CALCULATION OF PROTON IMPACT $2^1S$ AND $n^1P$ ($n = 2, 3, 4$) EXCITATIONS OF ATOMIC HEliUM USING THE GLAUBER APPROXIMATION

5.1. Introduction

Extensive investigations have so far been made about direct excitation of neutral atomic targets in fundamental ion-atom collision processes\textsuperscript{1}. However, almost all the calculations and measurements concentrate on the determination of cross sections integrated over the scattering angles. Cross sections for the $1^1S \rightarrow n^1P$ ($n = 2, 3, 4$) excitations of atomic helium under proton impact have been investigated over recent years theoretically\textsuperscript{2-13} as well as experimentally by a number of authors. However, substantial variation exists among the absolute values of various calculated and measured data, especially in the intermediate-energy region. In such cases it may be useful to look for agreement in terms of the physically consistent feature of various data like the functional dependence of cross sections on energy rather than their absolute values. Moreover, a systematic study of
excitations to states having the same total angular momentum, but different principal quantum number (n) have also been found to provide us with a knowledge of the consistency in various data. It, thus, appears worthwhile to undertake a systematic study of the $1^3S \rightarrow n^1P$ excitations in $H^+-$He collisions.

The spread over a wide range of absolute values which greatly exceeds the estimated uncertainty limits of the individual data calls for a more detailed study of the underlying physical processes. It is well known that a knowledge of the differential cross sections can furnish important information in this respect and can provide a better test of the theoretical models against experimental observations than do the total cross section data. However, the data of differential cross sections for excitation in fundamental ion-atom collisions in the intermediate energy region have so far been very scanty. The lack of experimental data is a testament to the severe problems of measurement of the angular differential cross sections at intermediate ion energies. On the other hand, the usually applied theoretical methods are based on the semiclassical impact-parameter formulation which can not furnish exact quantal differential cross sections. For $H^+-$H collisions, the theoretical calculations which report the differential cross sections for direct excitation of the $2s$ and $2p$ states of the target hydrogen atom employ the Glauber approximation (GA), the Coulomb-projected-Born (CPB) approximation, the Vainshtein-Presnyakov-Sobel'man approximation (VPSA), and the many-state close-coupling (CC) model. Similarly, differential
cross sections in H\(^+\)-He collisions have been calculated employing only the multistate eikonal\(^{10}\), the GA\(^9\), and the CPB\(^{29}\) theories. For both the systems, however, the differential cross sections predicted by the first Born approximation (FBA) have been compared. In the absence of any experimental data, one could only note from these calculations the wide difference of the FBA cross sections from the GA, CPB, and multistate eikonal predictions except at very high incident energies.

However, of late, the situation has changed very much. Park et al.\(^{30,22}\) have been able to overcome the difficulties of measurement of the angular differential cross sections for the scattering of intermediate-energy protons from atomic hydrogen and helium. They have presented the center-of-mass differential cross sections for excitation of the n=2 level of the target helium atom by 25-, 50-, and 100-keV protons. For H\(^+\)-H collisions, the measurement of Park et al.\(^{30}\) are in remarkable agreement with the predictions of the GA method in respect to both absolute magnitude and angular dependence. The FBA differential cross sections differ appreciably from the measurements and the GA results except at the highest energy in the forward direction. The GA method\(^{31-34}\); however, in spite of its superiority over the first-order methods and the relative simplicity it possesses in comparison with higher-order methods, has found very few applications in H\(^+\)-He collisions. For excitation of atomic helium, although a number of GA calculations\(^{35-38}\) with electrons as projectiles are available, proton impact excitation of helium has been studied only by Chan and Chang\(^9\) and by Sur and Mukherjee\(^{23}\) using the GA method. However,
whereas Chan and Chang\textsuperscript{9} present the total and differential GA cross sections for the $1^1S\rightarrow 2^1P$ excitations of He. Sur and Mukherjee\textsuperscript{23} report the total Glauber cross sections for $1^1S\rightarrow n^1S$ ($n = 2, 3, 4$) transitions. The measurement of Park et al.\textsuperscript{22} of the angular differential cross sections for proton impact $n\rightarrow 2$ excitation of He, hence, can not be compared with either of the above calculations. The multistate eikonal calculation of Flannery and McCann\textsuperscript{10} gives the proton impact angular differential cross sections for $1^1S\rightarrow 2^1S$ and $1^1S\rightarrow 2^1P$ excitations of He. Since the contribution to the cross sections from the triplet state is negligible because of spin conservation, the eikonal results\textsuperscript{10} are readily comparable to the measurement.

In view of the remarkable success of the GA theory in predicting the observed differential cross sections for excitation in $H^+ - H$ collisions, we undertake in the present chapter a study\textsuperscript{39} of the GA predicted angular differential cross sections for $1^1S\rightarrow 2^1S$ and $1^1S\rightarrow 2^1P$ excitations of He by the impact of intermediate energy protons and compare the sum with the measurement of Park et al.\textsuperscript{22}. We also employ the GA method for studying\textsuperscript{40} $1^1S\rightarrow n^1P$ ($n = 3, 4$) excitations of helium by the impact of protons. The present systematic study allows us to make a consistency check with other similar investigations by comparing the relative physical features of the data, like the ratio of various $n$-excitation cross sections and also the $n^{-3}$ law.

The present chapter is organized as follows. In section 5.2, we describe the derivation of the Glauber approximation.
for scattering of charged particles by a neutral target. Section
5.3 deals with the wave functions we have used for the target
states. The separation of the Glauber scattering amplitude into
single-scattering and double-scattering parts has been shown in
section 5.4. Sections 5.5, 5.6, 5.7, and 5.8 are devoted to the
reduction of the single-scattering amplitudes and double-scatt-
ering amplitudes for $1^1S \rightarrow 2^1S$ and $1^1S \rightarrow n^1P$ transitions. Evaluation
of cross sections are given in section 5.9. We give an outline
of the numerical procedure in section 5.10 used in the present
calculation. Section 5.11 presents our calculated results followed
by discussions. Conclusions are given in section 5.12.

5.2. The Glauber approximation

For a direct collision, the Hamiltonians are the same in
the initial and final arrangement channels and we may decompose
the total Hamiltonian $H$ of the system as

$$H = H_d + V_d$$

Here $H_d$ describes the unperturbed Hamiltonian of the two colliding
bodies and $V_d$ is the interaction potential.

Let $\vec{r}$ denote the relative coordinates of the two bodies,
i.e. the coordinates of the incident particle with respect to the
center of mass (CM) of the target atom (nucleus) and $\vec{s}$ is used to
collectively denote the coordinates of the target particles
(electrons) with respect to the target CM (nucleus). We may write

$$H_d = -\frac{1}{2\mu} \nabla_\vec{r}^2 + h$$

$$V_d(\vec{r}, \vec{s}) = \frac{Ze^2}{r} - \sum_{j=1}^{Z'} \frac{1}{|\vec{r} - \vec{r}_j|}$$
Here \( h = h(s) \) denotes the internal Hamiltonian of the target, and let the complete set of eigenstates of \( h(s) \) be given by \( \{ u_n \} \), so that
\[
h u_n = \omega_n u_n ,
\]
\( \omega_n \) being the eigenenergy corresponding to \( u_n \). We also denote the unperturbed wave function of the system in the initial state \( a \equiv (i, \alpha) \) by \( \phi_a(r, s) \) and that in the final state \( b \equiv (f, \beta) \) by \( \phi_b(r, s) \), with
\[
H_a \phi_a = E_a \phi_a ,
\]
\[
H_b \phi_b = E_b \phi_b ,
\]
where
\[
E_a = \frac{k_i^2}{2\mu} + \omega_\alpha ,
\]
\[
E_b = \frac{k_f^2}{2\mu} + \omega_\beta .
\]
Here \( \alpha \) and \( \beta \) represent the sets of quantum numbers of the target in the initial and final states, \( \mu \) be the reduced mass of the system and \( k_i \) and \( k_f \) are the incident and final momenta respectively.

The explicit form of the function \( \phi_a \) and \( \phi_b \) can be written in the coordinate representation as
\[
\phi_a(r, s) = (2\pi)^{-3/2} \exp(ik_i \cdot r)u_\alpha(s) ,
\]
\[
\phi_b(r, s) = (2\pi)^{-3/2} \exp(ik_f \cdot r)u_\beta(s) ,
\]
where we have chosen the normalization
\[
\langle \phi_a | \phi_a \rangle = \delta_{n' n} \delta(k - k') .
\]
The on-shell transition matrix (T-matrix) element corresponding to a transition \( a \rightarrow b \) becomes:\(^{41}\)
\[
T_{ab} = \langle \Phi_a | V_d | \Phi_b^+ \rangle 
\]
\( \Psi_a^+ \) is shown to satisfy the Lippmann-Schwinger\(^{42,43}\) equation:
\[
\Psi_a^+ = \Phi_a + G^+_d V_d \Psi_a^+ \\
G^+_d = (E - H_d + i\epsilon)^{-1} .
\]
It is implied that the limit \( \epsilon \rightarrow 0^+ \) is always to be taken after the appropriate operation. We also obtain an alternative form of the Lippmann-Schwinger equation as
\[
\Psi_a^+ = \Phi_a + G^+_d V_d \Phi_a + G^+_d V_d G^+_d V_d \Phi_a + \cdots .
\]
The scattering amplitude for the transition \( a \rightarrow b \) is given by the usual formula:
\[
f_{ab} = -(2\pi)^2 \mu T_{ab} ,
\]
so that
\[
f_{ab} = \sum_{n=1}^{\infty} f_{Bn} ,
\]
where
\[
f_{B1} = -(2\pi)^2 \mu \langle \Phi_b | V_d | \Phi_a \rangle .
\]
\[
f_{Bn} = -(2\pi)^2 \mu \langle \Phi_b | V_d G^+_d V_d \cdots G^+_d V_d | \Phi_a \rangle .
\]
In Eq.(19), \( V_d \) appears \( n \) times and \( G^+_d \) appears \( (n-1) \) times. Thus the scattering amplitude in the first Born approximation (FBA) is given by
\[
f_{B1} = f_{B1} = -(2\pi)^2 \mu \langle \Phi_b | V_d | \Phi_a \rangle .
\]
Similarly, the second Born approximation (SBA) to the scattering.
amplitude can be written as
\[ f_{B2} = f_{B1} + f_{B2} \]
\[ = f_{B1} - (2\pi)^2 \mu \langle \phi_b | V_d | c_d \phi_d | \phi_a \rangle . \] (21)

For the nonrelativistic potential scattering, the Born series converges at sufficiently high energies for central potentials \( V(r) \) less singular than \( r^{-2} \) at \( r=0 \) and decreasing to zero faster than \( r^{-3} \) as \( r \to \infty \).

The replacement of \( V_a^+ \) in the \( T \)-matrix element (12) by the first one or two terms on the right-hand side of Eq. (15) yields the first Born and second Born approximations, respectively.

Another quite different high-energy approximation for \( V_a^+ \) can be found by a many-body generalization of the eikonal approximation which is applicable for scattering under a potential \( V(r) \). The method consists in expanding the phase of the stationary scattering wave function \( \psi^+ \) of the system in powers of \( \hbar \) and retaining only the leading term in the expansion. This is valid when

(i) the energy \( E \) of the incident particle greatly exceeds the magnitude of the potential:
\[ V/E \ll 1 \] (22)
and (ii) the particle wavelength (proportional to \( k^{-1} \)) is much smaller than the range \( 'a' \) of the potential:
\[ k \cdot a \gg 1 . \] (23)

The eikonal approximation first suggested by Molière has attracted considerable interest in recent years, particularly
after Glauber formulated a many-body generalization of the method. Application of the Glauber method, however, remained confined to the domain of nuclear physics for a long time. It is only recently that the scope of its application in the field of atomic collisions has been realized. This has been possible through the pioneering work of Franco who first recognized that the electron-hydrogen elastic scattering amplitude could be reduced to a one-dimensional integral without further assumption. Numerous applications have since been made to study the collision of charged particles with neutral atomic targets. For inelastic collisions at intermediate energies, the reliability of the Glauber method in predicting various physical features of the experimentally observed cross sections is by now well established. A number of review articles which discuss the theory and applications of the Glauber approximation along with its various eikonal modifications are available (Gerjuoy and Thomas, Byron and Joachain, Joachain and Quigg, Ishihara).

Various improvements over the conventional Glauber approximation have been proposed to overcome its limitations. However, considering the ease of calculation concerned, the Glauber theory still remains very attractive and its application and extension to newer fields, e.g. ionization, electron exchange and charge-transfer processes, excitation of ionic targets, excitation of molecules etc., still continue. Special mention may be made of a very recent work by Franco on collisions between composite systems each of which may be an ion or a neutral atom, by employing an expansion of the Glauber approximation for particle-atom collisions.
5.2.1. Derivation of the Glauber approximation:

The many-body Green's function corresponding to the Green's operator $G_d^+$ may be written as

$$G_d^+(\vec{r}, \vec{s}, \vec{r}', \vec{s}') = \frac{-1}{4\pi^2} \sum_n \exp \left( \frac{ik_n}{2} \frac{1}{r-r'} \right) u_n(\vec{s}) u_n^*(\vec{s}') , \quad (24)$$

the summation on $n$ running over all the discrete and continuum target states $u_n(\vec{s})$. Here $\vec{s}$ collectively represents the internal coordinates of the bound target particles, $\mu$ is the reduced mass of the system and $k_n$ is defined via

$$k_n = \left[ k_i^2 - 2\mu(\omega_n - \omega_\alpha) \right]^{1/2} . \quad (25)$$

For sufficiently large values of the incident energy $(k_i^2/2\mu)$, we can replace the energy difference $(\omega_n - \omega_\alpha)$ by a single value $\bar{\omega}$ of the average excitation energy for all $n$. Thus we put

$$k_n \approx \bar{k} = (k_i^2 - 2\mu \bar{\omega})^{1/2} . \quad (26)$$

Substitution from (26) into (24) gives

$$G_d^+(\vec{r}, \vec{s}, \vec{r}', \vec{s}') = \frac{-1}{4\pi^2} \sum \frac{\exp(ik_i(\vec{r}'-\vec{r}'))}{|\vec{r}'-\vec{r}'|} \delta(\vec{s}' - \vec{s}) , \quad (27)$$

where we have used the closure property of $u_n(\vec{s})$:

$$\sum_n u_n(\vec{s}) u_n^*(\vec{s}') = \delta(\vec{s}' - \vec{s}) . \quad (26)$$

If, for large $k_i$, we further assume that for a sufficiently long range of intermediate states, $k_n \approx k_i$, then $\bar{\omega} = 0$ and from (26) we have

$$\bar{k} = k_i . \quad (29)$$
Equation (27) then gives
\[ G_+^t(\mathbf{r}, \mathbf{r'}, \mathbf{s}, \mathbf{s'}) = 2\mu G_0^t(\mathbf{r}, \mathbf{r'}) \delta(\mathbf{s} - \mathbf{s'}) , \quad \{30\} \]
where
\[ G_0^t(\mathbf{r}, \mathbf{r'}) = -\frac{1}{4\pi} \frac{\exp(ik_d|\mathbf{r} - \mathbf{r'}|)}{|\mathbf{r} - \mathbf{r'}|} , \quad \{31\} \]
is the free-particle Green's function satisfying the equation
\[ (V^2 + k_i^2) G_0^t(\mathbf{r}, \mathbf{r'}) = \delta(\mathbf{r} - \mathbf{r'}) , \quad \{32\} \]
and is given by
\[ G_0^t(\mathbf{r}, \mathbf{r'}) = -(2\pi)^{-3} \int \frac{\exp[ip \cdot (\mathbf{r} - \mathbf{r'})]}{p^2 - k_i^2 - i\epsilon} dp \quad \{33\} \]
Here it is implied that on the right hand side of (33) the limit \( \epsilon \to 0^+ \) should be taken after performing the \( p \)-integration. Substituting (30) in the Lippmann-Schwinger equation (13), we have
\[ \Psi_+^t(\mathbf{r}, \mathbf{s}) = \phi_a(\mathbf{r}, \mathbf{s}) + 2\mu \int G_0^t(\mathbf{r}, \mathbf{r'}) V_d(\mathbf{r'}, \mathbf{s'}) \Psi_a^t(\mathbf{r'}, \mathbf{s'}) \delta(\mathbf{s} - \mathbf{s'}) d\mathbf{r'} d\mathbf{s'} \quad \{34\} \]
After performing the \( \mathbf{s'} \)-integration in (34), we can write
\[ \Psi_+^t(\mathbf{r}, \mathbf{s}) = (2\pi)^{-3/2} \exp(ik_1 \cdot \mathbf{r}) u_\alpha(\mathbf{s}) + 2\mu \int G_0^t(\mathbf{r}, \mathbf{r'}) V_d(\mathbf{r'}, \mathbf{s'}) \Psi_a^t(\mathbf{r'}, \mathbf{s'}) d\mathbf{r'} , \quad \{35\} \]
where we have put the explicit coordinate representation for the free state function \( \phi_a \). Equation (35) for \( \Psi_+^t(\mathbf{r}, \mathbf{s}) \) involves the internal coordinates \( \mathbf{s} \) of the target only parametrically. Moreover, the \( \mathbf{s} \)-dependence of \( \Psi_a^t(\mathbf{r}, \mathbf{s}) \) is determined only by the initial target eigenstate \( u_\alpha(\mathbf{s}) \) and the interaction potential \( V_d \). Hence it is convenient to simplify (35) by the substitution (cf. Gerjuoy and Thomas) \( \{33\} \)
\[ \Psi_a^t(\mathbf{r}, \mathbf{s}) = \Psi(\mathbf{r}, \mathbf{s}) u_\alpha(\mathbf{s}) , \quad \{36\} \]
Equation (37) for $\psi$, except for its parametric dependence on $\xi$, is formally identical to the Lippmann-Schwinger equation for the scattering of a particle of mass $1$ and momentum $\mathbf{k}_1$ by a potential $V_d$. A solution of Eq. (37) can be obtained by linearizing the free Green's function $G^+_0$ of Eq. (33). We introduce a new integration variable $\zeta = \mathbf{r} - \mathbf{k}_1$, whence Eq. (33) gives

$$G^+_0(\mathbf{r}, \mathbf{r}) = -(\frac{2\pi}{\mathbf{k}_1})^3 \exp\left(\frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{\mathbf{k}_1}\right) \times$$

$$\int \exp\left(-\frac{i\mathbf{r}_1 \cdot \mathbf{r}_2}{\mathbf{k}_1}\right) \frac{d^3 \mathbf{t}}{2\mathbf{k}_1 - i\mathbf{e}} \left(\mathbf{r}_1 \cdot \mathbf{r}_2\right)$$

where we have linearized the denominator by neglecting the term in $t^2$. The Green's function $G^+_0$ may be considered to propagate like a free plane wave between successive interactions with the potential $V_d$. Under the assumptions (22) and (23), significant propagation occurs mostly in the forward direction, at values of $\mathbf{k}_1$ which are unmodified by the interaction $V_d$. The significant values of $\mathbf{p}$ then lie near $\mathbf{k}_1$ and the above linearization is permissible.

The integral in Eq. (38) can be readily evaluated by choosing the polar $z$-axis along the direction $\mathbf{k}_1$. The $t_x$- and $t_y$-integrations yield $\delta$-functions. For $z < z'$, the $t_z$-integral vanishes. For $z > z'$, the $t_z$-integration can be performed by choosing a contour closed at infinity in the upper half $t_z$-plane so as to include the only first-order pole at $t_z = i\mathbf{e}/2\mathbf{k}_1$. This gives

$$G^+_0(\mathbf{r}, \mathbf{r}') = -(i/2\mathbf{k}_1) \exp\left[ -ik_1 (z-z') \right] \delta^2(\mathbf{r}-\mathbf{r}') \delta(z-z')$$
where we have adopted a cylindrical coordinate system:

\[ \mathbf{r} = \mathbf{b} + \hat{n} z , \]
\[ \mathbf{r}' = \mathbf{b}' + \hat{n} z' , \]

\( b \) and \( b' \) being the respective projections of \( \mathbf{r} \) and \( \mathbf{r}' \) on a plane perpendicular to the polar \( z \)-axis in the direction \( \hat{n} \parallel \hat{k}_i \) and \( \theta \) is the step function given by

\[ \theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \]

Inserting (39) into the Lippmann-Schwinger equation (37), and performing the \( b' \)-integration, we obtain

\[ F(r,s) = \frac{(2\pi)^{-3/2} \exp(ik \cdot r)}{k_i} \int \exp(-ik \cdot z') \int \mathcal{V}(\mathbf{b}, z', s) \mathcal{V}(\mathbf{b}', z', s') dz' , \]

where the \( z' \)-integral is to be evaluated along a straight line parallel to \( \hat{k}_i \).

We now write

\[ \mathcal{V}(\mathbf{b}, z, s) = \left( \frac{2\pi}{k_i} \right)^{-3/2} \exp(i k \cdot r) F(r,s) , \]

where \( F(r,s) \) is a factor which modulates the incident plane wave and assume that the product \( \mathcal{V}F \) varies slowly within a particle wave length \((\sim 1/k_i)\). On substitution of (45) in (44), we obtain the following integral equation for \( F(r,s) \):

\[ F(r,s) \equiv F(b, z, s) \]

\[ = 1 - \frac{i}{v} \int \mathcal{V}(\mathbf{b}, z', s) F(\mathbf{b}', z', s') dz' , \]

which incorporates the boundary condition \( F(b', -\infty, z) = 1 \). Here,
we have put
\[ k_1 / \mu = v \]  
the incident relative velocity of the projectile with respect to the target. The solution of (46) can be readily obtained as
\[ \Psi(\vec{r}, \vec{z}, \vec{s}) = \exp \left[ -\frac{i}{v} \int_{-\infty}^{z} V_d(\vec{r}', \vec{z}', \vec{s}') d\vec{z}' \right] \]  
where the z-integral is evaluated along the direction \( \hat{n} \parallel \vec{r} \) [cf. Eqs. (40) and (41)]. Substitution of (48) into (45) gives
\[ \Psi(\vec{r}, \vec{s}) = \Psi(\vec{r}, \vec{z}, \vec{s}) = (2\pi)^{-3/2} \exp \left[ i k_1 \cdot \vec{r} - \frac{i}{v} \int_{-\infty}^{z} V_d(\vec{r}', \vec{z}', \vec{s}') d\vec{z}' \right] \]  
The expression (49), except for the parametric dependence on \( \vec{s} \), is equivalent to the eikonal approximation to the wave function for scattering by a potential \( V_d \). Inserting the expression (49) for \( \Psi(\vec{r}, \vec{s}) \) into Eq. (36) for \( \Psi_a^+ \), we obtain the Glauber approximation to the wave function
\[ \Psi_a^G(\vec{r}, \vec{s}) = (2\pi)^{-3/2} \exp \left[ i k_1 \cdot \vec{r} - \frac{i}{v} \int_{-\infty}^{z} V(\vec{r}', \vec{z}', \vec{s}') d\vec{z}' \right] u_\alpha(\vec{s}) \]  
It is worth noting that the approximate wave function (50) is not accurate at large distances from the scattering region. This requires that the target particles should be contained within a distance \( R \) from one another such that
\[ R \ll x_1 a^2 \]  
where 'a' is the force range. Since the above procedure is applicable for \( k_1 a \gg 1 \), condition (51) is usually satisfied.

The on-shell T-matrix element corresponding to a transition from an initial state \( a \equiv (i, \alpha) \) to a final state \( b \equiv (f, \beta) \) is obtained
on substitution for $\psi^G_a = \psi^+ a$ from (50) into (12):

$$T_{ab} = \langle \phi_b | V a | \psi^G_a \rangle$$

$$= (2\pi)^{-3} \int u_b^*(s) \exp(\frac{iq.r}{\hbar}) V_a(b, z, b') \exp\left[-\frac{i}{\hbar} \int_{-\infty}^{z} V_d(b', b', s') dz'\right]$$

$$u_a(s) d^2 b \ dz \ ds \ , \quad (52)$$

where $\vec{q} = \vec{k}_i - \vec{k}_f$ is the momentum transfer and we have used the explicit coordinate representation for $\phi_b(\vec{r}, s)$.

Now, for elastic scattering, energy conservation requires that $\vec{k}_i = \vec{k}_f$, so that for small angle scattering, the momentum transfer $\vec{q}$ is nearly perpendicular to $\vec{k}_i$. This error in the approximation $\exp(i \vec{q}.\vec{r}) \approx \exp(i \vec{q}.\vec{r}_0)$ [or, equivalently $\exp(i \vec{q}.\vec{r}_1) \approx 1$] is only of order

$$(1 - \cos \theta) k_i d \sim \theta^2 k_i d \ , \quad (53)$$

where $\theta$ is the scattering angle and $d$ is the distance within which $V_{df}$ varies appreciably. Under the assumptions already made the approximation (53) is permissible [Glauber$^3$]. For inelastic collisions, at high incident energies, the target excitation energy

$$| \omega_f - \omega_i | \ll \hbar \ , \quad (54)$$

and at small scattering angles, again, we may use the approximation

$$\vec{q} \perp \vec{k}_i \ , \quad (55)$$

$$\exp(i \vec{q}.\vec{r}) \approx \exp(i \vec{q}.\vec{r}_0) \ . \quad (56)$$

With the above assumptions (55) and (56), the $z$-integration in Eq. (52) is simply that of an exact differential and can be readily evaluated. This gives for the scattering amplitude corresponding
to the transition $a \rightarrow b$

$$f_{ab} = -(2\pi)^2 \mu T_{ab}$$

$$= \frac{ik_i}{2\pi} \left[ u_b^*(\hat{s}) \left[ 1 - \exp \left\{ i \chi^G(\hat{z}, \hat{s}) \right\} \right] u_a(\hat{s}) \exp(iq \cdot \hat{z}) d^2\hat{z} \right] \exp(iq \cdot \hat{z}) , \quad (57)$$

where we define the phase-shift function $\chi^G(\hat{z}, \hat{s})$ by

$$\chi^G(\hat{z}, \hat{s}) = -(1/v) \int_{-\infty}^{\infty} V_a(\hat{z}, z, \hat{s}) dz . \quad (58)$$

The above derivation of the Glauber approximation shows that it may be viewed as an eikonal approximation to a model, in which the particles of the target are assumed to be frozen into a given configuration $\hat{s}$. Thus when the Green's function $G_a^+(\hat{r}, \hat{s}, \hat{r}', \hat{s}')$ given by (30) is substituted into the Eqs. (17), (18), and (19) for the Born series, the $s$-function in $G_a^+$ causes each term in the Born series to reduce to potential scattering by a frozen target potential where the target coordinates appear only as parameters. The underlying idea was originally developed by Chase for application in nuclear physics and is valid when the relative velocity between the colliding systems is large in comparison with the internal velocities of the individual systems.

It should be noted that the above derivation is a combination of the eikonal and closure approximations along with the additional assumption that the average excitation energy of the target $\bar{\omega}$ is zero. This last assumption should be contrasted to some forms of the second Born approximation (Moiseiwitsch) employing closure relations, but where $\bar{\omega}$ is nonzero.

Byron has derived the Glauber approximation in a different way which is very useful in depicting its relation to the close-
coupling approximation. Whereas in the close-coupling approximation one retains in Eq. (24) for the Green's function only a small number of channels, but treats them exactly, in the Glauber method all channels are included at the expense of making certain approximations in each. At low incident energies, where only a small number of states are likely to be important, the close-coupling approximation is well-founded. But at intermediate energies, where a large number of states may have to be coupled, it may be more useful to employ the Glauber approximation.

5.2.2. Glauber scattering amplitude for charged particle-neutral atom collisions

The interaction potential \( V_d(\vec{r}, \vec{s}) \) for an atom with \( z \) electrons having coordinates \( \vec{s} = \vec{s}(\vec{r}_1, \vec{r}_2, ..., \vec{r}_z) \) is given by

\[
V_d(\vec{r}, \vec{s}) = \frac{Z' e}{r} - \sum_{j=1}^{Z'} \frac{1}{|\vec{r} - \vec{r}_j|}.
\]

The total Glauber phase-shift function given by (58) thus becomes:

\[
\chi^G(\vec{b}, \vec{s}) = -\frac{Z'}{\nu} \sum_{j=1}^{\infty} \left( \frac{1}{r^2} - \frac{1}{|\vec{r} - \vec{r}_j|^2} \right) dz = \sum_{j=1}^{Z} \chi_j(\vec{b}, \vec{r}_j),
\]

where we define \( \chi_j(\vec{b}, \vec{r}_j) \) as

\[
\chi_j(\vec{b}, \vec{r}_j) = -\frac{Z'}{\nu} \int_{-\infty}^{\infty} \left( \frac{1}{r^2} - \frac{1}{|\vec{r} - \vec{r}_j|^2} \right) dz.
\]

\( \chi_j \) represents the phase shift caused by the \( j \)-th electron of the target atom. Thus (60) shows that the total Glauber phase-shift function can be obtained by summing the individual phase-shifts contributed by each target scatterer. This property of the additivity of phase-shifts is an important feature of the Glauber theory.
Now, since according to our choice of coordinates (40) and (41): \( \vec{r} = \vec{b} + \hat{n}z \) and hence \( \vec{r}_j = \vec{b}_j + \hat{n}z_j \), where \( \vec{b}_j \) is the projection of \( \vec{r}_j \) onto the plane perpendicular to \( \hat{n} \), we have

\[
\vec{r} = (b^2 + z^2)^{1/2}, \quad |\vec{r} - \vec{r}_j| = (\vec{b}_j^2 + \vec{z}_j^2)^{1/2},
\]

where \( \vec{b}_j = |\vec{b} - \vec{b}_j| \) and \( \vec{z}_j = (z - z_j) \). The integral in Eq. (61) can be readily evaluated as

\[
\chi_j(b, \vec{r}_j) = -\frac{z'}{v} \lim_{a \to 0} \left[ \ln \left( \frac{a + \sqrt{a^2 + b^2}}{a + \sqrt{a^2 + b_j^2}} \right) - \ln \left( \frac{-a + \sqrt{a^2 + b^2}}{-a + \sqrt{a^2 + b_j^2}} \right) \right],
\]

so that from (60)

\[
\chi^G(b, \vec{z}) = \sum_{j=1}^{z} \ln \left[ \left| \vec{b} - \vec{b}_j \right| / b \right]^2,
\]

and

\[
\exp \left[ i\chi^G(b, \vec{z}) \right] = \sum_{j=1}^{z} \left( |\vec{b} - \vec{b}_j| / b \right)^2 i\eta,
\]

with

\[
\eta = -z'/v.
\]

On substitution from (65) in (57), we have the Glauber scattering amplitude for the transition \( \alpha \to \beta \) of an atomic target with \( z \) electrons:

\[
f_{\alpha\beta}(q) = \frac{ik}{2\pi} \left( u_{\beta}^* (\vec{r}_1, \ldots, \vec{r}_z) \Gamma (\vec{b}, \vec{r}_1, \ldots, \vec{r}_z) u_{\alpha} (\vec{r}_1, \ldots, \vec{r}_z) \right)
\exp (i \vec{q} \cdot \vec{b}) d^2 b \, d\vec{r}_1 \cdots d\vec{r}_z,
\]
where the Glauber profile function $\Gamma'(\vec{b}, \vec{r}_1, ..., \vec{r}_z)$ is given by

\[
\Gamma'(\vec{b}, \vec{r}_1, ..., \vec{r}_z) = 1 - \prod_{j=1}^{z} \left( |b - b_j| / b \right)^{2 \eta}.
\]  

(68)

with

\[
\Gamma_j'(b_j, r_j) = 1 - \left( |b - b_j| / b \right)^{2 \eta}.
\]  

(69)

It is worth noting that Eq. (68) can be written in the following alternative form:

\[
\Gamma'(\vec{b}, \vec{r}_1, ..., \vec{r}_z) = 1 - \prod_{j=1}^{z} \left[ 1 - \Gamma_j'(b_j, r_j) \right],
\]  

(70)

which gives an expansion of $\Gamma'(\vec{b}, \vec{r}_1, ..., \vec{r}_z)$:

\[
\Gamma = \sum_{j=1}^{z} \Gamma_j' - \sum_{j \neq l} \Gamma_j \Gamma_l + \ldots (-1)^{z-1} \prod_{j=1}^{z} \Gamma_j.
\]  

(71)

This expression, when substituted in (67), gives a multiple scattering expansion of the scattering amplitude. The term linear in $\Gamma_j$ can then be interpreted as the 'single scattering' contribution, the term in $\Gamma_j \Gamma_l$ as the 'double scattering' contribution and so on. It is interesting to note that the maximum order of such a Glauber multiple scattering series can be $z$ (number of electrons in the target).

An alternative Glauber series may be obtained by expanding the term $\exp \left[ i \chi^G(\vec{b}, \vec{s}) \right]$ in (57) as a series in powers of the interaction potential $V_d$:

\[
f_{ab} = \sum_{n=1}^{\infty} f_{Gn}.
\]  

(72)

\[
f_{Gn} = - \frac{ik}{2\pi} \frac{1}{n!} \int u_\beta^*(\vec{s}\vec{z}) [\chi^G]^n u_\beta(\vec{s}) \exp(i\vec{s}\cdot\vec{b}) b^2 d\vec{s}.
\]  

(73)

The terms $f_{Gn}$ of the Glauber series (72) take alternatively purely
real or purely imaginary values depending on the symmetry of the bound state wave functions $u_\alpha$ and $u_\beta$. Thus the Glauber series (72) is to be contrasted to the Born series (41), whose terms have both real and imaginary parts for $n > 1$.

If the integration of the Glauber phase (58) is performed along $\hat{n}_2 \hat{q}$, we obtain

$$\bar{f}_{\text{Gl}} = \bar{f}_{\text{Bl}}$$

for all scattering angles.

In the case of elastic collisions $|k_1^\perp| = |k_2^\perp|$, one is led to the choice $n_1 \perp q$ in Eq. (52), allowing its reduction to the form (57). Thus since $\hat{q}$ is perpendicular to the direction of the bisector of the scattering angle $(\hat{k}_1 + \hat{k}_2)$, one has $\hat{n} \perp (\hat{k}_1 + \hat{k}_2)$. For inelastic collisions, however, $|k_1^\perp| \neq |k_2^\perp|$, and the above choice $(\hat{n}_1 \perp \hat{q})$ does not coincide either with the direction $(\hat{k}_1 + \hat{k}_2)$ or with that of the bisector of the scattering angle. Actually, in the forward direction, $\hat{q}$ lies along $k_1^\perp$ and $\hat{n}$ is perpendicular to the corresponding classical trajectory. Yet, the choice $\hat{n}_1 \perp \hat{q}$ is supported by the following considerations.

Only for the integration-path $\hat{n}_1 \perp \hat{q}$, one has $\bar{f}_{\text{Gl}} = \bar{f}_{\text{Bl}}$, which assures exact coincidence of the FBA and Glauber amplitudes as $k_1 \to \infty$. With the choice $\hat{n} \parallel k_1^\perp$, Eq. (57) is valid only for small scattering angles. Moreover, calculations with this path, though considerably more complicated, fail to show consistent improvement in all cases.\footnote{54,79}

In the limit $k_1 \to \infty$, the Glauber series is dominated by the leading term $\bar{f}_{\text{Gl}}$, which exactly equals the FBA amplitude $\bar{f}_{\text{Bl}}$.\footnote{54,79}
for all angles [cf. Eq. (74)] with the choice of path \( \hat{n} \). Since the Glauber amplitude (57) or (67) approximately includes higher order multiple scattering terms also, at intermediate energies, the Glauber method is usually superior to the Born approximation. However, quantitative estimate of this superiority requires detailed comparison of the Glauber and the Born series, which can be obtained from the work of Byron and Joachain.\(^{34,55}\)

5.3. Wave function

For the ground state of the helium atom, we take the product form of wave function due to Green et al.\(^{80}\):

\[ \Psi_1 = \Psi(1S| r_1', r_2') = N_1 u_0(r_1)u_0(r_2) \quad (75) \]

where

\[ u_0(r) = \exp(-\alpha r) + a \exp(-\beta r) \quad (76) \]

while the final \( 2^1S \) state wave function of Winter and Lin\(^{81}\) is given in the form

\[ \Psi_2 = \Psi(2^1S| r_1', r_2') = N_{fs} \left[ u_2(r_1)g(r_2) + u_2(r_2)g(r_1) \right] \quad (77) \]

with

\[ g(r) = \exp(-2r), \quad u_2(r) = \exp(-\gamma r) \sum_{p=1}^{2} (-1)^{p-1} C_p r^{p-1} \quad (78) \]

Here \( \alpha, \beta, \gamma, a, \) and \( C_p \) are wave function parameters, and \( N_1 \) and \( N_{fs} \) are the normalization constants.

For the final \( n^1P \) state wave function of helium atom we take the following form given by Winter and Lin.\(^{81}\)

\[ \Psi_f(n^1P_m| r_1', r_2') = N_f \left[ u_n(r_1)g(r_2) + u_n(r_2)g(r_1) \right] \quad (79) \]
with
\[ g(r) = \exp(-2r), \]
\[ u_n(r) = r Y_{lm}(r) \exp(-Sr) \sum_{k=1}^{n-1} (-1)^{k-1} b_k r^{k-1}. \]  

Here \( m = 0, \pm 1 \) and \( \delta, b_k \) are the wave function parameters, and \( N_{fp} \) is the normalization constant.

5.4. Separation of the Glauber scattering amplitude into single-scattering and double-scattering parts

For the excitation of a He atom from an initial state \( \Psi_i(r_1, r_2) \) to a final state \( \Psi_f(r_1', r_2') \) under the incidence of a structureless particle with charge \( z_1 \) and relative velocity \( v_1 \), the scattering amplitude in the center-of-mass system is given, according to the Glauber approximation, by

\[ F(i \to f, q) = \frac{i k_1}{2\pi} \int \gamma(B, r_1', r_2') \Psi_f(r_1', r_2') \Psi_i(r_1, r_2) \exp(ia \cdot B) d^2B dr_1 dr_2, \]  

where
\[ \gamma(B, r_1', r_2') = 1 - \prod_{j=1}^{2} (|B - \vec{s}_j|/b)^{2\eta_j}, \]  

with \( \eta = -z_1/v_1 \). Here \( k_1 \) and \( k_f \) are, respectively, the incident and final momenta of the incident particle in the center-of-mass system and \( q = k_f - k_i \) is the momentum transfer vector. The vectors \( \vec{B} \) and \( \vec{s}_j \) represent the respective projections of the position vectors of the incident particle and the \( j \)-th bound electron \( (r_j) \) onto the plane perpendicular to the direction \( \vec{q} \) along which the Glauber path integration is performed. Moreover, \( \vec{q} \perp \vec{B} \), so that \( \vec{q}, B \) and \( \vec{s}_j \) are coplanar.
The composite two-particle function \( \Gamma(\vec{E}, \vec{r}_1, \vec{r}_2) \) given by (83) has the multiple scattering expansion:

\[
\Gamma(\vec{E}, \vec{r}_1, \vec{r}_2) = \Gamma(\vec{E}, \vec{r}_1) + \Gamma(\vec{E}, \vec{r}_2) - \Gamma(\vec{E}, \vec{r}_1) \Gamma(\vec{E}, \vec{r}_2)
\]

where

\[
\Gamma(\vec{E}, \vec{r}_j) = 1 - \left(1 - \frac{\vec{E} \cdot \vec{r}_j}{b}\right)^2 \eta
\]

On substitution from (84) in (82) we can express the Glauber amplitude in the form

\[
P(\vec{k} \rightarrow \vec{r}; \vec{q}) = P_1(\vec{q}) + P_2(\vec{q})
\]

where

\[
P_1(\vec{q}) = \left(\frac{i k}{\pi}\right) \int \psi^*(\vec{r}_1, \vec{r}_2) \Gamma(\vec{E}, \vec{r}_1) \psi_1(\vec{r}_1, \vec{r}_2) \exp(i \vec{q} \cdot \vec{b}) d^2 \vec{r}_1 d^2 \vec{r}_2
\]

\[
P_2(\vec{q}) = -\left(\frac{i k}{2\pi}\right) \int \psi^*(\vec{r}_1, \vec{r}_2) \Gamma(\vec{E}, \vec{r}_1) \Gamma(\vec{E}, \vec{r}_2) \psi_1(\vec{r}_1, \vec{r}_2) \exp(i \vec{q} \cdot \vec{b}) d^2 \vec{r}_1 d^2 \vec{r}_2
\]

Here \( P_1(\vec{q}) \) is called the single-scattering amplitude. The target He atom may be considered to consist of two hydrogenic atoms, each being composed of an electron and a unit positive charge at the nucleus, and thereby taking part in scattering as a single target. The sum of such contributions is contained in \( P_1(\vec{q}) \). The remaining part \( P_2(\vec{q}) \) of the total amplitude may then be interpreted as the double-scattering amplitude.
5.5. Reduction of the single-scattering amplitude for $1^S \rightarrow 2^S$ transition

We obtain from Eqs. (75) - (73)

$$
\Psi_{2S}^*(\vec{r}_1, \vec{r}_2) \Psi_{1S}^1(\vec{r}_1, \vec{r}_2) = \sum_j c_j \left[ X_j(\vec{r}_1, \vec{r}_2) + X_j(\vec{r}_2, \vec{r}_1) \right], \quad (89)
$$

with

$$
X_j(u, v) = u^{\tau_j} v^{p_j} \exp(-\alpha_j u - \beta_j v), \quad (90)
$$

where $\alpha_j$ and $\beta_j$ involve wave-function parameters and $\tau_j$ and $p_j$ take integral values.

On substitution for $\Psi_{2S}^*(\vec{r}_1, \vec{r}_2) \Psi_{1S}^1(\vec{r}_1, \vec{r}_2)$ from Eq. (89) in (87), we obtain the single-scattering amplitude for $1^S \rightarrow 2^S$ transition as

$$
F_{11}^S(q) = \frac{ik}{\pi} \sum_j c_j F_{11}^S(j, q), \quad (91)
$$

where

$$
F_{11}^S(j, q) = \sum_j \left[ X_j(\vec{r}_1, \vec{r}_2) + X_j(\vec{r}_2, \vec{r}_1) \right] \Gamma(\vec{b}, \vec{r}_1) \exp(iq \cdot \vec{b}) \, d^2b \, dr_1 \, dr_2. \quad (92)
$$

Using the expression (90) in Eq. (92), we can express $F_{11}^S(q)$ as

$$
F_{11}^S(j, q) = A(\beta_j, p_j) D(\vec{q}, \alpha_j, \tau_j) + A(\alpha_j, \tau_j) D(\vec{q}, \beta_j, p_j), \quad (93)
$$

where

$$
A(x, y) = \frac{\int \exp(-xr) x^y \, dx}{x^y + 3}, \quad (94)
$$

and

$$
D(q, x, y) = \int \exp(iq \cdot \vec{b}) \exp(-xr) r^y \, \Gamma(\vec{b}, \vec{r}) \, d^2b \, dr. \quad (95)
$$
We now introduce the generating function
\[ I^S_1(\vec{q}, x) = \int \exp(iq.b) \frac{\exp(-xr)}{r} d^2b \, dr, \quad (96) \]
in terms of which \( D(\vec{q}, x, y) \) becomes
\[ D(\vec{q}, x, y) = \left( \frac{\partial}{\partial x} \right)^{y+1} I^S_1(\vec{q}, x). \quad (97) \]

The function \( I^S_1(\vec{q}, x) \) can be reduced to a closed form following the procedure employed by Thomas and Gerjuoy for \( l_s \rightarrow n_s \) transitions in hydrogen giving
\[ I^S_1(\vec{q}, x) = -16 \pi^2 i^{l+2} \Gamma(1+\eta) \Gamma(1-\eta) q^{2\eta-2} x^{-2\eta-2} \]
\[ \frac{2F_1(1-\eta, l-\eta; 1; -x^2/q^2)}. \quad (98) \]

Combining Eqs. (91), (93), and (97), we obtain for the single-scattering amplitude for \( l^1S \rightarrow 2^1S \) transition in He
\[ f^S_1(q) = \sum_j c_j \left[ A(\beta_j, \rho_j) \left( -\frac{\partial}{\partial \alpha_j} \right)^{t_j+1} I^S_1(\vec{q}, \alpha_j) \right. \]
\[ + A(\alpha_j, \tau_j) \left( -\frac{\partial}{\partial \beta_j} \right)^{l_j+1} I^S_1(\vec{q}, \beta_j) \left]. \quad (99) \]

\( A(x, y) \) and \( I^S_1(\vec{q}, x) \) being given by Eqs. (94) and (98), respectively.

5.6. Reduction of the double-scattering amplitude for \( l^1S \rightarrow 2^1S \) transition

For evaluating the double scattering amplitude \( f^S_2(q) \) for \( l^1S \rightarrow 2^1S \) transition, we substitute from Eq. (89) in Eq. (88), obtaining
\[ f^S_2(q) = -\frac{ik}{\pi} \sum_j c_j f^S_2(q), \quad (100) \]
Performing the azimuthal integration on the \( b \) plane, we have

\[
F_{2,1}^{S}(q) = 2\pi \int_{0}^{\infty} db \, b \, J_{0}(qb) R_{0}(\alpha_{j}, t_{j}, b) R_{0}(\beta_{j}, p_{j}, b),
\]

where

\[
R_{0}(x, y, b) = \int \exp(-xr) \Gamma(x, r) dr.
\]

We now introduce the generating function

\[
\Gamma_{0}(x, \eta, b) = \left( -\frac{\partial}{\partial x} \right)^{y+1} \Gamma_{0}(x, \eta, b).
\]

The function \( \Gamma_{0}(x, \eta, b) \) can be reduced as in Ref. 37 to give

\[
\Gamma_{0}(x, \eta, b) = -16\pi \eta^{2} K_{0}(x, \eta, b),
\]

where \( K_{0}(x, \eta, b) \) is given by

\[
K_{0}(x, \eta, b) = x^{-2}(ixb)^{-2i\eta} L_{2i\eta-1, 0}(ixb),
\]

the function \( L_{\mu, \nu}(iu) \) being a modified version of the Lommel function introduced by Thomas and Chan.\(^{37}\) The double-scattering amplitude for \( 1^{1}S \rightarrow 2^{1}S \) transition obtained by combining Eqs. (100), (102), and (105) - (107) can be written as

\[
F_{2,1}^{S}(q) = -512\pi^{2} \eta^{4} k_{1} \int_{0}^{\infty} db \, b \, J_{0}(qb) \sum_{j} \frac{C_{j}}{t_{j}} \left[ \left( -\frac{\partial}{\partial \alpha_{j}} \right)^{t_{j}+1} K_{0}(\alpha_{j}, \eta, b) \right] \left( -\frac{\partial}{\partial \beta_{j}} \right)^{p_{j}+1} K_{0}(\beta_{j}, \eta, b).
\]
5.7. Reduction of the single-scattering amplitude for $1^S \rightarrow n^P$ transitions

We obtain from Eqs. (75) and (79):

$$\psi_{n^P}^{1S}(\vec{r}_1, \vec{r}_2) \psi_{1S}^{n^P}(\vec{r}_1, \vec{r}_2) = \sum_j c_j' \left[ X_j(\vec{r}_1, \vec{r}_2) + X_j(\vec{r}_2, \vec{r}_1) \right], \quad (109)$$

with

$$X_j(u, v) = u \, j \, P_j \exp(-u \, \beta_j \, v) P_m^{\alpha}(\cos \theta) \exp(i m \delta) \quad (110)$$

where $P_m^{\alpha}$ represents the usual Legendre polynomial of order $(1-m)$, other notations being the same as in section 5.5. Using the expression (109) in Eq. (87), we have

$$\mathcal{F}_1(q) = \frac{ik}{\pi} \sum_j c_j' \mathcal{F}_{1j}(q), \quad (111)$$

where

$$\mathcal{F}_{1j}(q) = \frac{1}{\pi} \left[ X_j(\vec{r}_1, \vec{r}_2) + X_j(\vec{r}_2, \vec{r}_1) \right] \Gamma(\vec{r}_1, \vec{r}_2) \exp(iq \cdot \vec{r}) d^2b \, dr_1 \, dr_2. \quad (112)$$

The angular part of $\vec{r}_2$ in $X_j(\vec{r}_2, \vec{r}_1)$ makes the integral involving the second term in Eq. (112) vanish. Whence on substitution from Eq. (110), the $\vec{r}_2$ integral can be readily evaluated. This gives

$$\mathcal{F}_{1j}(q) = A(\beta_j, p_j) G(q, x, y), \quad (113)$$

where

$$G(q, x, y) = \int \exp(-\lambda r) \mathcal{Y} \Gamma(\vec{r}, \vec{r}) \exp(iq \cdot \vec{r}) P_m^{\alpha}(\cos \theta) \exp(-im \delta) d^2b \, dr, \quad (114)$$

and

$$A(\lambda, y) = \int \exp(-\lambda r) r^y dr = 4\pi \frac{y+2)!}{\lambda^{y+3}} \quad (115)$$
G(q, x, y) can be expressed in terms of a generating function
I^P_1(q, x) as

\[ G(q, x, y) = \left( -\frac{\partial}{\partial x} \right)^y I^P_1(q, x) , \]  

(116)

with

\[ I^P_1(q, x) = \int \frac{\exp(-xr)}{r} \Gamma(\beta, \beta) \exp(iq \cdot \beta) \exp(-im\phi) d^2 b \ dr . \]  

(117)

The function I^P_1(q, x) is similar to the generating function used by Thomas and Gerjuoy for 1s \rightarrow np transitions in hydrogen and can be reduced to a closed form following their procedure. This gives

\[ I^P_1(q, x) = -2\pi \eta \exp(-im\phi) q^{-3+2i\eta} x^{-2-2i\eta} \Gamma(1+i\eta) \Gamma(2-i\eta) \]  

\[ \left[ 2F_1(2-i\eta, 1-i\eta, 2, -x^2/q^2)-(1-i\eta) \right]_2F_1(2-i\eta, 2-i\eta, 2, -x^2/q^2) \]  

(118)

Combining Eqs. (111), (113), and (118), we have for the single-scattering part of the Glauber amplitude:

\[ F^P_1(q) = \frac{ik}{n} \sum_j c_j A(\beta_j, p_j) \left( -\frac{\partial}{\partial x_j} \right)^j I^P_1(q, \alpha_j) , \]  

(119)

A(\lambda, y) and I^P_1(q, x) being given by (115) and (118) respectively.

5.8. Reduction of the double-scattering amplitude for 1s \rightarrow np transitions

The double-scattering amplitude is obtained by substitution from Eqs. (109) and (110) in Eq. (88) as

\[ F^P_2(q) = -\frac{ik}{n} \sum_j c'_j F^P_2j(q) , \]  

(120)

with

\[ F^P_2j(q) = \int x_j(r_1, r_2) \Gamma(\beta, \beta) \Gamma(\beta, \beta) \exp(1q \cdot \beta) d^2 b \ dr_1 dr_2 . \]  

(121)
Performing the azimuthal integration on the \( \hat{z} \)-plane, we obtain
\[
R_{2j}^*(q) = 2\pi i \exp(-im\phi_q) \int_0^\infty db \, b J_1(qb) R_0(\beta_j, \eta_j, b) R_1(\alpha_j, t_j, b), \tag{122}
\]

with
\[
R_1(x, y, b) = \int \exp(-xr) \gamma(\hat{z}, \hat{r}) P^m_1(\cos\theta) \exp(-im\phi) dr. \tag{123}
\]

We now introduce the generating function
\[
\Gamma_1(x, \eta, b) = \int \frac{\exp(-xr)}{r} \gamma(\hat{z}, \hat{r}) P^m_1(\cos\theta) \exp(-im\phi) dr, \tag{124}
\]
in terms of which \( R_1(x, y, b) \) becomes
\[
R_1(x, y, b) = \left(- \frac{\partial}{\partial x}\right)^y \Gamma_1(x, \eta, b). \tag{125}
\]
The generating function \( \Gamma_1(x, \eta, b) \) can be reduced as in Ref.37 to give
\[
\Gamma_1(x, \eta, b) = -16\pi \left[(1+i\eta) K_1(x, \eta, b) + \eta K_0(x, \eta, b)\right], \tag{126}
\]
where \( K_1(x, \eta, b) \) is defined by
\[
K_1(x, \eta, b) = x^{-3} b^{-1} (ixb)^{-2m} J_{2m-1}(ixb). \tag{127}
\]
Here \( J_{\mu,\nu}(iu) \) is the modified Lommel function introduced by Thomas and Char.37

The double-scattering amplitude for \( ^1S \rightarrow n^3P \) transitions are obtained by combining Eqs. (120), (122), (125), and (126):
\[
R^F_2(q) = 512\pi^2 \eta^3 k_1 \exp(-im\phi_q) \int_0^\infty db \, b J_1(qb) \sum_{j=1}^\infty C_j \left[ \left(-\frac{\partial}{\partial \beta_j}\right) \left(-\frac{\partial}{\partial \eta_j}\right)^{p_j+1} K_0(\beta_j, \eta_j, b) \right] \left[ \left(-\frac{\partial}{\partial t_j}\right) \left(1+i\eta\right) K_1(\alpha_j, \eta_j, b) + \eta K_0(\alpha_j, \eta_j, b) \right]. \tag{128}
\]
5.9. Differential cross sections:

The center-of-mass differential cross section for \(^1S\rightarrow^2S\) excitation of helium atom is given by

\[
\frac{d\sigma(1^S \rightarrow 2^S)}{d\Omega} = \frac{k_F}{k_1} |F(1^S \rightarrow 2^S ; \frac{\mathbf{q}}{\lambda})|^2
\]  

(129)

Similarly, the differential cross section in the center-of-mass system for the \(^1S\rightarrow^1P\) transitions is obtained by summing over the contribution from the various magnetic sub-levels:

\[
\frac{d\sigma(1^S \rightarrow 1^P)}{d\Omega} = \frac{k_F}{k_1} \sum_{m} |F(1^S \rightarrow 1^P_m ; \frac{\mathbf{q}}{\lambda})|^2
\]  

(130)

Since \(F(1^S \rightarrow 1^P_0 ; \frac{\mathbf{q}}{\lambda}) = 0\)

(131)

and

\[ |F(1^S \rightarrow 1^P_m ; \frac{\mathbf{q}}{\lambda})| = |F(1^S \rightarrow 1^P_{-m} ; \frac{\mathbf{q}}{\lambda})| \]

(132)

we have

\[
\frac{d\sigma(1^S \rightarrow 1^P)}{d\Omega} = 2 \frac{k_F}{k_1} |F(1^S \rightarrow 1^P_1 ; \frac{\mathbf{q}}{\lambda})|^2
\]  

(133)

The total cross sections can be obtained as usual.

5.10. Numerical procedure:

Evaluation of \(K_0(x, \eta, b), K_1(x, \eta, b)\), and their derivatives

The function \(J_\mu(\lambda u)\) in \(K_0(x, \eta, b)\) has been expressed by Thomas and Chan\(^37\) as a combination of two hypergeometric functions. Direct evaluation of these functions is easy for small values of
the argument \( u \). For large values of the argument \( u \), the computations are to be performed entirely in double precision for an accurate evaluation of \( \phi_{\mu}(iu) \), because the number of terms required for the convergence of each of the two hypergeometric series become very large. However, an asymptotic expansion for \( \phi_{\mu}(iu) \) has been used by Thomas and Chan\(^{37} \) for sufficiently large values of \( u \).

We have followed the numerical procedure of Sur et al.\(^{83,84} \) for the evaluation of \( R_0(x, \eta, b) \), \( R_1(x, \eta, b) \) and their various derivatives required for the evaluation of double scattering amplitudes in Eqs. (108) and (128). This method has the advantage that only normal precision is required during computation and is applicable for a wide range of values of \( u = xb \), except for very small values. At such small values, the direct series for \( \phi_{\mu}(iu) \) [Eq. A(6) of Ref. 37] has been used to compute \( R_0 \) and \( R_1 \) and their various derivatives by the help of Eqs. (107) and (27).

We find the following expressions for \( R_0 \), \( R_1 \) as:

\[
R_0(x, \eta, b) = A(\eta) x^{-2} \left[ M_1(\eta, u) + M_2(\eta, u) \right] \\
R_1(x, \eta, b) = B(\eta) x^{-2} \left[ M_1'(\eta, u) + M_2'(\eta, u) \right] \\
A(\eta) = -2^{2i\eta} \frac{\Gamma(i\eta)}{\Gamma(1-i\eta)} \\
B(\eta) = \frac{1}{2} 2^{2i\eta} \frac{\Gamma(i\eta)}{\Gamma(-i\eta)} \\
u = xb \\
M_1(\eta, u) = B(\eta) \sum_{K=0}^{6} a_K G_{K,1}(\eta, v) \\
M_1'(\eta, u) = \frac{1}{2} t_0^{-2i\eta} \\
M_2(\eta, u) = B(\eta) \sum_{K=0}^{6} a_K G_{K,1}(\eta, v) \\
M_2'(\eta, u) = \frac{1}{2} t_0^{-2i\eta} \]
\[ v = (u/t_o)^2 \tag{141} \]

\[ G_{p,m}(\eta, v) = \int \frac{dy}{(y+v)^m} \tag{142} \]

\[ M_2(\eta, u) = \frac{1}{2} \left[ F_+(\eta, u, 1) + F_-(\eta, u, 1) \right] \tag{143} \]

\[ F_\pm(\eta, u, s) = \mp i \exp \left[ \mp i(t_o + d_o) \right] \int \frac{dz \exp(-z) N_\pm(z, s)}{(t^2 + u^2)^{s/2}} \tag{144} \]

\[ N_\pm(z, s) = \frac{t^{2s} - 1/2}{(t^2 + u^2)^s} f(t) \exp \left[ \pm i \sum_{K=1}^{6} g_K \left( \frac{t^2}{t_o} \right)^K \right] \tag{145} \]

\[ M'_1 = B'(\eta) \sum_{K=0}^{6} a'_K G_{K+1}(\eta, v) \tag{146} \]

\[ M'_2 = \frac{1}{2} \left[ F'_+(\eta, u, 1) + F'_-(\eta, u, 1) \right] \tag{147} \]

\[ F'_\pm(\eta, u, s) = \mp i \exp \left[ \mp i(t_o + g_o) \right] \int \frac{dz \exp(-z) N'_\pm(z, s)}{(t^2 + u^2)^s} \tag{148} \]

\[ N'_\pm(z, s) = \frac{t^{2s} - 1/2}{(t^2 + u^2)^s} f(t) \exp \left[ \mp i \sum_{K=1}^{6} g_K \left( \frac{t^2}{t_o} \right)^K \right] \tag{149} \]

\[ f(t) = \sum_{K=0}^{6} b_K \left( \frac{t^2}{t_o} \right)^K + O(10^{-8}) \tag{150} \]

\[ f'(t) = \sum_{K=0}^{6} a_K \left( \frac{t^2}{t_o} \right)^K + O(10^{-8}) \tag{151} \]

where \( t_o = 3 \) and \( a_K, b_K, d_K, a'_K \), and \( g_K \) being numerical constants. The integrals in Eqs. (144) and (148) have been evaluated numerically by the Gauss-Laguerre quadrature method, whereas Gauss-Legendre quadrature method has been used for Eq. (142).

The double-scattering amplitude requires derivatives of various orders of \( M' \) and \( M'_1 \) and can be obtained by using the relation

\[ \left( \frac{d}{dv} \right)^L G_{p,m} = (-1)^L (m)_L G_{p, m+L} \tag{152} \]

\[ G_{p-1,m-1} = \frac{1}{(1+\nu)^{m-1}} \left[ (m-1) G_{p,m} + \frac{1}{(1+\nu)^{m-1}} \right] \tag{153} \]
Derivatives of $F^\pm$ and $F^\pm$, and hence of $M_2$ and $M'_2$, can be obtained by using the recurrence relation

\begin{align}
\left(\frac{1}{x} \frac{d}{dx}\right)^{s} F^\pm (\eta, x, s) &= (-2)^{s} (s)_{L} F^\pm (\eta, x, s + L), \\
\left(\frac{1}{x} \frac{d}{dx}\right)^{s} F^\pm (\eta, x, s) &= (-2)^{s} (s)_{L} F^\pm (\eta, x, s + L)
\end{align}

where $s, p, \text{and } m$ are integers.

5.11. Results and discussions

5.11.1. Differential cross sections for $1S \rightarrow 2S$ excitations

Our results of the center-of-mass differential cross sections in the Glauber approximation for $1S \rightarrow 2S$ excitations of He at 25- and 100-keV proton energies are plotted in Figs. 5.1 and 5.2, respectively. We have also included our FBA calculations in these figures. The only other available calculation is due to Flannery and McCann, who have used multistate eikonal approximations and the FBA method in their work. These results are also compared in Figs. 5.1 and 5.2 with our FBA and Glauber results.

From a comparison of the curves for the present full Glauber (QA) and the corresponding single-scattering Glauber (SSG) results of Figs. 5.1 and 5.2, it is seen that the relative...
Fig. 5.1. Angular differential cross sections (in the center-of-mass system) for 2\(^1\)S excitation of atomic helium by the impact of 25-keV (laboratory energy) protons. Present calculations: (----), Glauber approximation (GA); (—•—), single scattering Glauber (SSG) approximation; (----), first Born approximation (FBA). Calculation of Flannery and McCann (Ref. 10); (----), two-state eikonal (T); (— — —), four-state eikonal (F).
Fig. 5.2. Angular differential cross sections (in the center-of-mass system) for $2^1S$ excitation of atomic helium by the impact of 100-keV (laboratory energy) protons. Present calculations: (----) Glauber approximation (GA); (—*—), Single scattering Glauber (SSG) approximation; (——), First Born approximation (FBA). Calculation of Flannery and McCann (Ref.10): ('———), two-state eikonal (T); (----), four-state eikonal (F).
contribution of the double-scattering term \( F_2 \) in the Glauber scattering amplitude \( F \) is appreciable at low energies and inclusion of this term lowers the GA cross sections significantly at small scattering angles from the corresponding SSG results. Since the contribution to the integrated Glauber cross sections comes predominately from the small angle region, this causes a consequent reduction of the total Glauber cross sections from the corresponding single scattering results. With the increase of energy, the GA and SSG curves, however, approach each other in the forward direction, explaining the agreement of the integrated GA and SSG results at high energies. The common agreement of both of these Glauber results also with the integrated FBA results in the high-energy region can be similarly explained. At intermediate energies, however, the behaviours of the differential FBA cross sections are completely different from those of the GA or SSG results, as may be seen from Fig. 5.1.

An interesting feature of the differential (GA) cross sections at 25 keV is the minimum occurring at a scattering angle of 0.026° in Fig. 5.1. At a proton energy of 50 keV also, this minimum in the differential cross sections (not presented here) is observable and occurs at 0.0344°. With increasing energy, however, this minimum flattens and the GA curve at 100 keV in Fig. 5.2 shows no such minimum. The overall nature of the Glauber differential cross sections of Figs. 5.1 and 5.2, especially, the minima in cross-sections-scattering-angle curve at lower energies, resemble the corresponding trends obtained in
the $^6$H-$^6$H differential Glauber cross sections for $ls$-$ns$ excitations. The SSG curve of Fig. 5.1 also shows a broad minimum near $0.04^\circ$.

As regards the relative behaviour of our present Glauber differential cross sections and the multistate eikonal results of Flannery and McCann, Figs. 5.1 and 5.2 show that there is appreciable difference between the predictions of the two calculations for small-angle scattering. In particular, the small-angle minima occurring in the Glauber curves of Fig. 5.1 being absent from the eikonal curves, the absolute values of the two sets of results differ by orders of magnitude near the scattering angles concerned. In the forward direction also, the absolute values of the Glauber and the multistate eikonal cross sections show a large difference. It should be noted, however, that the two eikonal calculations themselves appreciably differ from each other in absolute values in the forward direction—the four-state (F) results overestimating the two-state (T) ones almost by an order. The agreement between the results of the present Glauber and the multistate eikonal calculations improves with the increase of incident proton energy when, as mentioned already, the minima in the Glauber curves gradually flatten. The angular dependence of the Glauber and the eikonal cross sections becomes similar also with the increase of the scattering angle at any given proton energy. We would like to mention here that although the wave functions used in the present work are different from those employed in the
eikonal calculations, our present FBA cross sections agree closely with those of Flannery and McCann. This indicates that the difference in absolute values as well as in the angular dependence between the two sets of results as discussed above may not be attributed only to the difference in the wave functions employed. Hence the full elucidation of the matter should await an absolute measurement of the \( l^1S \rightarrow 2^1S \) differential cross sections.

There is no experimental measurement of the differential cross sections for \( l^1S \rightarrow 2^1S \) excitation in \( H^+ - \text{He} \) collisions. However, Byron and Joachain have made a detailed analysis of the Glauber and Born series for electron-atom collisions. They have shown that whereas the FBA breaks down outside the small angle region for inelastic s-s transitions, the Glauber method can give correctly the leading term of the differential cross sections \([0(k_1^{-2})]\) obtained from the Born series expansion. The eikonal Born series (EBS) method, which has been proposed by Byron and Joachain for improving over the Glauber approximation, also predicts similar results. These analysis are also justified by actual comparison of the theory and experiment for \( l^1S \rightarrow 2^1S \) excitation in e-\( \text{He} \) collisions, when the Glauber and the EBS results are found to agree well with the measured data over the entire angular range. Considering all these facts we may expect that our present \( H^+ - \text{He} \) Glauber differential results of Figs. 5.1 and 5.2 are reliable so far as the angular dependence of the cross sections are concerned.
5.11.2. Differential cross sections for $1^1S \rightarrow n^1P$ transitions

In Figs. 5.3 and 5.4, we present our Glauber (GA), single-scattering Glauber (SSG), and FBA differential cross sections for $1^1S \rightarrow 2^1P$ excitations of He by 25- and 100-keV protons at various scattering angles and compare the results with the two-state and four-state eikonal calculations of Flannery and McCann. It is worth mentioning that our present results agree closely with the earlier Glauber calculations of Chan and Chang employing a different set of He wave functions.

As in the case of $1^1S \rightarrow 2^1S$ transition, the effect of the double scattering term ($F_2$) on the full Glauber amplitude ($F$) for $1^1S \rightarrow 2^1P$ transition also is found to be appreciable at intermediate energies for the range of scattering angles shown and lowers the GA cross sections from the corresponding SSG results at small scattering angles. With increasing energy, however, the GA and SSG curves of Figs. 5.3 and 5.4 approach each other in the forward direction. This corresponds to the agreement of both the results at high energies where the contribution of $F_2$ is negligible. The FBA cross sections agree with the GA and SSG results at high energies in the forward direction.

At 25 keV, although the Glauber (GA and SSG) and the multi-state eikonal (T and P) differential cross sections show some spread in absolute values, the angular dependence of these results in the range of scattering angles shown in Fig. 5.3 are in reasonable
Fig. 5.3. Angular differential cross sections (in the center-of-mass system) for $2^1P$ excitation of atomic helium by the impact of 25-keV (laboratory energy) protons.

Present calculations: (-----), Glauber approximation (GA); (- - -), single-scattering Glauber (SSG) approximation; (---), first Born approximation (FBA).

Calculation of Flannery and McCann (Ref.10): (-----), two-state eikonal (T); (-----), four-state eikonal (F).
Fig. 5.4. Angular differential cross sections (in the center-of-mass system) for $2^\text{1}_\text{p}$ excitation of atomic helium by the impact of 100-keV (laboratory energy) protons.

Present calculations: (---), Glauber approximation (GA); (--), single scattering Glauber (SSG) approximation; (---), first Born approximation (FBA).

Calculation of Flannery and McCann (Ref. 10): (· · ·), two- and four-state eikonal (T, F).
agreement. The FBA cross sections, however, considerably overestimate the other theoretical predictions. At 100 keV, on the other hand, the behaviour of the Glauber and the eikonal curves of Fig. 5.4 (the $T$ and $F$ curves are now coincident) appreciably differ at larger scattering angles. The FBA results are now in better absolute agreement with the Glauber results in the forward direction, but fall much more quickly in complete disagreement with the GA and SSG predictions as the scattering angle increases.

Our results of the center-of-mass differential cross sections obtained by applying the Glauber approximation for $1^1S \rightarrow 1^1P$ and $1^1S \rightarrow 4^1P$ excitations of He at 50- and 100-keV proton energies are displayed, respectively, in Figs. 5.5 and 5.6. Since no other theoretical or experimental results are available, we display our calculated results in the SSG and FBA methods for comparison in the above figures. It appears that for small values of scattering angles considered, the GA curve lies below the SSG curve. This demonstrates that the relative contribution of the double scattering term ($F_2$) in the Glauber scattering amplitude is appreciable at small scattering angles. The overall effect of $F_2$ is to reduce the integrated GA cross sections from the corresponding SSG results at intermediate energies. With the increase of energy, the GA and SSG curves are seen to approach each other in the forward direction, although they differ appreciably at larger angles. Since the contribution to the integrated GA cross sections come predominantly from the small angle region, this explains the reason for the agreement of the integrated GA and
Fig. 5.5 Angular differential cross sections (in the centre-of-mass system) for $3^1P$ excitation of atomic helium by the impact of 50- and 100-keV (laboratory energy) protons. Present calculations: (---), Glauber approximation (GA); (-----), single-scattering Glauber (SSG) approximation; (---), first Born approximation (FBA).
Fig. 5.6. Angular differential cross sections (in the center-of-mass system) for $^4\!_{1}P$ excitation of atomic helium by the impact of 50- and 100-keV (laboratory energy) protons.

Present calculations: (---), Glauber approximation (GA); (-----), single scattering Glauber (SSG) approximation; (——-), first Born approximation (FBA).
SSG results at high energies. The common agreement of both of these GA results also with the integrated FBA results in the high-energy region can be similarly explained. However, as the angle increases, the FBA results fall quite rapidly in complete disagreement with the GA and SSG predictions. For scattering angles from $0^\circ$ to $0.15^\circ$, we find that differential-cross-section ratios for the $3^1P$ and $4^1P$ levels are in the range from 2.0 to 2.6. Since the most important contribution to the total cross section comes from the forward direction, the $n^{-3}$ rule for the total cross section predicts a value of $4^3/3^3 \approx 2.4$ for the ratio of cross sections in the present case and is hence approximately valid.

### 5.11.3. Excitation of the n = 2 levels

We now come to the case of sum of the differential cross sections for proton impact excitation of He of the n = 2 levels. As mentioned earlier, the contribution from the triplet states in the excitation of n = 2 levels of He is negligible because of spin conservation. Hence we compare the sum of our calculated $1^1S \rightarrow 2^1S$ and $1^1S \rightarrow 2^1P$ differential cross sections with the corresponding eikonal results of Flannery and McCann, and with the measurements of Park et al. at 25- and 100-keV proton energies. At 50 keV, the multistate eikonal results are not available.

In the range of scattering angles considered, the present GA and SSG curves of the differential cross sections displayed in Figs. 5.7 and 5.8 do not differ much in shape and both agree qualitatively with the observed angular distributions.
Fig. 5.7. Angular differential cross sections for 25- and 50-keV (laboratory energy) proton impact excitation of atomic helium to the n = 2 state. Present calculations: (----), Glauber approximation (GA); (-----), single scattering Glauber (SSG) approximation; (----), first Born approximation (FBA). Calculation of Flannery and McCann (Ref. 10): (------), four state eikonal (F). Experiment: •, Park et al. (Ref. 22)
Fig. 5.8. Angular differential cross sections for 100-keV (laboratory energy) proton impact excitation of atomic helium to the n=2 state.

Present calculations: (---), Glauber approximation (GA); (-----), single-scattering Glauber (SSG), approximation; (-----), first-Born approximation (FBA).

Calculation of Flannery and McCann (Ref. 10): (---), four-state eikonal (F).

Experiment: O, Park et al. (Ref. 22).
The present GA results in general underestimate the observed cross sections. But the absolute agreement between the two results in the forward direction improves at higher energies and is within the accuracy of measurement at 50 and 100 keV. On the other hand, although the present FBA results show some agreement with the measurement in the forward direction at higher energies, they fail to predict the observed angular distributions. The four-state (F) eikonal results of Flannery and McCann\textsuperscript{10} are in excellent agreement with the experiment in both curve shape and absolute magnitude.

It is worth noting here that Park et al.\textsuperscript{22} make no estimate of the possible systematic errors in their measurement, which may arise chiefly from the data-analysis method and also from the absolute measurement of the interaction length and pressure. As a result, the shapes of the curves are more reliable, according to the authors, than the absolute magnitude and the authors comment that the excellent agreement of their measured cross sections with the four-state eikonal results is perhaps fortuitous, especially with respect to magnitude. We may also note from the work of Park et al.\textsuperscript{22} that their integrated results which correspond to the measured angular differential cross sections overestimate the earlier calculations and measurements of the total cross sections. In view of these facts, the underestimation shown by our differential Glauber (GA) results of Figs. 5.7 and 5.8, in comparison with the eikonal calculations and measurement, is probably not serious. The reasonable agreement of the GA results in the small angle region thus shows the usefulness of the Glauber theory.
Finally, we note that the above facts in general conform to the earlier results with electron impact, as also to the theoretical considerations of Byron and Joachain.

5.11.4. Total excitation cross sections for \( l^1S \rightarrow n^1P \) transitions

Our results for the total Glauber (GA) cross sections for \( l^1S \rightarrow n^1P \) \((n = 2, 3, 4)\) excitations of He by proton impact are displayed in Figs. 5.9 - 5.11 along with the available theoretical and experimental results. The single-scattering \( (F_1) \) and the double-scattering \( (F_2) \) parts of the Glauber amplitude have been calculated separately and the single-scattering Glauber (SSG) cross sections are included in these figures along with the total Glauber (GA) cross sections. We also include in the same figures the present FBA cross sections of the above mentioned transitions obtained by using the wave functions of Winter and Lin. We compare in tables 5.1 - 5.3 our results for the integrated cross sections obtained by applying the GA, SSG and FBA methods with the experimental data of Hippler and Schartner. In table 5.1, we also present the GA and SSG predicted cross sections for \( l^1S \rightarrow 2^1P \) excitation of He by proton impact calculated by Chan and Chang employing a different wave function from ours.

There is general agreement in various physical features predicted by both sets of results in table 5.1. The energy dependence of the cross sections in the present GA and SSG methods is similar to those of Chan and Chang. Peaks in our GA and SSG results.
Table 5.1. Comparison of total cross sections \(10^{-20} \text{ cm}^2\) in the present Born (FBA) and Glauber calculations (SSG and GA) with the earlier Glauber calculation and the experimental data for \(\text{H}^+\text{He}(1^1S)\rightarrow\text{H}^+\text{He}(2^1P)\).

| Energy keV | Present calculations | Calculation of Chan- and Ghang, Ref. 9 | Experimental results of Hippler and Schartner (Ref. 2)
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</table>

* SSG Single-scattering Glauber approximation
* GA Glauber approximation.
Table 5.2. Comparison of total cross sections \(10^{-20}\) cm\(^2\) in the present Born (FBA) and Glauber calculations (SSG and GA) with experiment for \(H^+He(1s) \rightarrow H^+He(3p)\).

<table>
<thead>
<tr>
<th>Energy (keV)</th>
<th>Present calculations</th>
<th>Experimental results of Hippler and Schartner (Ref. 20)</th>
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* SSG  Single-scattering Glauber approximation

GA  Glauber approximation
Table 5.3. Comparison of total cross sections (10^{-20} cm^2) in the present Born (FBA) and Glauber calculations (SSG and GA) with experiment for H^+He(1^S) \rightarrow H^+He(4^1P).

<table>
<thead>
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<th>Energy keV</th>
<th>Present calculations</th>
<th>Experimental results of Hippier and Schartner (Ref. 20)</th>
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* SSG Single-scattering Glauber approximation

* GA Glauber approximation
occur around the same energy values (120 and 100 keV, respectively) as in the calculation of Chan and Chang. It is found that the present GA and SSG results converge to each other at energies exceeding 225 keV approximately, whereas in the work of Chan and Chang this occurs around 150 keV. This difference may be attributed to the different set of approximate He wave functions used by the authors. From the tables it appears that the cross sections obtained by the present GA and FBA methods approach each other in the high-energy region, almost coinciding above the incident energy of 625 keV. Above these energies, the significant contribution to the cross sections comes from the single-scattering part \( F_1 \) of the Glauber amplitude \( F \). The effect of the double-scattering term \( F_2 \) in the Glauber amplitude \( F \) becomes appreciable at intermediate energies, where it substantially reduces the total Glauber cross sections from the corresponding SSG results.

In Fig. 5.9, we compare the present Glauber cross sections with some of the existing theoretical calculations and the experimental findings. From this figure, we find that like the existing other theoretical calculations, the present GA results give an excellent agreement with the observed data in the energy range 150–1000 keV. The GA results also agree well with the second order diagonalization method and both the results are in excellent agreement with the observed data for proton impact energies of 25–126 keV, whereas the nine-state impact parameter calculation of Van den Bos, though agree well
Fig. 5.9. Total cross sections for $^{1}S_{1} \rightarrow ^{2}P_{1}$ excitation of atomic helium by proton impact.

Theory: (-----), present Glauber approximation (GA); (------), present single-scattering Glauber (SSG) approximation; (....), present first Born approximation (FBA); (———), first Born approximation Bell et al., Ref.2; (—*—), second-order diagonalization method, Baye and Heenen, Ref.7; (------), second-order potential model, Begum et al., Ref.8; (———), nine-state close coupling approximation, Van den Bos, Ref. 5.

Experiment: •, Hippier and Schartner, Ref.20; 0, Park and Schowengerdt, Ref.21.
with the observed data always underestimate the theoretical results obtained by using the GA, second-order diagonalization and second-order potential methods. The present FBA cross sections agree well with the FBA results of Bell et al., obtained by using the many-parameter correlated wave functions. These results also exhibit an excellent agreement with the observed data for proton impact energies of 100 – 1000 keV. However, these FBA results fail to predict the energy dependence of the observed data in the intermediate energy region (E < 100 keV).

For proton impact excitation of He to the 3P state, the present GA results are displayed in Fig. 5.10 along with the existing theoretical and experimental results. An excellent agreement is obtained between the GA and other existing theoretical calculations for proton impact energies of 150 – 1000 keV. The present GA results are also in good agreement with the experimental findings of Van den Bos et al. and Thomas and Bent in the intermediate-energy region. However, in this energy region, the existing theoretical and observed results show an appreciable amount of variation among themselves in absolute values. Throughout the energy region considered, the GA, the second-order diagonalization method, and the coupled state calculations of Van den Bos can predict well the energy dependence of the observed data, whereas the FBA results fail to predict the energy dependence of the observed findings.
Fig. 5.10. Total cross sections for $^1S\rightarrow^3P$ excitation of atomic helium by proton impact.
Theory: (----), present Glauber approximation (GA); (-----), present single-scattering Glauber (SSG) approximation; (—-—), present first Born approximation (FBA); (-----), first Born approximation, Bell et al., Ref.2; (—-—), second-order diagonalization method, Baye and Heenen, Ref.7; (-----), three-state close coupling calculation of Van den Bos, Ref.5; (-----), six-state close coupling calculation of Van den Bos, Ref.5; (-----), nine-state close coupling calculation of Van den Bos, Ref.5.
Experiment: *, Hippler and Schartner, Ref.20; ., Scharman and Schartner, Ref.18; X, Thomas and Bent, Ref.15; ©, Van den Bos et al., Ref.16; 0, Denis et al., Ref.14; ✰, Dodd and Hughes, Ref.17.
In Fig. 5.11, we compare our results for the proton impact excitation of He to the $^1P$ state, with the existing theoretical calculations\textsuperscript{2,7} and the experimental findings.\textsuperscript{16,18,20} The present GA results always overestimate the observed data in the intermediate energy region, whereas the second-order diagonalization method\textsuperscript{7} agrees rather well with the experimental findings. Except the FBA method, the GA as well as the second-order diagonalization methods can predict well the functional dependence of the observed findings. The theoretical results obtained by applying the GA, FBA\textsuperscript{2} and second order diagonalization method\textsuperscript{7} give an excellent agreement with the experimental findings in the high-energy region.

The wide variation in absolute values among the experimental data, which persists in the intermediate and high-energy region and arises from using a variety of absolute calibration techniques in different measurements, indicates that no rigorous comparison of absolute values of cross sections obtained from various theories and experiment is very much meaningful. The theoretical results also show an appreciable amount of variation in the intermediate-energy region due to the different choice of approximate He wave functions by various authors. Under such circumstances, a reliable physical feature for comparison between theory and experiment is to study the ratio of cross sections $\sigma(n)/\sigma(n+1)$ with varying principal quantum number $n$ but of the same total angular momentum. In a comparison of the cross section ratios obtained by various experimental groups, the systematic errors in absolute calibration...
Fig. 5.11. Total cross sections for $^1S \rightarrow ^4P$ excitation of atomic helium by proton impact. 
Theory: (----), Present Glauber (GA) calculation; (-----), present single-scattering Glauber (SSG) calculation; (—---), present first Born (FBA) calculation; (----), second-order diagonalization calculation of Baye and Heenen (Ref.7); (------), first Born calculation of Bell et al (Ref.2).
Experiment: ∆, Hippier and Schartner (Ref.20); ○, Van den Bos et al. (Ref.16); •, Scharmann and Schartner (Ref.18).
should cancel out and hence should give similar values. To verify this, we compare in Table 5.4, the ratios of the present GA and FBA cross sections of different n values as obtained from the data of Van den Bos et al.\(^{16}\), Thomas and Bent,\(^{15}\) Denis et al.\(^{14}\), and Hippler and Schartner\(^{20}\), as also from the nine-state close-coupling calculation of Van den Bos\(^{5}\) and the second-order diagonalization calculation of Baye and Heenen\(^{7}\).

The asymptotic high-energy behaviour\(^{87-89}\) of the cross sections, which is proportional to \(n^{-3}\), are in close agreement with the present theoretical predictions. Except for the calculation of Baye and Heenen\(^{7}\), all other theoretical results show good agreement among themselves. On the other hand, the ratio of cross sections for \(\sigma(3^1P)/\sigma(4^1P)\) in different experimental data vary considerably. However, the latest measurements of Hippler and Schartner\(^{20}\) are in reasonable agreement with the calculations in a consistent manner, especially with the present calculation.

5.1.2. Conclusions

Differential cross sections for the proton-impact \(1^1S \rightarrow 2^1S\) and \(1^1S \rightarrow 2^1P\) excitations of atomic helium are calculated at, 25-, 50-, and 100-keV proton energies using the Glauber approximation. The sum of the results are compared with an earlier multistate eikonal calculation as well as with the very recent measurements
Table 5.4. Average ratio of cross sections $\sigma(n)/\sigma(n+1)$ at energies above 50 keV.

<table>
<thead>
<tr>
<th></th>
<th>Theory</th>
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<tbody>
<tr>
<td></td>
<td>Present</td>
<td>Present</td>
<td>Nine-state</td>
<td>Second order</td>
<td>n$^3$ law</td>
<td>Van den Bos</td>
<td>Thomas et al.</td>
<td>Denis and Bent et al.</td>
<td>Hippler and Schartner</td>
</tr>
<tr>
<td></td>
<td>Glabbor</td>
<td>first Born</td>
<td>close coupling</td>
<td>diagonalization</td>
<td>(Ref.5)</td>
<td>(Ref.6)</td>
<td>(Ref.15)</td>
<td>(Ref.14)</td>
<td>(Ref.20)</td>
</tr>
<tr>
<td>$\sigma(2^1p)$</td>
<td>3.7</td>
<td>3.8</td>
<td>3.3</td>
<td>4.3</td>
<td>3.4</td>
<td>3.9</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$\sigma(3^1p)$</td>
<td>9.0</td>
<td>8.9</td>
<td>11.3</td>
<td>8.0</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$\sigma(2^3p)$</td>
<td>2.4</td>
<td>2.4</td>
<td>2.6</td>
<td>2.4</td>
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<tr>
<td>$\sigma(4^3p)$</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>3.8</td>
<td>4.3</td>
<td>4.5</td>
<td>2.5</td>
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</table>

* Above 100 keV
** From 150 keV
of Park et al.\textsuperscript{22} The angular dependence of the cross sections predicted by the Glauber theory is in reasonable agreement with the eikonal results and the observed data.

For direct collisional excitation of ground state He by incident protons, a reliable physical feature for comparison between theory and experiment is the functional dependence of cross sections on energy rather than the absolute values of the cross sections. Furthermore, for excited states of He having same total angular momentum but varying principal quantum number, the average ratio of cross sections at high energy predicted by different theoretical data can show reasonable agreement among themselves, as also with the $n^{-3}$ law of cross sections. One may also observe from the work of Thomas\textsuperscript{1} that the cross section ratio furnishes a good consistency check on measured data for still higher $n$ states.

The present calculation for proton impact $2^1\text{P}$, $3^1\text{P}$ and $4^1\text{P}$ excitations of ground state He using the Glauber approximation conforms to the above observations. The energy dependence of the cross sections obtained by applying the Glauber approximation is predicted well from intermediate to high energy region. The Glauber method is also able to give a good estimate of the cross section ratios in agreement with other theories.
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