CHAPTER II

SPECTRAL DISTRIBUTION THEORY —— A REVIEW

2.1. EIGENVALUE DENSITY OF THE HAMILTONIAN IN LARGE SHELL MODEL SPACES AND ITS MOMENTS

The many-body spaces encountered in nuclear spectroscopic studies are, in principle of infinite dimensionality. But in practice one introduces the concept of effective operators and treats the problem in model spaces of finite dimensionality. The model spaces are generated by distributing m particles over N single particle states, belonging to spherical orbits of the coventional shell model (SM). This m particle model vector space (N;m) has dimensionality $d(m) = \binom{N}{m}$. For this space one can define an energy eigenvalue density $\rho(x)$ (or I(x) which is normalised to the total dimensionality d(m)) and the crude (M_p) and central (μ_p) moments by

$$\rho(x) = \langle\langle\delta(H-x)\rangle\rangle^m/d(m)$$

$$M_p = \int E^p \rho(E)\, dE = \langle H^p\rangle^m = d^{-1}(m)\, \langle\langle H^p\rangle\rangle^m$$

$$\mu_p = \int (E-M_1)^p \rho(E)\, dE = \langle(H-M_1)^p\rangle^m = d^{-1}(m)\langle\langle(H-M_1)^p\rangle\rangle^m \qquad (2.1\text{-}1)$$

In equation (2.1-1) and later throughout this thesis we use the notation that, for any operator A, $\langle\langle A\rangle\rangle^s$ and $\langle A\rangle^s$ denote respectively the trace and the average expectation value of the operator in the space s. It is now quite well established[1),2)] that the smoothed energy eigenvalue density

$\rho(E)$ (= $d^{-1}(m)$ $I(x)$) in large m-particle shell model space, is close to a Gaussian for almost all (1+2)-body realistic Hamiltonians when m is not too small. Thus in spectral distribution theory, for the smoothed densities one needs to know only the low order moments of the Hamiltonian. To construct the eigenvalue density $\rho(E)$ explicitly one needs to calculate the centroid ε and variance σ^2 of the Hamiltonian (these two define the Gaussian) and the third and fourth moments wherever practicable to take care of small departures from the Gaussian in finite spaces through an expansion of the density by the Edgeworth (EW), Gram-Charlier (GC) or a Cornish-Fisher form.

The fact that the eigenvalue distribution is close to a Gaussian, has been established both by numerical[1),2)] and analytical[3)] methods. Firstly, by constructing and diagonalising realistic (1+2)-body Hamiltonians in finite dimensional shell model spaces in sd and fp shell, one finds that the deviations, as measured by the low order shape parameters of the distribution (e.g. skewness and excess) are, in general, small. Also the smoothed forms of the exact shell model density is seen to be reproduced quite accurately by densities constructed using a few low order Hamiltonian moments going beyond the Gaussian form. Secondly, analytically, for the non-interacting case i.e. with 1-body Hamiltonians one can easily obtain the Gaussian form by direct use of the Central Limit Theorem (CLT)[2)] if one ignores "Pauli blocking" effects. Small departures from Gaussian due to Pauli blocking are also calculable. In the case of interacting particles, the complication arises from the fact that the energies here are not additive and the convolution argument breaks down. But it has been shown analytically in the framework of Random Matrix Theory (RMT) that the smoothed density $\rho(E)$ in the m-particle space of the k-body Hamiltonian averaged over an ensemble (Embedded Gaussian Orthogonal

Ensemble or EGOE) goes to a Gaussian asymptotically as the particle number (>>k) increases. Here one constructs[3] a k-body Hamiltonian with each (real) matrix element an independent random number chosen according to a Gaussian law for the off-diagonal matrix elements and another, with twice the variance, for the diagonal matrix elements. One then calculates ensemble averaged central moments or cumulants of order p and shows that in the asymptotic limit $m \longrightarrow \infty$, $N \longrightarrow \infty$ and $\frac{m}{N} \longrightarrow 0$, all the cumulants of order $p > 2$ vanish. This gives in the interacting particle case an extended CLT. It is also found that in the same limit , only a negligible fraction of the members of the ensemble give deviant densities.

The m-particle scalar space described above can be partitioned into subspaces according to different symmetries. The decomposition of the (N;m) space into subspaces is important for many reasons. On the one hand it itself carries a lot of structure information and on the other, sometimes it becomes necessary for improved accuracies in prediction. The most important decomposition is the configuration partitioning. Here one makes group of orbits out of single particle states such that $N \longrightarrow \tilde{N} \equiv (N_1, N_2, \ldots, N_l)$, where l is the number of orbits (we use spherical orbits of the conventional shell model) in the group and distributes m valence particles according to $m \longrightarrow \tilde{m} \equiv (m_1, m_2, \ldots, m_l)$ such that $N = \sum_{i=1,l} N_i$ and $m = \sum_{i=1,l} m_i$ [m_i is the number of particles in the ith orbit].

The dimension $\binom{N}{m} = \prod_i \binom{N_i}{m_i}$. For our purpose we shall use the following subspaces :

(a) scalar space, denoted by (m)

(b) scalar isospin space, denoted by (m,T) **or scalar–T**

(c) configuration isospin space, denoted by $(\tilde{m},T)$ (and also implicitly a space of the kind $(\tilde{m},\tilde{T})$).

(d) scalar proton-neutron space denoted by (m_p,m_n)

(e) configuration proton-neutron space denoted by $(\tilde{m}_p,\tilde{m}_n)$

For all these five cases, one may of course specify the total angular momentum value J or the configuration $\tilde{J}$ value. In such cases the evaluation of the traces becomes more involved.

The smoothed density $\rho(E)$ or I(E), under partitioning of the space decomposes similarly; for example, under configuration isospin partitioning $\rho(E) \longrightarrow \sum_{\tilde{m}} \rho_{\sim m,T}(E)$ and $I(E) \longrightarrow \sum_{\tilde{m}} I_{\sim m,T}(E) = \sum_{\tilde{m}} d(\tilde{m},T)\ \rho_{\sim m,T}(E)$, where $d(\tilde{m},T)$ is the dimensionality of the configuration isospin space $(\tilde{m},T)$. Some plausible arguments[4] using multivariate density distributions and their moments indicate that the asymptoticity of the density is also carried over to the symmetry partitioned subspaces. This is also seen by explicit construction of the densities in these subspaces when the dimensionalities are large.

2.2 POLYNOMIAL EXPANSION OF STRENGTH AND EXPECTATION VALUE

Let O be an excitation operator which induces transitions, $|E\rangle \xrightarrow{O} |E'\rangle$, where we label the states by their energies and where the initial states $|E\rangle$ span the initial model subspace of H and $|E'\rangle$ the final one (the two subspaces may or may not coincide). The microscopic strength is defined as the square of the off-diagonal matrix element of the transition operator O in the H-diagonal basis :

$$R(E,E') = \left|\langle E'|O|E\rangle\right|^2 \qquad (2.2\text{-}1)$$

which is the square of the expansion coefficient of the state $O|E\rangle$ in terms of the final Hamiltonian eigenstates $|E'\rangle$. Associated with each density $\rho(x)$, one can define a set of orthonormal polynomials $P_\mu(x)$ with $\rho(x)$ as

the weight function and is given by

$$\int P_\mu(x)\ P_\nu(x)\ \rho(x)\ dx = \delta_{\mu\nu}$$

$$\delta(x-y) = \rho(x) \sum_\mu P_\mu(x)\ P_\mu(y) \qquad (2.2\text{-}2)$$

These polynomials can be constructed explicitly in terms of the density moments M_p as given by equation (A-1) in Appendix A. The first two polynomials are $P_0(x) = 1$, $P_1 = (x-\varepsilon)/\sigma$ where $\varepsilon = M_1$ and $\sigma = (M_2-M_1)^{1/2}$ are the centroid and width of the distribution $\rho(x)$.

In spectral distribution theory one expresses[5] the microscopic strength given by (2.2-1) in terms of the polynomials $P_\mu(x)$, and gets

$$R(E,E') = [I'(E')\ I(E)]^{-1} \times \langle\langle O^+ \delta(H-E')O\ \delta(H-E)\rangle\rangle^s \qquad (2.2\text{-}3a)$$

$$= [dd']^{-1} \sum_{\mu,\nu} \langle\langle O^+ P'_\mu(H)\ OP_\nu(H)\rangle\rangle^s\ P'_\mu(E')P_\nu(E) \qquad (2.2\text{-}3b)$$

$$= d'^{-1} \sum_{\mu,\nu} \langle O^+ P'_\mu(H)\ OP_\nu(H)\rangle^s\ P'_\mu(E')P_\nu(E) \qquad (2.2\text{-}3c)$$

$$I'(E')\times R(E,E') = \rho'(E') \sum_{\mu,\nu} \langle O^+ P'_\mu(H)\ OP_\nu(H)\rangle^s\ P'_\mu(E')P_\nu(E) \qquad (2.2\text{-}4)$$

Integrating eq.(2.2-4) over E' we get

$$\int_{-\infty}^{\infty} I'(E')\ R(E,E')\ dE' = \sum_\nu \langle O^+ OP_\nu(H)\rangle^s P_\nu(E) \qquad (2.2\text{-}5)$$

The average expectation value of an arbitrary operator K is given by

$$K(E) = \frac{\sum_{\alpha} \langle E\alpha | K | E\alpha \rangle}{d \times \rho(E)} = \frac{\langle\langle K\ \delta(H-E)\rangle\rangle^{S}}{d \times \rho(E)} \qquad (\alpha \in E) \qquad (2.2\text{-}6)$$

$$= \frac{\langle K\ \delta(H-E)\rangle^{S}}{\rho(E)} \qquad (2.2\text{-}7)$$

Therefore,

$$K(E) = \rho^{-1}(E)\ \langle K\delta(H-E)\rangle = \sum_{\nu} \langle KP_{\nu}(H)\rangle^{S}\ P_{\nu}(E) \qquad (2.2\text{-}8)$$

where we have used for $\delta(H-E)$ the expression from equation (2.2-2).Equations (2.2-3c), (2.2-8) give respectively the strength and average expectation value of operators expanded in terms of the orthonormal polynomials. We note here that replacing K by $O^{+}O$ in equation (2.2-8), gives an expansion for the sum rule quantities (which is identical with eq.(2.2-5)).

The polynomial expansions for the average expectation value and the strength becomes very useful for practical applications if one can demonstrate rapid convergence of the expansions. The central limit theorem which produces an asymptotic form for the eigenvalue density $\rho_{H}(E)$ of the Hamiltonian also implies an asymptotic linearity in K(E). It can be shown[5] that, when $\rho_{H}(E)$ is Gaussian and remains so under the "deformation" $H \longrightarrow H + \alpha K$, for infinitesimal α, only the zero'th and the first order polynomials in the expansion of K(E) contribute. Then one has the CLT result

$$K(E) = \langle K\rangle^{S} + \langle K(H-E)\rangle^{S}(E-\epsilon)/\sigma^{2} \qquad (2.2\text{-}9)$$

This behaviour is to be expected for many K operators, though there are operators like J^{2}, Q.Q and pairing which do not have the asymptotic Gaussian spectra. The convergence properties of $R(E,E')$ is more difficult to study. One can do that by constructing the bivariate strength moments

and see that the convergence of (2.2-2c) is rapid for small values of ζ, (the strength correlation coefficient). Section 2.4 discusses more about this. Retaining terms upto the 1st order polynomials, one gets

$$R(E,E') = d^{-1}[\langle O^+O\rangle^s + \langle O^+(H-E')O\rangle^s (E'-\varepsilon')/\sigma'^2 + \langle O^+O(H-\varepsilon)\rangle^s (E-\varepsilon)/\sigma^2 + \langle O^+(H-\varepsilon')O(H-\varepsilon)\rangle^s (E'-\varepsilon')/\sigma'^2 (E-\varepsilon)/\sigma'^2] \qquad (2.2-9)$$

Here ε' and σ' are the centroid and width of the Hamiltonian in the final space.

2.3 CONFIGURATION PARTITIONING FOR STRENGTH AND EXPECTATION VALUE

The partitioning of the space into configurations ($m \longrightarrow \tilde{m}$, $N \longrightarrow \sum_i N_i$) is particularly useful in the calculation of strengths and expectation values in large spaces. To get improved accuracy for scalar spaces one can evaluate higher order terms beyond CLT (i.e.beyond 1st order terms) but this needs calculation of more and more complicated traces involving the excitation operator and powers of the Hamiltonian. On the other hand by introducing configuration partitioning and then evaluating the traces over many configurations one achieves improved agreement with the exact shell model, compared to the scalar case, even with low order polynomials (defined appropriately in the configuration spaces).

Thus with $I_m(x) = \sum_{\tilde{m}} I_{\tilde{m}_m}(x) = \sum_{\tilde{m}} d(\tilde{m})\, \rho_{\tilde{m}_m}(x) = \sum_{\tilde{m}} \langle\langle \delta(H-x)\rangle\rangle^{\tilde{m}}$, the expression for the average expectation value and strength corresponding to equations (2.2-8) and (2.2-3c) can be written as

$$K(E) = \sum_{\tilde{m}} \frac{I_{\tilde{m}}(E)}{I_{m}(E)} K(E;\tilde{m}) \qquad (2.3\text{-}1)$$

where
$$K(E;\tilde{m}) = \sum_{\mu} \langle KP_{\mu}^{(\tilde{m})}(H)\rangle^{\tilde{m}} \; P_{\mu}^{(\tilde{m})}(E) \qquad (2.3\text{-}2)$$

and
$$R(E',\tilde{m}';E,\tilde{m}) = \sum_{\tilde{m}',\tilde{m}} \frac{I_{\tilde{m}'}(E')I_{\tilde{m}}(E)}{I_{m'}(E')I_{m}(E)} R(E',\tilde{m}';E,\tilde{m}) \qquad (2.3\text{-}3)$$

where

$$R(E',\tilde{m}';E,\tilde{m}) = d^{-1}(\tilde{m}) \sum_{\mu,\nu} \langle O^{+}(\tilde{m}'-\tilde{m}) \; P_{\mu}^{(\tilde{m}')}(H) \; O \; P_{\nu}^{(\tilde{m})}(H)\rangle^{\tilde{m}} \; P_{\mu}^{(\tilde{m}')}(E') \; P_{\nu}^{(\tilde{m})}(E) \qquad (2.3\text{-}4)$$

Keeping $\mu = 0$ and 1 in equation (2.3-2) gives the CLT result for the configuration averaged expectation value. As the configuration traces formally are no harder to evaluate than the scalar ones, this procedure seems to be the natural one to follow. Only problem one sometimes encounters in configuration partitioning of large scalar spaces, is in taking care of a large number of configurations. We note here that there is no restriction on the intermediate states reached through the excitation operator.

2.4 STRENGTH AND EXPECTATION VALUE DENSITY FORMALISM

In recent years, a new formalism[6] has been developed to obtain asymptotic forms for the strength densities instead of the actual strengths, in the framework of SDM and this method has many advantages over

the earlier SDM methods to treat the strengths. Firstly by its explicit construction the strength density is positive definite and this formalism gives one a better insight into the convergence properties of the different expansions. This method starts with the strength density function $S(E,E')$, defined as

$$S(E,E') = \left\{ I_m(E') \left| \langle mE' | O | mE \rangle \right|^2 I_m(E) \right\} / \langle\langle O^+ O \rangle\rangle^m \qquad (2.4\text{-}1)$$

$$= \langle\langle O^+ \delta(H-E') O \delta(H-E) \rangle\rangle^m / \langle\langle O^+ O \rangle\rangle^m \qquad (2.4\text{-}2)$$

One then constructs the reduced central moments μ_{pq} of the bivariate distribution $S(E,E')$ by

$$\mu_{pq} = \langle O^+ ((H-\epsilon_2)/\sigma_2)^q \, O((H-\epsilon_1)/\sigma_1)^p \rangle^m / \langle O^+ O \rangle^m \qquad (2.4\text{-}3)$$

and then expresses the reduced cumulants K_{pq} of $S(E,E')$ in terms of μ_{pq}.

It has been shown[6] in the framework of random matrix theory, using EGOE ensembles that the CLT form of $S(E,E')$ is a bivariate Gaussian. Here the Hamiltonian H and the transition operator O^* (* actually one uses the transition operator with a part of the operator which is correlated with the Hamiltonian subtracted from it; for a discussion of the point we refer to the work of French et al[6]) are replaced by EGOE ensembles of k and k'-body operators and the ensemble averaged lower-order bivariate cumulants of $S(E,E')$ are studied in the dilute limit ($m \to \infty$, $N \to \infty$, $\frac{m}{N} \to 0$ and $m >> k, k'$). By construction all the odd cumulants (K_{pq} with p+q odd) vanish and for the even ones it is shown that the cumulants of order four

or larger vanish in the dilute limit. Defining $\zeta = \mu_{11}$ (given by equation(2.4-3))and the transformation, $X = \frac{E + E'}{\sqrt{2(1+\zeta)}}$ and $X' = \frac{E - E'}{\sqrt{2(1+\zeta)}}$, where we assume E and E' in S(E,E') are standardised variables, it is also shown that the centre of mass coordinate X and the difference coordinate X' become uncorrelated as m $\longrightarrow\infty$ i.e $S(X,X') \longrightarrow S_1(X)\, S_2(X')$ and that the marginal density $S_1(X) \longrightarrow S_{1G}(X)$ (a Gaussian form) and $S_2(X') \longrightarrow S_{2G}(X')$, another Gaussian form with the additional constraint of large k'. As the variance $2(1-\zeta)$ of the marginal density $S_2(X')$ approaches zero as $\zeta \longrightarrow 1$, for large ζ even with small k', one can safely approximate S_2 by a Gaussian. Thus with (1+2)-body Hamiltonian and with the usual excitation operators, one can approximate S(E.E') by a bivariate Gaussian form,

$$S(E,E') \xrightarrow{CLT} S_G(E,E') \qquad (2.4-4)$$

where

$$S_G(E,E') = \left[2\pi\sigma_1\sigma_2(1-\zeta^2)^{1/2}\right]^{-1} \exp\left\{-\left[\left(\frac{E-\epsilon_1}{\sigma_1}\right)^2 - 2\zeta\left(\frac{E-\epsilon_1}{\sigma_1}\right)\left(\frac{E'-\epsilon_2}{\sigma_2}\right) + \left(\frac{E'-\epsilon_2}{\sigma_2}\right)^2\right] / \left[2(1-\zeta^2)\right]\right\}$$

with the marginal centroids, variances and the strength correlation coefficient given respectively by

$$\epsilon_1 = \frac{\langle O^+ OH\rangle^m}{\langle O^+ O\rangle^m} \quad \text{and} \quad \epsilon_2 = \frac{\langle O^+ HO\rangle^m}{\langle O^+ O\rangle^m}$$

$$\sigma_1^{\,2} = \frac{\langle O^+ OH^2\rangle^m}{\langle O^+ O\rangle^m} - \epsilon_1^{\,2} \quad \text{and} \quad \dot{\sigma}^{\,2} = \frac{\langle O^+ H^2 O\rangle^m}{\langle O^+ O\rangle^m} - \epsilon_2^{\,2} \qquad (2.4-5a)$$

and

$$\zeta = \langle O^+ \left(\frac{H-\epsilon_1}{\sigma_1}\right) O \left(\frac{H-\epsilon_2}{\sigma_2}\right)\rangle^m / \langle O^+ O\rangle^m \qquad (2.4-5b)$$

The strength moments $M_p(E)$, originating from energy E, are defined by

$$M_p(E) = \sum_{E'} (E')^p \left|\langle E'|O|E\rangle\right|^2 = \int (E')^p \, R(E,E') \, I(E') \, dE'$$

$$= \int \langle O^+O\rangle \, \rho^{-1}(E) \, (E')^p \, S(E,E') \, dE' \qquad (2.4\text{-}6)$$

where we have used

$$I(E') \, R(E,E') = \langle O^+O\rangle \, S(E,E')/\rho(E) \qquad (2.4\text{-}7)$$

The quantity $I(E') \, R(E,E')$ can be shown to be related directly to the experimentally measured quantites.

Equation (2.4-6) gives the p-th energy-weighted strength sum. For p = 0 one gets the non-energy weighted strength sum as

$$M_0(E) = K(E) = \langle O^+O\rangle \frac{S(E)}{\rho(E)} \qquad (2.4\text{-}8)$$

S(E) is called the expectation value density and is defined as

$$S(E) = K(E) \, \rho(E) = \langle\langle K \, \delta(H-E)\rangle\rangle/d = \langle K \, \delta(H-E)\rangle \qquad (2.4\text{-}8a)$$

With the CLT approximation $S(E,E') \xrightarrow{CLT} S_G(E,E')$, one gets from equation(2.4-8)

$$M_0(E) = \langle O^+O\rangle \frac{S_G(E)}{\rho_G(E)} = \langle O^+O\rangle \left(\frac{\sigma_c}{\sigma_s}\right) \exp\left\{ -\frac{1}{2}\left[\left(\frac{E-\epsilon_s}{\sigma_s}\right)^2 - \left(\frac{E-\epsilon_c}{\sigma_c}\right)^2 \right] \right\} \qquad (2.4\text{-}9)$$

where ϵ_c and σ_c are the centroid and width of the Hamiltonian and ϵ_s and σ_s are the same for the marginal strength density $S_{1G}(E)$ (as defined in equations (2.4-5)). Similarly, the linear energy weighted sum rule (LEWSR) quantity or the strength centroid

$$\epsilon(E) \left(\equiv \frac{M_1(E)}{M_0(E)} = \int \frac{E' S(E,E')}{S_1(E)} \, dE' \right)$$

and the strength width

$$\sigma(E) \left(\equiv \left[\frac{M_2(E)}{M_0(E)} - \left(\frac{M_1(E)}{M_0(E)} \right)^2 \right]^{1/2} \right) ,$$

using the bivariate Gaussian form for $S(E,E')$ and the Gaussian form for $\rho(E)$ are given by

$$\epsilon(E) = \epsilon_s + \zeta(E-\epsilon_s) \qquad (2.4-10)$$

and $$\sigma(E) = \sigma_s(1-\zeta^2)^{1/2}$$

Thus the CLT result for $S(E,E')$ predicts that the NEW strength sum is a ratio of two Gaussians, the strength centroid is linear in energy and strength width is a constant. These predictions have been tested[7] in a shell model example as well as by other means[8] and are found to be good particularly for the strength sums and the strength centroids. In these comparisons one finds that for $\sigma(E)$, though there is an overall agreement with a constant value, but there is scope for improvment particularly at the extreme ends of the spectrum by making Edgeworth corrections to $S(E,E')$.

Throughout this thesis we shall refer to the method of polynomial expansion for the expectation values, sum rules and strength as the R-strength method whereas those by the strength or expectation value density formalism as the S-strength method.

2.5 STRENGTH AND EXPECTATION VALUES IN MULTIPOLE FORM

As the excitation operators usually encountered are non-scalar in (JT), one needs to convert the strengths and expectation values in multipole form. We discuss that in this section following Draayer et al[5]. Let us label the m-particle states by (m,Γ,M,x), where Γ stands for J or T or (J,T) as the case may be, M represents the z-component of Γ, so that the $[\Gamma]$ $(\equiv (2\Gamma+1))$ -dimensional set with fixed (m,Γ,x) constitutes a "level" and the non-angular momentum label x counts the levels. Let $d(m,\Gamma)$ be the state dimensionality for fixed (m,Γ) and $d_1(m,\Gamma)$ the level dimensionality, and similarly for $I(m,\Gamma;E)$ and $I_1(m,\Gamma;E)$. Then

$$d(m,\Gamma) = [\Gamma]\, d_I(m,\Gamma)$$

$$I(m,\Gamma;E) = [\Gamma]\, I_I(m,\Gamma;E) = d(m,\Gamma)\, \rho(m,\Gamma;E)$$

$$\sum_E [\ ] \longrightarrow \int I_I(m,\Gamma;E)\ [\ ]\ dE = d_I(m,\Gamma) \int \rho(m,\Gamma;E)\ [\ ]\ dE \qquad (2.5\text{-}1)$$

The double-barred traces are defined as

$$\langle\langle||O^\nu||\rangle\rangle^{m\Gamma} = d_I(m,\Gamma)\langle||O^\nu||\rangle^{m\Gamma} = \sum_x \langle m\Gamma x||O^\nu||m\Gamma x\rangle$$

$$= \int I_I(m,\Gamma;E)\ \langle m\Gamma E||O^\nu||m\Gamma E\rangle\ dE \qquad (2.5\text{-}2)$$

- in the last form of which we have used the Hamiltonian basis, taking for granted that we have a scalar Hamiltonian.

With this definition of double-barred trace the tensorial expectation value $K^\lambda(E,\Gamma)$ is given by

$$K^\lambda(E,\Gamma) = \langle m\Gamma E||K^\lambda||m\Gamma E\rangle = \sum_\nu \langle m\Gamma E||K^\lambda P_\nu^{(m\Gamma)}(H)||m\Gamma E\rangle\ P_\nu^{(m\Gamma)}(E) \qquad (2.5\text{-}3)$$

which gives tensorial extension of (2.2-8) and with ν = 0,1 we have only the CLT expression corresponding to (2.2-9). The polynomials introduced here are defined by the (m,Γ) partial density.

For a pure multipole transition $(m\Gamma ME) \xrightarrow{O^{\lambda}} (m'\Gamma' M'E')$, in which only the $(M' - M)$ -component of O^{λ} contributes to the transition, we define the pure multipole strength as

$$R^{\lambda}(m'\Gamma' M'E' ; m\Gamma ME) = \left| \langle m'\Gamma' M'E' | O^{\lambda}_{M'-M} | m\Gamma ME \rangle \right|^2$$

$$= [\Gamma']^{-1} \left[C^{\Gamma \quad \lambda \quad \Gamma'}_{M \; M'-M \; M'} \right]^2 \left| \langle m'\Gamma'E' || O^{\lambda} || m\Gamma E \rangle \right|^2$$

$$= [\Gamma']^{-1} \left[C^{\Gamma \quad \lambda \quad \Gamma'}_{M \; M'-M \; M'} \right]^2 R^{\lambda}(\Gamma' E' ; \Gamma E) \qquad (2.5\text{-}4)$$

where

$$R^{\lambda}(\Gamma' E' ; \Gamma E) = \left| \langle \Gamma' E' || O^{\lambda} || \Gamma E \rangle \right|^2 = \sum_{M,M'} R^{\lambda}(\Gamma M' E' ; \Gamma M E) \qquad (2.5\text{-}5)$$

We have used above the Wigner-Eckart theorem, and for convenience, have dropped the (M,M') labels. The unpolarised strength $R^{\lambda}(\Gamma' E' ; \Gamma E)$ $(= R^{\lambda}(\Gamma E ; \Gamma' E')$ for self-adjoint excitations) is, for electromagnetic and other particle-hole transitions, essentially the B(Eλ) or B(Mλ) value, differing by a statistical factor since we have summed over initial states instead of averaging.

If $P^{(m',\Gamma')}(E')$ and $P^{(m,\Gamma)}(E)$ stand for the orthonormal polynomials in the spaces (m',Γ') and (m,Γ) respectively, one can write

$$d_I(m',\Gamma')\ R^{\lambda}(m'\Gamma'E';\ m\Gamma E)\ d_I(m,\Gamma) = (-1)^{\Gamma'-\lambda-\Gamma}\ [\Gamma']^{1/2}$$

$$\sum_{\omega} (-1)^{\omega}\ U(\Gamma\lambda\Gamma\lambda\ :\ \Gamma'\omega)$$

$$\sum_{\mu,\nu} \langle\langle \Big|\Big|\ \Big(\bar{O}^{\lambda}\ P_{\mu}^{(m',\Gamma')}(H) \times O^{\lambda}\ P_{\nu}^{(m,\Gamma)}(H)\Big)^{\omega}\ \Big|\Big| \rangle\rangle^{m\Gamma}$$

$$P_{\mu}^{(m',\Gamma')}(E') \times P_{\nu}^{(m,\Gamma)}(E)$$

(2.5-6)

The ω -sum above in equation (2.5-6) is responsible for the Γ' projection. Equation (2.5-6) are the multipole form extension of the equation (2.2-3c).

In equations (2.5-4) and (2.5-6) we give the unpolarised and polarised strengths in spaces with fixed Γ and for states with fixed energy. In this thesis we shall treat excitation operators which are vector both in spin and isospin but shall present quantities polarized with respect to isospin only. Therefore we fix initial and final spaces by definite T, M_T and T', $M_{T'}$ values only. Then as is the usual practice, we sum over all the final states (for fixed mTM_TE) and average similarly over the initial ones. Thus we define the quantity

$$R^{\lambda}(m'T'M_{T'}E';\ mTM_TE) = \sum_{J} (2J+1)^{-1} \sum_{J'M_{J'}M_J} \frac{d_1(m,\Gamma)}{d_1(m,T)}\ R^{\lambda}(m'\Gamma'M';\ m\Gamma M)$$

(2.5-7)

For details of the procedure for J-averaging we refer to Kar[9]. We give here only the final results. We take example of the operator $O^{\lambda} \equiv O^{11}$ displaying explicitly its spherical tensorial rank in (J,T)-space and get,

$$R(m'T'M_T'E';mTM_TE) = [T']^{-1}\left[C^{T\ \ 1\ \ T'}_{M_T\ M_T'-M_T\ M_T'}\right]^2 d^{-1}(m',T')$$

$$(-1)^{T'-1-T}[T]^{1/2}\sqrt{3}\sum_{\omega_T}(-1)^{\omega_T}U(T1T1;T\omega_T')$$

$$\sum_{\mu,\nu}\langle mT\|\left(\bar{O}^{11}P_\mu^{(m',T')}(H)\times O^{11}P_\nu^{(m,T)}(H)\right)^{0,\omega_T}\|mT\rangle$$

$$P_\mu^{(m',T')}(E')\times P_\nu^{(m,T)}(E)$$

(2.5-8)

and similarly, for the average expectation value,

$$K(m'T'M_T'E';mTM_TE) = [T']^{-1}\left[C^{T\ \ 1\ \ T'}_{M_T\ M_T'-M_T\ M_T'}\right]^2$$

$$(-1)^{T'-1-T}[T]^{1/2}\sqrt{3}\sum_{\omega_T}(-1)^{\omega_T}U(T1T1;T\omega_T')$$

$$\sum_{\mu}\langle mT\|\left(K^{0,\omega_T}\times P_\mu^{(m,T)}(H)\right)^{0,\omega_T}\|mT\rangle P_\mu^{(m,T)}(E)$$

(2.5-9)

The extension of equations (2.5-8) and (2.5-9) to the configuration-partitioned spaces is quite straightforward now.

2.6 PROPAGATION OF OPERATOR AVERAGES AND TRACE EVALUATION

One of the most remarkable features of the SDM which makes it attractive and practical is that the averages in many particle spaces, can be expressed in terms of traces in some defining few particle spaces

resulting in propagation of information. The basic result gives the trace of a k -body operator G(k) over the m -particle states as

$$\langle\langle G(k)\rangle\rangle^{m} = \binom{N-k}{N-m} \langle\langle G(k)\rangle\rangle^{k} \qquad (2.6\text{-}1)$$

$$\langle G(k)\rangle^{m} = \binom{m}{k} \langle G(k)\rangle^{k} \qquad (2.6\text{-}2)$$

Equation (2.6-1) which can be derived by a combinatorial argument, shows clearly the propagation of information, from a few particle space to a many-particle space.

We observe that any spectroscopy, using a finite set of single particle states of number N, automatically introduces[10] the group U(N) of unitary transformation among the single particle states. Then the $\binom{N}{m}$ states for m fermions generate an irreducible representation of U(N). The traces of operators or operator products being invariant under basis transformations can be expressed as linear combination of the scalars of the group. It is clear that with fermions the only independent U(N) scalar is the number operator n, so that $\binom{n}{k}$, which has particle rank k is the only possible k-body scalar leading to equation (2.6-1). For an operator with mixed particle rank k = 0,1,2, ...,u, equation (2.6-2) generalises to [9]

$$O \underset{\text{equivalent}}{\overset{\text{trace}}{\equiv}} \sum_{k=0}^{u} \langle O\rangle^{k} \rho_{u}(k) \qquad (2.6\text{-}3)$$

$$\text{with } \rho_{u}(k) = \binom{u-n}{u-k} \binom{n}{k}.$$

The concept underlying in writing equations (2.6-3) can be extended to configuration spaces and to other subgroup structures. For our purpose, the density $\rho^{\nu}(\varkappa,\tau)$ with fixed $(\varkappa,\tau)$ and having the isospin rank ν

is of particular interest, and is defined as (in spherical tensor form)

$$\rho^{\nu}(x,\tau) = \left[\frac{\tau}{\nu}\right]^{1/2} (-1)^{2\tau} \sum_{\alpha} \left(\Psi^{\tau}_{\alpha}(x) \times \bar{\Psi}^{\tau}_{\alpha}(x)\right)^{\nu} \qquad (2.6\text{-}4)$$

Here α sums over all states $\Psi^{\tau}_{\alpha}(x)$ with x -particles and isospin τ. The trace equivalent for such an operator of (definite isospin rank ν and defined in a unique subspace (x,τ)) is then given by

$$O^{\nu}(x,\tau) \underset{\text{equivalent}}{\overset{\text{trace}}{\equiv}} [\tau]^{-1/2} \langle|| \; O^{\nu}(x,\tau) \; ||\rangle^{x,\tau} \rho^{\nu}(x,\tau) \qquad (2.6\text{-}5)$$

where as usual

$$\langle|| \; O^{\nu} \; ||\rangle^{x,\tau} = [\tau] \; d^{-1}(x,\tau) \sum_{\alpha} \langle \Psi^{\tau}_{\alpha}(x) || \; O^{\nu} \; || \bar{\Psi}^{\tau}_{\alpha}(x) \rangle \qquad (2.6\text{-}6)$$

and $d(x,\tau)$ is the dimensionality of the (x,τ) -space. The group involved here is $U(N/2) \times U(2) \subset U(N)$ and the group scalars in this case are functions of 1,n and $\vec{T}$.

An analytical form for $\rho^{\nu}(x,\tau)$ for arbitrary values of x, τ has been given by French[11]. In the configuration isospin space $(\tilde{m},\vec{T})$ where $\tilde{m} = (m_1,m_2, \quad ,m_l)$, $\vec{T} = (\vec{T}_1,\vec{T}_2, \quad \vec{T}_l)$, in which the group involved is $\sum_{i=1}^{l} U(N_i/2) \times U_i(2) \subset U(N)$, the density operators can be obtained by using $n = \sum_r n_r$ and $\vec{T} = \sum_r \vec{T}_r$ and then separating the terms according to the orbits they involve. All possible configuration densities for $x \leq 2$ are given in table 4.3-T3 in chapter IV. We are here not interested in averages over spaces with iscspin specified in each orbit but would average over all possible values for them and retain only the total isospin.

For the group theoretical formulation of unitary group decomposition of operators which is particularly useful for the evaluation of traces of operator products, we refer to Chang et al[1].

REFERENCES FOR CHAPTER II

1). Chang F S, French J B and Thio T H, Ann. Phys. (NY) 66 (1971) 137

2). Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M, Rev. Mod. Phys. 53 (1981) 385

3). Mon K K and French J B, Ann. Phys. (NY) 95 (1975) 90

4). Kota V K B and Kar K, Pramana 32 (1989) 647;
Kota V K B, Z. Phys. A315 (1984) 91;
Kota V K B and Potbhare V, Z. Phys. A322 (1985) 129

5). Draayer J P, French J B and Wong S S M,
Ann. Phys. (NY) 106 (1977) 472, 503

6). French J B, Kota V K B, Pandey A and Tomsovic S, Phys. Rev. Lett. 58 (1987) 2400 ; Ann. Phys. (NY) 181 (1988) 198, 235

7). Kota V K B and Kar K, Pramana, 32 (1989) 647

8). Kar K, Nucl. Phys. A368 (1981) 285

9). Kar Kamales, Ph.D. Thesis, University of Rochester, (1979)

10). Parikh J C, 1978, Group Symmetries in Nuclear Structure (New York : Plenum)

11). French J B, 1969, in Isospin in Nuclear Physics, edited by, Wilkinson D H (Amsterdam : North Holand) p259