say that "q can be deduced from p" is to say that "p entails q", and not "p materially implies q". Thus the notion of entailment cannot be defined in terms of the notion of material implication.

Again, according to the paradoxes of material implication a true proposition is materially implied by any proposition whatever, and a false proposition materially implies any proposition. If material implication is considered to be the converse of deducibility then, it has to be said that a true proposition is deduced or inferred from any proposition whatever, and any proposition is deduced or inferred from a false proposition. However, it may be pointed out that, no inference is made from any proposition that is known to be false in order to obtain a true one, nor is there any requirement of inferring any proposition which is already known to be true. But this seems to be the inevitable consequence of material implication. What seems to be problematic with material implication is that its implicans may be totally irrelevant to its implicate and again the implicate may not be derivable from the implicans, i.e. material implication can hold between propositions having completely unrelated subject-matter.

Some logicians like Charles E. Gauss in his 'The Interpretation of Implication', holds that material implication is flawed in the sense that it does not consider the condition of necessity and more so that its truth is considered to be dependent on the truth of its parts, which according to him should not be the case.

Charles E. Gauss enumerates the errors of material implication thus:

> Material implication is erroneous on two counts. It fails to see that there is a necessary connection between P and Q, something inherent in the content of these respective propositions which permits us to infer the second from the first. There is implication between two propositions only when we can see a valid logical

> dependence of the one upon the other. Material implication, in the second place, fails to understand that the truth of the whole "if ... then" proposition cannot be determined by an inspection of the elements of which it is composed alone. The whole is a unit which is not only the summation of its parts. The parts are only associated by virtue of the inferential connection which they bear to each other and that connection is independent of the parts in themselves. Implication is synonymous with entailment, and means that the deduction of Q is possible from P. This is the meaning of strict implication."[14]

2.3.2 Lewis' Criticisms of Russell's Theory of Material Implication

C. I. Lewis strongly opposes Russell's theory of material implication. He maintains that any system of logic that accepts the principle that if either p is false or q is true then q is deducible from p, is not useful at all. In fact, Lewis' approach becomes more stringent as he goes on to call all the theorems of *Principia Mathematica* unproved, because, according to him, their deductions are based on the notion of material implication.

Lewis further points out that if the notion of 'material implication' is equated to that of 'entailment' then any random pair of propositions cannot be said to be both consistent and independent at the same time. To elucidate: by 'p is consistent with q'

Lewis means ‘p does not imply the falsity of q’, i.e. ~(p⊃~q), and by ‘q is independent of p’ he means ‘p does not imply q’, i.e. ~(p⊃q). Lewis in his S*ymbolic Logic*, shows that ~(p⊃~q) entails (p⊃q) i.e. ‘p is consistent with q’ entails ‘it is not the case that q is independent of p’, and again that ~(p⊃q) entails (p⊃~q) i.e. ‘q is independent of p’ entails that ‘it is not the case that q is consistent with p’.

Thus, it is obvious that equating the notion of “deducibility” and, in turn, that of “entailment” with that of “material implication” will make the two concepts of consistency and independence contrary to each other. As a result, it cannot be said that in any deductive system, the set of postulates are both consistent and independent.

As Lewis observes:

> “If material implication, p⊃q, should be taken as equivalent to “q is deducible from p”, then no pair of propositions could be at once consistent and independent. Because if p and q are consistent, then p cannot imply that q is false and if q is independent of p, then p cannot imply that q is true. Thus the ordinary procedure of deduction – as, for instance, in the usual mathematical requirement that the members of a set of postulates should be consistent and independent—would not be possible if p⊃q and “q is deducible from p” should be synonymous”. [15]

Lewis at first raises a question as to whether material implication is at all a relation for inference.[16] But then he himself goes on to state that it "is not the relation which we ordinarily have in mind when we say that q can be inferred from p".[17] According to Lewis, what *Principia Mathematica* sets forward is an extensional logic, while deductions in general are intensional as they are based on the meanings of the propositions involved. Thus he refuses to accept material implication as a relation for inference on the grounds that "*inference* depends upon meaning, logical import, *intension*. a $\subset$ b is a, relation purely of extension".[18] Lewis accordingly writes that the paradoxes of material implication do not have '*any application to valid inference'*.[19] He goes to the extent of calling the system of P.M. defective[20] in so far as it contains these so called paradoxes which he thinks are "peculiar" and "useless".[21] From the above arguments it is clear that the definition of 'entailment' in terms of 'material implication' is untenable.

However, notwithstanding these difficulties, some logicians have made an effort to defend the notion of material implication. The following section deals with some of the efforts.

2.3.3 A Defense of Russell's Position

In the previous section we have seen how Lewis has subjected Russell's theory of material implication to vehement criticism. It can, however, be said that not all of Lewis' objections against Russell are acceptable. According to Lewis, all the theorems of *Principia* are unproved as their deductions are based on the illogical concept of material implication. This seems to be an extreme denigration of Russell and Whitehead's work. Lewis appears to believe that the proofs of *Principia* do have '$p, p \supset q, \therefore q$,' as the schematic form; this notion may have led Lewis to the conclusion that all the proofs of *Principia* are basically dependent on material implication and so all the theorems are unsound.

As stated earlier, it follows from Russell's definition of material implication that a true proposition is implied by any proposition, and a false proposition implies any proposition. Even though Russell regarded these to be the consequences of the definition of material implication and never considered them to be paradoxical, but he never applied them for the construction of the proofs of *Principia*. None of the proofs of *Principia* has a conclusion drawn merely on the basis of its premises being false nor merely on the basis of the conclusion being true. It can be said that Lewis exaggerated by saying that all the theorems of *Principia* are unproved and unsound. Moreover, the fact cannot be ignored that Russell's theory of material implication is what inspired Lewis to initiate the valuable system of modal logic. Thus, Russell's system apart from having much value of its own can nonetheless be given the credit for being an inspiration for the formation of other valuable systems as well.

2.3.3.1 I. M. Copi's Attempt to Defend Russell

Further, some logicians, like Irving M. Copi, have made an attempt at solving the issue. In his book *Introduction to Logic*, he says that, some of the theorems of the *Principia Mathematica* appear to be paradoxical since it is held that irrelevant statements, be it true or false, cannot actually imply each other. Yet it is established through truth tables that any statement is implied by a false statement, and a true statement is implied by any statement. I.M. Copi, tries to safeguard Russell by saying that the word "implies" is highly ambiguous. The point that propositions with unrelated subject matter, or propositions which are irrelevant to each other, do not imply each other holds good in the case of logical, definitional and causal implications, (construing the notion of relevance in a broader sense) but doesn't hold in case of material implication. For, what is important in the latter case is just truth and falsehood. Copi goes on to say that no paradox is involved in saying that any disjunction with at least one true disjuncts is true, and in fact this is all that is asserted by statements of the forms $p \supset (\sim q \vee p)$ and $\sim p \supset (\sim p \vee q)$, which in turn are logically equivalent to the paradoxes $p \supset (q \supset p)$ and $\sim p \supset (p \supset q)$ respectively. According to Copi 'subject matter or meaning' is irrelevant in case of material implication which is a truth function.

These theorems are not contradictions, they are merely counterintuitive. They are not paradoxical; but are the unavoidable consequences of the definition of implication in terms of negation and disjunction. Russell himself says: “When the special meaning which we have given to implication is remembered, it will be seen that this proposition…(i.e., Anything implied by a true elementary proposition is true, 2.02.⊢:q.⊃.p⊃q),…is obvious”.[22]

2.3.3.2 Peter Suber Favours Material Implication

Peter Suber[23] speaks much in favour of material implication. According to him, material implication being truth-functional can be successfully applied to test the validity of arguments containing implication statements. It captures the essential meaning of implication as ordinarily used. He argues that even though the paradoxes fail to fulfill our ordinary expectations, they, nevertheless, do not violate any logical principle. Moreover, material implication sufficiently captures the logically essential core of the meaning of implication in ordinary uses. Suber goes on to say that: “At least take comfort from this. (1) The perversity of material implication is deliberate, for it is the only way to get truth-functionality. (2) But the perversity violates only our ordinary expectations, not any logical principles. (3) For material implication, despite its perversity, suffices to capture the logically essential core of meaning in ordinary uses of implication. The nuances of English that it does not translate do not affect validity (in standard logic). (4) Hence, the price is worth paying, for the counter-intuitive material implication distorts nothing essential and, because it is truth-functional, allows us to test the validity of arguments that contain implication statements”.[24]

2.3.3.3 Johnson's Solution to the Paradoxes of Material Implication

Again, William Ernest Johnson doesn't consider these to be paradoxes at all. Johnson, in his *Logic*, suggests a solution for the paradoxes of material implication by stating that the two paradoxes of material implication, i.e. '$q \supset (p \supset q)$'and '$\sim p \supset (p \supset q)$', are inferentially harmless, as no inference can be successfully made from this.

Johnson points out that inferring 'q' from $p \supset q$ and 'p', when $p \supset q$ itself has been inferred from '~p', leads to contradiction. To elucidate: p materially implies q may be validly inferred from the negation of 'p'; again, from the affirmation of 'p' in combination with p materially implies q, 'q' can be validly inferred. However, if the material implication of 'q' by 'p' has been inferred from the negation of 'p', then this material implication cannot further be used for drawing inferences in conjunction with the affirmation of 'p' without leading to contradiction. This kind of inference, according to Johnson, is impossible as it involves contradiction.

Similarly, inferring q from $p \supset q$ and p, when $p \supset q$ itself is arrived at by the affirmation of q, involves circularity. In other words, p materially implies q may be validly inferred from the affirmation of q. However, if the material implication of q by p itself has been inferred from the affirmation of q, then this material implication in conjunction with the affirmation of p, cannot further be used for inferring q without involving circularity. Such an inference, according to Johnson, is again not possible on the ground that it involves circularity.

Johnson thus tries to find a solution for the paradox by stating that even though an implicative may be properly inferred from any one of the following i.e. from the negation of its implicans, or from the affirmation of its implicate, or again a disjunctive from the negation of one of its disjuncts, but, he insists, the implicative or disjunctive arrived at in such a manner cannot be used for the purpose of any further inference. For doing so will lead to the logical fallacy of either contradiction or that of circularity.

2.3.3.3.1 Further Elucidation of Johnson's Solution to the Paradoxes on the Basis of the Difference Between Hypothesis and Assertion

Johnson further elucidates the difference between implicative propositions which can be used for the purpose of inference and those which cannot be so used, with the help of the difference between hypothesis and assertion. According to him, only those implicative propositions may be used for inference, in which the implicans and the implicate are taken hypothetically.

Johnson urges that in the case of modus *ponendo ponens,* a valid inference, what is done is that the implicate is *asserted* on the basis of the *assertion* of the implicans; and in doing so, the propositions which were entertained hypothetically in the implicative, become assertive in the process of inference. Similar is the case with the other modes of inference, namely Modus tollendo tollens, Modus ponendo tollens, Modus tollendo ponens etc.

However, in case of the paradoxes, an implicative is inferred either from the affirmation of its implicate or from the negation of its implicans. Here the two components of the implicative, so arrived at, cannot be said to have been entertained hypothetically. So, "the principle, according to which inference is a process of passing from propositions entertained hypothetically to the same propositions taken assertorically, would be violated",[25] if the composite is used for inference. Johnson seeks to solve the problem of paradoxes in this way. According to him, since the paradoxes violate the above principle of inference, so, they cannot be used for the purpose of inference. He regards these paradoxes to be 'inferentially harmless'.

Johnson goes on to demonstrate how the above principle of inference holds good with respect to each of the four valid forms of inference. He proceeds to give a symbolic representation of the above, by placing the sign ├ to stand for assertorically entertained and the sign H for hypothetically taken, under each letter that stands for a proposition.

So following Johnson, the fundamental formulae for correct inference may be formulated as:

From 'p would imply q' with p; q may be inferred

H H ├ ├

Here both implicans and implicate in the implicative premise are entertained hypothetically. Johnson urges that the following inferences, that lead to paradoxical consequences, may be considered to be correct:[26]

"(a) From q, we may infer 'p would imply q'

$\vdash$ **H** $\vdash$

(b) From $\bar{p}$, we may infer 'p would imply q'"

$\vdash$ $\vdash$ **H**

But Johnson points out that the implicative conclusions arrived at cannot be utilized for inference any further: i.e.[27]

"(c) From 'p would imply q' with p; we cannot infer q

H $\vdash$ $\vdash$ $\vdash$

(d) From 'p would imply q' with p; we cannot infer q"

$\vdash$ **H** $\vdash$ $\vdash$

Because in the case of (c) the implicate, and in the case of (d) the implicans are taken to be asserted; as such these inferences violate the aforementioned fundamental formula that demands that both the implicate and the implicans be entertained hypothetically. Johnson thus argues that even though one can accept (a) i.e. a true proposition is implied by *any* proposition', but one cannot accept (c) i.e. from any proposition, a true proposition can be inferred. In the same way, it can be said that one can accept (b) i.e., a false proposition implies any proposition; however, one cannot accept (d) i.e. from a false proposition, any proposition can be inferred.

Johnson points out that, the attempted inference (c), leads to circularity, as the conclusion has already been asserted here; similarly, the attempted inference (d) leads to contradiction, as the premise has already been denied here.

Johnson, while maintaining the equivalence of the composite propositions, which are expressed through the implicative, the counter implicative, the alternative or the disjunctive form, goes on to show that each of these forms lead to a similar kind of paradox. He shows with the help of the following table, all those cases in which a Paradoxical Composite is arrived at; i.e., a composite which cannot be utilized for the purpose of inference, either in the modus ponendo ponens, tollendo tollens, ponendo tollens, or tollendo ponens form. He points out that the sign of assertion, in each of the composites, must be understood as asserted to be true in the case when the terms to which it is attached is in agreement with the premise, and when it contradicts the premise, then it must be understood as asserted to be false.

Following Johnson the Table of Paradoxical Composites can be shown thus.

<u>Table of Paradoxical composites</u> [28]

(a) From q we may properly infer

$\vdash$

(1) $\bar{p}$ **or q = If p then q = If** $\bar{q}$ **then** $\bar{p}$ **= Not both p and** $\bar{q}$**,**

H $\vdash$ H $\vdash$ $\vdash$ H H $\vdash$

(2) p or q = If $\bar{p}$ then q = If $\bar{q}$ then p = Not both $\bar{p}$ and $\bar{q}$,

H ⊦ H ⊦ ⊦ H H ⊦

(b) From $\bar{q}$ we may properly infer

⊦

(3) p or $\bar{q}$ = If $\bar{p}$ then $\bar{q}$ = If q then p = Not both $\bar{p}$ and q,

H ⊦ H ⊦ ⊦ H H ⊦

(4) $\bar{p}$ or $\bar{q}$ = If p then $\bar{q}$ = If q then $\bar{p}$ = Not both p and $\bar{q}$,

H ⊦ H ⊦ ⊦ H H ⊦

These composites cannot be used for the purpose of inference any further. To explain: In the first line, i.e., *in line* (1), the attempted inference, 'p ∴ q' leads to circularity, while, 'not – q ∴ not – p' leads to contradiction. Again, *in line* (2), it is clear that, the attempted inference 'not – p ∴ q' leads to circularity, and 'not – q ∴ p' leads to contradiction. While, *in line* (3), the attempted inference, 'not – p ∴ not -q' leads to circularity, and 'q ∴ p' leads to contradiction. Lastly, *in line* (4), it can be seen that, the attempted inference 'p ∴ not - q' leads to circularity, and 'q ∴ not - p' leads to contradiction.

Johnson urges that a composite and an implicative proposition in particular, cannot be applied for the process of further inference, 'without qualification'. It is only when the composite has been arrived at irrespective of any *assertion* of the truth or falsity of its components, that it can be used for further inference. To put it differently, a necessary condition for further inference is that, when asserting a composite, its components have to be entertained hypothetically.

It can be noted here, that however ingenious Johnson's argument may be, these paradoxes of material implication cannot go scot free on the ground that they are harmless. As pointed out by Georg Henrik von Wright, it cannot be said following Johnson's theory that entailment is equivalent to material implication. These paradoxes, he thinks, are the ones which hinder such equivalence, and that such impediment cannot be escaped by calling the paradoxes inferentially harmless. von Wright says:

> …the paradoxes of material implication constitute an obstacle to an identification of entailment with material implication and that this obstacle cannot be removed by considerations about the 'inferential harmlessness' of those paradoxes.[29]

Further, Robert K. Meyer also thinks that these paradoxes cannot be accepted even if they are considered by some to be inferentially harmless. In his article 'Entailment', he says: "…Because intuitions are after all vague, and because no outright contradiction results, it is often held that the paradoxes of implication do no harm. One might as well argue that arson does no harm because it is not murder, as though burning down a man's house is of no import if one does not kill him".[30]

Thus, we can see that although efforts on the part of some logicians to defend material implication have been made, none of them is found to be unsatisfactory.

Russell himself, however, was aware of the paradoxical consequences of identifying material implication with entailment. In order to overcome these paradoxes, he formulated the notion of formal implication. The next chapter will deal with Russell's notion of formal implication and his definition of entailment in terms of it.

NOTES AND REFERENCES

1. Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, Cambridge University Press, Volume I, 1910, p. 98.
2. Russell reiterates in *Principia Mathematica* thus: “Now in order that one proposition may be inferred from another, it is necessary that the two should have that relation which makes the one a consequence of the other. When a proposition q is a consequence of a proposition p, we say that p implies q. Thus deduction depends upon the relation of implication, and every deductive system must contain among its premises as many of the properties of implication as are necessary to legitimate the ordinary procedure of deduction”. *Ibid*., p. 94.
3. Bertrand Russell, *The Principles of Mathematics,* George Allen and Unwin, London, 1979, p. 33.
4. As Russell reiterates in *Principia Mathematica*: “The Implicative Function is a propositional function with two arguments p and q, and is the proposition that either not-p or q is true, that is, it is the proposition ~p v q. …the proposition ~*p* v *q* 'will be quoted as stating that *p* implies *q*… But "implies" as used here expresses nothing else than the connection between *p* and *q* also expressed by the disjunction "not-p or *q*." The

symbol employed for *"p* implies *q," i.e.* for *"~pvq,"* is *"p⊃q."*" Bertrand Russell and Alfred North Whitehead, *Principia Mathematica*, *op. cit.,* p. 7.

5. Ibid., p. 98.
6. Ibid., p. 103.
7. *Ibidem*.
8. *Ibidem*.
9. W. E. Johnson, *Logic*: Part I, Cambridge University Press, London, 1921, p.39.
10. Independence condition, unlike the other two conditions, however, is not a formal condition; it is required for the elegance of any system.
11. Russell holds: "Provided our use of words is consistent, it matters little how we define them". Bertrand Russell, *Introduction to Mathematical Philosophy*, Allen & Unwin Ltd., London, May 1919, p. 154.
12. John G. Slater, ed., *Logical and Philosophical Papers, 1909-13*, London, Routledge, 1992, p. 350.
13. G.E., Moore, 'External and Internal Relations', in *Proceedings of the Aristotelian Society*, New Series, Vol. 20, 1919 - 1920, p. 58, (of pp. 40-62).
14. Charles E., Gauss, 'The Interpretation of Implication', in *Philosophy of Science*, Vol. 10, No. 2, Apr., 1943, The University of Chicago Press p. 97, (of pp. 95-103).

15. Clarence Irving Lewis, and Cooper Harold Langford, *Symbolic Logic,* Dover Publication, INC., New York, 1959, p**.** 144.

16. Lewis asks: "Is this material implication, a⊂b, a relation which can validly represent the logical nexus of proof and demonstration?". C. I. Lewis, *A Survey of Symbolic Logic,* Berkeley, University of California Press, 1918, p.328.

17. C.I., Lewis, 'The issues concerning material implication', *The Journal of Philosophy*, Vol. 14, No. 13, Jun.21, 1917, p. 350, (of pp. 350-356).

18. C. I. Lewis*, A Survey of Symbolic Logic, op. cit.,* p. 328.

19. Lewis maintains "...there are any number of such "peculiar" theorems in any calculus of propositions based on material implication...*These theorems do not admit of any application to valid inference... There are, then, in the system of Material Implication, a class of propositions, which do not admit of any application to valid inference". Ibid., p.* 326.

20. C.I., Lewis, 'The issues concerning material implication', op. cit., p. 352, (of pp. 350-356).

21. C. I. Lewis, *A Survey of Symbolic Logic, op. cit.,* p. 319.

22. Bertrand Russell, *Principia Mathematica, op. cit.,*p. 103.

23. Peter Suber, http://legacy.earlham.edu/~peters/courses/log/mat-imp.htm (in electronic hand-out for the course on symbolic Logic, at Philosophy Department, Earlham College.
24. *Ibidem*.
25. W. E. Johnson, *Logic*: Part I, op. cit., p. 44.
26. *Ibid*., p. 45.
27. *Ibidem*.
28. *Ibid*., p. 46.
29. Georg Henrik von Wright, *Logical Studies,* Routledge and Kegan Paul, London, 1967, p. 171ff.
30. Robert K. Meyer 'Entailment', in *The Journal of Philosophy*, Vol. 68, No. 21, Nov. 4, 1971, p. 812, (of pp. 808-818).