CHAPTER – III

THE ANALYTIC HIERARCHY PROCESS (AHP) IN
GOAL PROGRAMMING

In solving problems by explicit logical analysis, three principles can be distinguished: the principle of constructing hierarchies, the principles of establishing priorities, and the principle of logical consistency.

3.1 STRUCTURING HIERARCHIES

Humans have the ability to perceive things and ideas, to identify them, and to communicate what they observe. For detailed knowledge our minds structure complex reality into its constituent parts, and these in turn into their part, and so on hierarchically. The number of parts usually ranges between five and nine. By breaking down reality into homogeneous clusters and subdividing these clusters into smaller ones, we can integrate large amounts of information into the structure of a problem and form a more complete picture of the whole system.

3.2 SETTING PRIORITIES

Humans also have the ability to perceive relationships among the things they observe, to compare pairs of similar things against certain criteria, and to discriminate between both members of a pair by judging the intensity of their preference for one over the other. Then they synthesize their judgements – through imagination or with the AHP, through a new logical process and gain a better understanding of a whole system.

3.3 LOGICAL CONSISTENCY
The third principle of analytical thought is logical consistency. Humans have the ability to establish relationships among objects or ideas in such a way that they are coherent – that is, they relate well to each other and their relations exhibit consistency. Consistency means two things. The first is the similar ideas or objects are grouped according to homogeneity and relevance. The second meaning of consistency is that the intensities of relations among ideas or objects based on a particular criterion justify each other in some logical way.

In utilizing these principles, the analytic hierarchy process incorporates both the qualitative and the quantitative aspects of human thoughts: the qualitative to define the problem and its hierarchy and the quantitative to express judgements and preferences concisely.

3.4 AHP: A FLEXIBLE MODEL FOR DECISION MAKING

Basic observations on human nature, analytic thinking, and measurement have led to the development of the analytic hierarchy process as a useful model for solving problems quantitatively. The AHP incorporates judgements and personal values in a logical way. To define a complex problem and to develop sound judgements, the AHP must be progressively repeated, iterated, overtime; one can hardly expect instant solutions to complicated problems with which one has wrestled for a long time. The AHP also provides framework for group participation in decision making or problem solving. Fig.3.1 below summarizes the advantages of using the AHP as a new approach to problem solving and decision making.

A decision-making approach should have the following characteristics.
• be simple in construct.
• be adaptable to both groups and individuals
• be natural to our intuition and general thinking
• encourage compromise and consensus building and
• not require inordinate specialization to master and communicate

In addition, the details of the processes leading up to the decision-making process should be easy to review.

The AHP reflects the way we naturally behave and think. But it improves upon nature by accelerating our thought processes and broadening our consciousness to include more factors than we would ordinarily consider. The AHP is a process of “systemic rationality”. It enables us to consider a problem as a whole and to study the simultaneous interaction of its components within a hierarchy.

Complex systems can best be understood by breaking them down into their constituent elements, structuring the elements hierarchically, and then composing, or synthesizing, judgments on the relative importance of the elements at each level of the hierarchy into a set of overall priorities.

Hierarchies can be divided into two kinds: structural and functional. In structural hierarchies, complex systems are structured into their constituent parts in descending order according to structural properties such as size, shape, color, or age. A structural hierarchy of the universe would descend from galaxies to constellations to solar systems to planets, and so on, down to atoms, nuclei, protons, and neutrons.
Advantages of the AHP

Fig (3.1)

**Units:**
The AHP provides a single, easily understood, flexible model for a wide range of unstructured problems.

**Process Repetition:**
The AHP enables people to refine their definition of a problem and to improve their judgement and understanding through repetition.

**Complexity**
The AHP integrates deductive and systems approaches in solving complex problems.

**Judgement & consensus:**
The AHP does not insist on consensus but synthesizes a representative outcome from diverse judgements.

**Interdependence:**
The AHP can deal with the interdependence of elements in a system and does not insist on linear thinking.

**Trades offs:**
The AHP takes into consideration the relative priorities of factors in a system and enables people to select the best alternative based on their goals.

**Hierarchie Structuring:**
The AHP reflects the natural tendency of the mind to sort elements of a system into different levels and to group like elements in each level.

**Synthesis:**
The AHP leads to an overall estimate of the desirability of each alternative.

**Measurement:**
The AHP provides a scale for measuring intangibles and a method for establishing priorities.

**Consistency:**
The AHP tracks the logical consistency of judgements used in determining priorities.
In contrast, functional hierarchies decompose complex systems into their constituent parts according to their essential relationships. It helps people to steer a system toward a desired goal – like conflict resolution, efficient performance, or overall happiness. Each set of elements in a functional hierarchy occupies a level of the hierarchy. The top level, called the focus, consists of only one element: the broad, overall objective. Subsequent levels may each have several elements, although their number is usually small – between five and nine.

One’s approach to constructing hierarchy depends on the kind of decision to be made. If it is a matter of choosing among alternatives we could start from the bottom level by listing the alternatives.

Suppose we want to select a school from three choices, the criteria are given in figure 3.2

There is no limit to the number of levels in a hierarchy.

Hierarchy may sometimes be more complicated.

**Example:** One related to projected planning and repeated later should have:

(i) Uncontrollable environmental constraints

(ii) Risk scenarieos

(iii) Controllable systemic constraints.

(iv) Overall objectives of the system
(v) Stake holders

(vi) Stake holder’s objectives (separate one for each stake holder)

(vii) Stake holders policies (separate one for each stake holder)

(viii) Exploratory scenarios (out comes)

Fig. 3.2.

Hierarchy for selecting a school

3.5 STRUCTURING A HIERARCHY
The basic principle to follow in structuring a hierarchy is to see if one can answer the question “can you compare the elements in a lower level in terms of some or all the elements in the next higher level”. Some suggestions in this are:

1. Identify the overall goal. What are we trying to accomplish? What is the main question?
2. Identify the subgoals of the overall goal. If relevant, identify the time horizons that affect the decision.
3. Identify criteria that must be satisfied to fulfill the subgoals of the overall goal.
4. Identify subcriteria under each criterion. Note that criteria or subcriteria may be specified in terms of ranges of values of parameters or in terms of verbal intensities such as high, medium, low.
5. Identify the actors involved.
6. Identify the actors’ goal.
7. Identify actors’ policies.
8. Identify options or outcomes.
9. For yes-no decisions take the most preferred outcome and compare the benefits and costs of making the decision with those of not making it.
10. Do a benefit/cost analysis using marginal values since we are dealing with dominance hierarchies, ask which alternative yields the greatest benefit; for costs, which alternative costs the most, and for risks, which alternative is more risky.

3.6 MAKING JUDGEMENTS
The first step in establishing the priorities of elements in a decision problem is to make pairwise comparisons – that is, to compare the elements in pairs against a given criterion. For pairwise comparisons, a matrix is a preferred form. It is a simple well-established tool that offers a framework for testing consistency, obtaining additional information through making all possible comparisons, and analyzing the sensitivity of overall priorities to changes in judgement. This approach uniquely reflects to dual aspects of priorities dominating and dominated.

To begin the pairwise comparison process, start at the top of the hierarchy to select the criterion C, on property, that will be used for making the first comparison. Then, from the level immediately below, take the elements to be compared: A₁, A₂, A₃… and so on. Let us suppose that there are seven elements. Arrange these elements in a matrix as in Matrix 3.1.

Matrix 3.1

<table>
<thead>
<tr>
<th></th>
<th>A₁</th>
<th>A₂</th>
<th>A₃</th>
<th></th>
<th></th>
<th></th>
<th>A₇</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁</td>
<td>1</td>
<td>5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>A₂</td>
<td>1/5</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
Now compare the elements $A_1$ in column on the left with the elements $A_1$, $A_2$, $A_3$... and so on in the row on top with respect to property $C$ in the upper left-hand corner. Then repeat with column element $A_2$ and so on. To compare elements, ask: How much more strongly does the element (or activity) possess – or contribute to, dominate, influence, satisfy, or benefit – the property than does the element with which it is being compared.

To fill in the matrix of pairwise comparisons, we use numbers to represent the relative importance of one element over another with respect to the property. Table 3.1 contains the fundamental scale of the AHP for pairwise comparisons.

This scale defines and explains the values 1 through 9 assigned to judgements in comparing pairs of like elements in each level of a hierarchy against a criterion in the next higher level. Experience has confirmed that a scale of nine units is reasonable and reflects the degree to which we can discriminate the intensity of relationships between elements. The numerically translated judgements are approximations: their validity can be evaluated by a test of consistency. When tradeoffs is to be made among several criteria, the problem of ranking becomes complex. It is no longer sufficient to simply to assign arbitrary numbers. We must select with care the numbers used to express the strength with which each element possesses or contributes to the property in question.
Such care ensure that in the end we obtain the correct overall priorities for the elements by considering all tradeoffs. (these priorities can also then be used to allocate resources).

Table 3.1

The fundamental scale for pairwise comparisons.

<table>
<thead>
<tr>
<th>Intensity of Importance</th>
<th>Definition</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Equal Importance</td>
<td>Two activities contribute equally to the objective</td>
</tr>
<tr>
<td>3</td>
<td>Moderate importance</td>
<td>Experience and judgement slightly favour one activity over another</td>
</tr>
<tr>
<td>5</td>
<td>Strong importance</td>
<td>Experience and judgement strongly favour one activity over another</td>
</tr>
<tr>
<td>7</td>
<td>Very strong or demonstrated importance</td>
<td>An activity is favoured very strongly over another; its dominance demonstrated in practice</td>
</tr>
<tr>
<td>9</td>
<td>Extreme importance</td>
<td>The evidence favouring one activity over another is of the highest possible order of affirmation</td>
</tr>
<tr>
<td>2,4,6,8</td>
<td>For compromise between the above values</td>
<td>Sometimes one needs to interpolate a compromise judgement numerically because there is no good word to</td>
</tr>
</tbody>
</table>
Reciprocals of above

If activity i has one of the above non zero numbers assigned to it when compared with activity j, then j has the reciprocal value when compared with i.

A comparison mandated by choosing the smaller element as the unit to estimate the larger one as a multiple of that unit

Rationals

Ratios arising from the scale

If consistency were to be forced by obtaining ‘n’ numerical values to span the matrix

1.1 to 1.9

For tied activities

When elements are close and nearly indistinguishable; moderate is 1.3 and extreme is 1.9

3.7 DERIVING PRIORITIES

To derive the priorities for a group of pairwise comparisons such as for the matrix in 3.1 a decision problem, we have to synthesize the judgements made in the pairwise comparisons, that is we have to do some weighting and adding to give us a single number to indicate the priority of each element. The following example explains how to derive priorities from judgements.

Suppose we want to decide which of three new two wheelers – B₁, B₂, B₃ to buy on the basis of comfort, mileage, road grip etc. We draw a matrix with “comfort” listed in the upper left – hand corner and the bikes listed in the column on the left and in a row on top (Matrix 3.2). We then put 1’s in the diagonal positions.

Matrix 3.2 Simple matrix comparing 3 brands of two wheelers for comfort.
<table>
<thead>
<tr>
<th>Brand</th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1</td>
<td>1</td>
<td>1/2</td>
<td>1/4</td>
</tr>
<tr>
<td>B2</td>
<td>2</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>B3</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

This matrix has nine entries to fill. Three are already committed to 1’s. Only the three judgements above the diagonal of 1’s needs to be made. The judgements below are their reciprocals. In general, if the matrix deals with, say, seven elements, the number of judgements needed to fill the entries is $(7 \times 7) - 7 \div 2 = 21$. As seen from the matrix B2 is more comfortable than B1 and B3 is the most preferable.

Now we synthesize the judgements to get an appropriate estimate of the relative priorities of these vehicles with respect to comfort. To do so we first add the values in each column. This is presented in matrix 3.3. Now we divide each entry in each column by the total of that column to obtain the normalized matrix which permits meaningful comparison among elements (matrix 3.4).

**Matrix 3.3 Synthesizing the judgements**

<table>
<thead>
<tr>
<th>Comfort</th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1</td>
<td>1</td>
<td>1/2</td>
<td>1/4</td>
</tr>
<tr>
<td>B2</td>
<td>2</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>B3</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

**Matrix 3.4 Normalized Matrix**
Finally, we average over the rows by adding the values in each row of the normalized matrix and dividing the rows by the number of entries in each. This synthesis yields the percentages of the overall relative priorities, or preference for B$_1$, B$_2$, B$_3$: 14, 29 and 57 percent respectively. As for as the comforts are considered B$_2$ and B$_3$ are about 2 times and 4 times more preferable than B$_1$.

### 3.7.1 Calculation of priorities using the Exact method

Here the priorities are obtained from the matrix of paired comparisons and its accompanying Fig 3.3 by calculating the total dominance of each of the activities A, B, C, represented by the judgements in a row, the first row represents activity A and so on.

Let the consistent matrix of judgements be:

<table>
<thead>
<tr>
<th>Comfort</th>
<th>B$_1$</th>
<th>B$_2$</th>
<th>B$_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B$_1$</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{1}{7}$</td>
</tr>
<tr>
<td>B$_2$</td>
<td>$\frac{2}{7}$</td>
<td>$\frac{2}{7}$</td>
<td>$\frac{2}{7}$</td>
</tr>
<tr>
<td>B$_3$</td>
<td>$\frac{4}{7}$</td>
<td>$\frac{4}{7}$</td>
<td>$\frac{4}{7}$</td>
</tr>
</tbody>
</table>

$$\frac{\frac{1}{7} + \frac{1}{7} + \frac{1}{7}}{3} = \frac{1}{7} = 0.14$$

$$\frac{\frac{2}{7} + \frac{2}{7} + \frac{2}{7}}{3} = \frac{2}{7} = 0.29$$

$$\frac{\frac{4}{7} + \frac{4}{7} + \frac{4}{7}}{3} = \frac{4}{7} = 0.57$$
<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

The value of 2 in the first row and second column represents the dominance of A on the left over B on the top. It is equal to 4 in the first row and fourth column comparing A with D multiplied by the value $\frac{1}{2}$ in the fourth row and second column comparing D with B. In otherwords, here the dominance of A over B also be obtained indirectly from A to D to B. The dominance of A over B can also be obtained by first taking the dominance of A over C in the first row and third column which is 2 multiplied by the dominance of C over B which is one in the $3^{rd}$ row, $2^{nd}$ column.
When the matrix is consistent, all the entries satisfy this relation for all intermediate dominances through one other element. To check all such dominances in two steps we need to multiply the matrix of judgements by itself which gives us all the products necessary by passing through intermediate activities and adding these products. But that is not all the ways in which A dominates B. We can also consider three step dominance. For example, the same value 2 for comparing A with B is equal to three step dominance by first taking, for example, the dominance of A over C, then C over D and finally D over B. If we multiple all the three we have $2 \times 2 \times \frac{1}{2}$ which is again 2. All such three step dominances are captured by multiplying the matrix of judgements by itself three times. This process can be repeated by multiplying the matrix by itself 4 times, 5 times and so on. We note that in this process we can for example compare A with A, then A with C and finally C with B (a path of length three) or we can compare A
with C, C with B, (a path of length three) or we can compare A with C, C with A, A with D, D with D and the back to A. In otherwards we cannot exclude any way of repeating a segment. To ensure that all possible dominance paths are covered, we need to consider all powers of the matrix of judgements. When this matrix is consistent all its powers give the same dominance multiplied by a constant. To illustrate, the square of our matrix is

\[
\begin{bmatrix}
1 & 2 & 2 & 4 \\
\frac{1}{2} & 1 & 1 & 2 \\
\frac{1}{2} & 1 & 1 & 2 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 2 & 4 \\
\frac{1}{2} & 1 & 1 & 2 \\
\frac{1}{2} & 1 & 1 & 2 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1
\end{bmatrix} = 
\begin{bmatrix}
4 & 8 & 8 & 16 \\
2 & 4 & 4 & 8 \\
2 & 4 & 4 & 8 \\
1 & 2 & 2 & 4
\end{bmatrix} = 4 
\begin{bmatrix}
\frac{1}{2} & 1 & 1 & 2 \\
\frac{1}{2} & 1 & 1 & 2 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1
\end{bmatrix}
\]

The RHS matrix is 4 times the original. In relative terms, the dominance of each activity relative to the other activities is the same as in the original matrix.

In fact it is easy to prove that any power k of the matrix gives all paths of length k between two activities and is a constant n^{k-1} times the matrix, n is the number of activities being compared.

3.7.2 Consistency measurement
In any decision-making problem, it is very important to note the consistency, since we do not want the decision to be based on judgements that have such a low consistency that they appear to be random. On the other hand, perfect consistency is hard to live up to; our judgements on B₁, B₂ and B₃ were consistent, but in real life specific circumstances often influence preferences, and circumstances change. The violations of relative preferences within may some time lead to inconsistency which will damage the entire results.

We have seen that the exact priorities (collectively) known as a vector in mathematical language are derived by raising the judgement matrix to large powers by, for example, squaring it, the squaring that result and so on. The rows of the resulting matrix are added and then normalized. The computer is instructed to quit when the normalized vector from the previous power is within a prescribed decimal accuracy from the next power. This process yields what is known as the principal eigen value $\lambda_{\text{max}}$ (lambda max) used to calculate the consistency of judgements. It is obtained by summing each column, thus getting n numbers, multiplying each by its corresponding priority, which is the priority given in the principal eigen vector just described, and adding the results. The principal eigen vector correctly captions the rank inherent in the Judgements from a tolerable level of inconsistency.

If suppose we change some priority values in the original matrix considered in (3.2) and the new inconsistent matrix is
Matrix 3.5  Inconsistent matrix

<table>
<thead>
<tr>
<th>Comfort</th>
<th>B₁</th>
<th>B₂</th>
<th>B₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>B₁</td>
<td>1</td>
<td>½</td>
<td>¼</td>
</tr>
<tr>
<td>B₂</td>
<td>2</td>
<td>1</td>
<td>¼</td>
</tr>
<tr>
<td>B₃</td>
<td>4</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Column Total</td>
<td>7</td>
<td>5.5</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Now the normalized Matrix, Rowsum and overall priorities are

Matrix 3.6  Normalized matrix

<table>
<thead>
<tr>
<th>Comfort</th>
<th>B₁</th>
<th>B₂</th>
<th>B₃</th>
<th>Row sum</th>
<th>Average sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>B₁</td>
<td>1/7</td>
<td>1/11</td>
<td>1/6</td>
<td>0.40</td>
<td>0.13</td>
</tr>
<tr>
<td>B₂</td>
<td>2/7</td>
<td>2/11</td>
<td>1/6</td>
<td>0.63</td>
<td>0.21</td>
</tr>
<tr>
<td>B₃</td>
<td>4/7</td>
<td>8/11</td>
<td>4/6</td>
<td>1.97</td>
<td>0.66</td>
</tr>
</tbody>
</table>

With inconsistency all the values are changed. To know the level of inconsistency with the random judgements we first multiply the first column of the inconsistency matrix (matrix 3.5), changed to decimal form, by the relative priority of the B₁ (0.13), the second column by that of the B₂ (0.21) and the third by that of B₃ (0.66). Then total the entries in the rows (matrix 3.7)
### Matrix 3.7 (a) Totaling the entries

<table>
<thead>
<tr>
<th>Comfort</th>
<th>B₁ (0.13)</th>
<th>B₂ (0.21)</th>
<th>B₃ (0.66)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B₁</td>
<td>1</td>
<td>0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>B₂</td>
<td>2</td>
<td>1</td>
<td>0.25</td>
</tr>
<tr>
<td>B₃</td>
<td>4</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

### Matrix 3.7 (b) Totaling the entries

<table>
<thead>
<tr>
<th>Comfort</th>
<th></th>
<th></th>
<th></th>
<th>Row Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>B₁</td>
<td>0.13</td>
<td>0.11</td>
<td>0.17</td>
<td>0.41</td>
</tr>
<tr>
<td>B₂</td>
<td>0.26</td>
<td>0.21</td>
<td>0.17</td>
<td>0.64</td>
</tr>
<tr>
<td>B₃</td>
<td>0.52</td>
<td>0.84</td>
<td>0.66</td>
<td>2.02</td>
</tr>
</tbody>
</table>

Now take the column of row totals and divide each of its entries by the corresponding entry from the priority vector (matrix 3.8). Now find the average of the three entries in the last column of matrix 3.8

\[
\frac{3.15 + 3.05 + 3.06}{3} \approx 3.09
\]

This is a way to approximate the \( \lambda_{\text{max}} \). The consistency index (CI) where the number of elements, n, is 3 is
\[
\frac{\lambda_{\text{max}} - n}{n - 1} = \frac{3.09 - 3}{3} = \frac{0.09}{3} = 0.045
\]

The random value of the CI for \(n=3\) is 0.52. The consistency ratio (CR) is \(0.045/0.52=0.09\), which is more than the good consistency of 5%.

**Matrix 3.8 Determining \(\lambda_{\text{max}}\)**

\[
\begin{bmatrix}
0.41 \\
0.64 \\
2.02
\end{bmatrix}
\div
\begin{bmatrix}
0.13 \\
0.21 \\
0.66
\end{bmatrix}
=
\begin{bmatrix}
3.15 \\
3.05 \\
3.06
\end{bmatrix}
\]

We can improve the consistency by noting the judgement that is closest to the derived ratio of the priorities it is intended to estimate, or by multiplying by the reciprocal of this ratio and noting its closeness to one. That judgement which gives the largest result should be made smaller, if desired, to improve consistency. It should not be done mechanically. It is better to be approximately right than precisely wrong.

A second approximation procedure to obtain priorities and \(\lambda_{\text{max}}\) is to compute the geometric mean (GM) of the elements in each row – that is, to multiply the elements and then take the \(n\)th root. This is followed by normalizing the resulting vector’s so that comparands add to unity. In general, the G.M. is good approximation, particularly when the consistency is high. The calculation of \(\lambda_{\text{max}}\) is as before. The G.M. of for the inconsistent matrix for the comfort vehicles B1, B2, B3 gives 0.16, 0.20 and 0.64.

* If numerical judgements were taken at random from the scale \(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \ldots, \frac{1}{2}, \ldots, 1, \ldots, 9\), then using a reciprocal matrix we do the following average consistencies for different-order random matrices (recalculated):
The exact solution by the computation is 0.13, 0.21, 0.66 and $\lambda_{\text{max}} = 3.09$, which is almost close to the results obtained earlier. The GM always gives the true priority vector when $n=3$ but not for larger values of $n$ and can in fact give a different ranking of the elements and should not be used for $n>3$. Note that row averaging followed by a normalization of the resulting vector yields 0.13, 0.23 and 0.64. Many examples can be constructed in which this last process yields unsatisfactory results when the matrix is inconsistent.

One way to improve consistency when it turns out to be unsatisfactory is to rank the activities by a simple order based on the weights obtained in the first run of the problem. A second pairwise comparison matrix is then developed with this knowledge of ranking in mind. The consistency should generally be better.

3.8 EXTENDING THE PROCESS

Now we extend the AHP to fuzzy. These methods are completely different from what has been described earlier. Here a synthetic extent value $S_i$ of the pairwise comparison is introduced, by applying the principle of the comparison of fuzzy numbers. We have the following basic concepts.

3.8.1 PRIORITY THEORY - General case

Let us now consider a decision problem with $n$ factors $F_1, F_2, \ldots, F_n$. Now we have to obtain estimates of the positive weights $w_1, w_2, \ldots, w_n$ of these factors which assumed
to be normalized in the sense $\sum_{i=1}^{n} w_i = 1$. Suppose a matrix $R=(r_{ij})$ is available where $r_{ij}$ is an estimate for the relative significance of the factors $F_i$ and $F_j$.

ie, for $w_i/w_j$. We assume that $R$ is a reciprocal matrix,

$\text{ie, } r_{ij}, r_{ji} = 1, \quad i, j=1, 2, \ldots n$

Now there are several methods of obtaining estimates for the weights $w_1, w_2, \ldots, w_n$ from the matrix $R$ as already discussed. Another method for this is the method of logarithmic regressions. It is suitable for extension to the situation in which multiple comparisons for pairs of factors are available. The method is as follows. We estimate the vector $\mathbf{w}$ by the normalized vector $\mathbf{\alpha}$, which minimizes

$$\sum_{i<j} (\ln r_{ij} - \ln (\alpha_i / \alpha_j))^2$$

under additional conditions (normalization) this results in estimating $w_i$ by the corresponding normalized row mean. Suppose the matrix $R$ has empty cells ($r_{ij}$ not available) or cells with more than one entry (several $r_{ij}$ is available). Such situation can occur, when several decision-makers express their opinion on the relative significance of a pair of factors. In this case we estimate $\mathbf{w}$ by the normalized vector, which minimizes

$$\sum_{i<j} \sum_{k=1}^{\delta_{ij}} (\ln r_{ijk} - \ln (\alpha_i / \alpha_j))^2$$

where $r_{ijk}$ ($k=1, 2, \ldots \delta_{ij}$) are $\delta_{ij}$ estimates for $w_i/w_j$ ($\delta_{ij}$ can be equal to 0, if no comparisons are available, equal to one or greater than one, in which case there are multiple comparisons) and where we have taken (1) in to account.

If we put $y_{ijk} = \ln r_{ijk}$, we minimize
\[ \sum_{i<j} \sum_{k=1}^{\delta_{ij}} (y_{ijk} - x_i + x_j)^2 \]

by solving the associated normal equations.

\[ x_i \sum_{j=1}^{n} \delta_{ij} - \sum_{j=1}^{n} \delta_{ij} x_j = \sum_{j=1}^{n} \sum_{k=1}^{\delta_{ij}} y_{ijk}, \quad i=1,2,\ldots,n. \quad \text{for } x_i \quad \ldots \quad (A) \]

Taking the exponentials of the \( x_i \) and normalizing them, we obtain estimates for the \( w_i (i=1, 2, \ldots n) \)

In a concrete situation, the \( r_{ij} \)'s are usually taken between 5 and 1/5: \( r_{ij} \) is set to 5, if \( F_i \) is felt to be much more important than \( F_j \), it is set to 3 if \( F_i \) is felt to be more important than \( F_j \) and if \( r_{ij}=1 \), \( F_i \) and \( F_j \) are considered to be equally important. Intermediate values can be assigned to \( r_{ij} \) in case of doubt between two adjacent values.

If \( F_i \) is less important than \( F_j \), we have \( r_{ij}<1 \) and, as was already mentioned, we assume \( r_{ij} \) \( r_{ji}=1 \) (\( \forall \ i,j \)). Now, the essence of the method is to replace the \( r_{ij} \)'s by fuzzy numbers, since it is more realistic to set \( r_{ij} \) to “about three” – if \( F_i \) is more important than \( F_j \) - than to put \( r_{ij}=5 \). Thus we have to solve a linear system like (A) with fuzzy right – hand sides.

Now we shall pay a little attention to fuzzy numbers before presenting the method of solving.

**Triangular fuzzy numbers.**

**Definition 1.**

Let \( M \in F(R) \) be called a fuzzy number if,

a) there exists \( x_0 \in R \) such that \( \mu_M(x_0)= 1 \)

b) for any \( \alpha \in [0,1] \),

\[ A = [x, \mu_{A_{\alpha}}(x_0) \geq \alpha] \] is a closed interval.
Here $F(R)$ represents all fuzzy sets and $R$ is the set of all real numbers.

**Definition 2.** We define a fuzzy number $M$ on $R = (-\infty, +\infty)$ to be a triangular fuzzy number if its membership function $\mu_M: R \to [0,1]$ is equal to,

$$
\mu_M(x) = \begin{cases} 
\frac{1}{m-l}x - \frac{l}{m-l}, & x \in [l, m], \\
\frac{1}{m-u}x - \frac{u}{m-u}, & x \in [m, u], \\
0, & \text{otherwise}
\end{cases}
$$

with $l \leq m \leq u$, $l$ and $u$ stand for the lower and upper value of the support of $M$, respectively, and $m$ for the modal value. The triangular number as given in (1), will be denoted by $(l, m, u)$, see Fig.3.4. The support of $M$ is the set of elements $\{x \in R \mid l < x < u\}$.

![Fig 3.4. Membership function of a triangular fuzzy number, M=(l, m, u).](image)
For any two fuzzy numbers $M$ and $N$, defined by their membership functions $\mu_M$ and $\mu_N$, the membership function of the fuzzy number $T = f(M,N)$ is calculated according to the extension principle as

$$\mu_T(z) = \sup_{(x,y) \in \mathbb{R}^2: z = x+y} \left( \min(\mu_M(x), \mu_N(y)) \right)$$

...(2)

Using the above we present the operations on the triangular fuzzy numbers.

a) Addition and multiplication

Consider two triangular fuzzy numbers $M_1 = (l_1, m_1, u_1)$ and $M_2 = (l_2, m_2, u_2)$.

Equation (2) implies

- for addition

$$\mu_{M_1 \oplus M_2}(z) = \sup_{(x,y) \in \mathbb{R}^2: z = x+y} \left( \min(\mu_{M_1}(x), \mu_{M_2}(y)) \right)$$

$$= \sup_{x \in \mathbb{R}} \left( \min(\mu_{M_1}(x), \mu_{M_2}(z-x)) \right)$$

...(3)

- for multiplication

$$\mu_{M_1 \otimes M_2}(z) = \sup_{(x,y) \in \mathbb{R}^2: z = xy} \left( \min(\mu_{M_1}(x), \mu_{M_2}(y)) \right)$$

$$= \sup_{x \in \mathbb{R}} \left( \min(\mu_{M_1}(x), \mu_{M_2}(\frac{z}{x})) \right)$$

...(4)
Since the sum $M_1 \oplus M_2$ and product $M_1 \odot M_2$ are continuous fuzzy numbers, whose membership functions are ‘onto; we can carryout the construction separately on the increasing and decreasing parts of the membership function. Thus for addition,

- for increasing parts and fixed $w \in [0,1]$ there exists $x, y \in \mathbb{R}$, satisfying the equations $w = \mu_{M_1}(x) = \mu_{M_2}(y)$

Thus we obtain

$$z = x + y = w(m_1 - l_1) + l_1 + w(m_2 - l_2) + l_2$$

$$w(m_1 + m_2 - l_1 - l_2) + l_1 + l_2$$

... (6)

- for decreasing parts equations (5) result in

$$z = x + y = w(m_1 + m_2 - u_1 - u_2) + u_1 + u_2$$

... (7)

Now, using the same technique as in original triangular fuzzy, we have

via $w = \mu_{M_1 \oplus M_2}(Z)$:

- if $m_1 - l_1 + m_2 - l_2 \leq z \leq m_1 + m_2$:

$$\mu_{M_1 \oplus M_2}(Z) = \frac{1}{m_1 - l_1 + m_2 - l_2} z - \frac{l_1 + l_2}{m_1 - l_1 + m_2 - l_2}$$

... (8)

- if $m_1 + m_2 \leq z \leq m_1 + u_1 + m_2 + u_2$:

$$\mu_{M_1 \oplus M_2}(Z) = \frac{1}{m_1 - u_1 + m_2 - u_2} z - \frac{u_1 + u_2}{m_1 - u_1 + m_2 - u_2}$$

... (9)

Hence we get

$$(l_1, m_1, u_1) \oplus (l_2, m_2, u_2) = (l_1 + l_2, m_1 + m_2, u_1 + u_2)$$

... (10)

For multiplication of fuzzy numbers we have:

- for increasing part ($w \in [0,1]$):
\[
\begin{align*}
  z &= xy = F_1(w) = (l_1 + w(m_1 - l_1)) (l_2 + w(m_2 - l_2)) \\
  &= l_1 l_2 + w l_2 (m_1 - l_1) + w l_1 (m_2 - l_2) + w^2 (m_1 - l_1) (m_2 - l_2) \quad \ldots (11)
\end{align*}
\]

For decreasing part \((w \in [0, 1]):\)
\[
\begin{align*}
  z &= xy = F_2(w) = (u_1 + w(m_1 - u_1)) . (u_2 + w(m_2 - u_2)) \\
  &= u_1 u_2 + w u_2 (m_1 - u_1) + w u_1 (m_2 - u_2) + w^2 (m_1 - u_1) (m_2 - u_2) \quad \ldots (12)
\end{align*}
\]

This implies that:
- if \(l_1 l_2 \leq z \leq m_1 m_2\) then \(\mu_{M_1 \odot M_2}^{(Z)}(z) = \bar{F}_1(z);\)
- if \(m_1 m_2 \leq z \leq u_1 u_2\) then \(\mu_{M_1 \odot M_2}^{(Z)}(z) = \bar{F}_2(z);\)

It is obvious that \(M_1 \odot M_2\) is, in fact, not a triangular fuzzy number. Accepting the following approximation formula,
\[
(l_1, m_1, u_1) \odot (l_2, m_2, u_2) \approx (l_1, l_2, m_1 m_2, u_1 u_2) \quad \ldots (13)
\]
it provides a triangular fuzzy number, which coincides with \(M_1 \odot M_2\) at the interval \((-\infty, l_1 l_2], \{m_1, m_2\}, [u_1 u_2, \infty);\) see fig. 3.5
Fig: 3.5 Membership functions of the triangular fuzzy numbers $M_1$ and $M_2$ and of $M_1 \odot M_2$ and its approximation.

b) Inverse: Bearing in mind the equation

$$\mu_{M^{-1}}(x) = \mu_M(x^{-1}) \quad (x \neq 0)$$ … (14)

and equation (1), we get

$$\mu_M(x) = \begin{cases} \frac{1}{(m-u)x} - \frac{u}{(m-u)} , & x \in \left[\frac{1}{u}, \frac{1}{m} \right] \\ \frac{1}{(m-u)x} - \frac{l}{(m-l)} , & x \in \left[\frac{1}{m}, \frac{1}{l} \right] \\ 0 & \text{otherwise} \end{cases}$$ … (15)

Calculating values $x, x_{01}$ and $x_{02}$ which satisfy

$$\mu_{M^{-1}}(x_{01}) = \mu_{M^{-1}}(x_{02}) = 0,$$ we have $x_{01} = \frac{1}{u}$ and $x_{02} = \frac{1}{l}$

and thus we obtain the following approximation formula,

$$(l,m,u)^{-1} \approx \left(\frac{1}{u}, \frac{1}{m} \frac{1}{l}\right)$$ … (16)

c) Logarithm

In the same way as before, we get

$$\mu_{l\ln M}(x) = \begin{cases} \frac{1}{(m-l)} e^x - \frac{l}{(m-l)} , & x \in [l_{ln}, l_{nm}] \\ \frac{1}{(m-u)} e^x - \frac{u}{(m-u)} , & x \in [l_{nm}, l_{nu}] \\ 0 & \text{otherwise} \end{cases}$$ … (17)

and $ln(l,m,u) \approx (l_{nl}, l_{nm}, l_{nu})$ … (18)
d) **Exponential**

Taking into account the equation

\[ \mu_{eM}(x) = \mu_{em}(lnx) \quad \ldots \ (19) \]

we can calculate

\[
\mu_{eM}(x) = \begin{cases} 
\frac{1}{(m-l)} \ ln x - \frac{l}{(m-l)}, & x \in [e^l, e^m] \\
\frac{1}{(m-u)} \ ln x - \frac{u}{(m-u)}, & x \in [e^m, e^u] \\
0 & \text{otherwise}
\end{cases} \quad \ldots \ (20)
\]

An approximation formula takes the form

\[ e^{(l,m,u)} \simeq (e^l, e^m, e^u) \quad \ldots \ (21) \]

e) **Multiplication by constant**

\[ (\lambda, \lambda, \lambda) \odot (l_1, m_1, u_1) = (\lambda l_1, \lambda m_1, \lambda u_1), \lambda > 0, \lambda \in \mathbb{R} \quad \ldots \ (22) \]

Consider two triangular fuzzy numbers M_1 and M_2, M_1=(l_1, m_1, u_1) and M_2=(l_2, m_2, u_2).

Their operational laws are as follows:

1. \[ (l_1, m_1, u_1) \oplus (l_2, m_2, u_2) \simeq (l_1 + l_2, m_1 + m_2, u_1 + u_2) \]
2. \[ (l_1, m_1, u_1) \odot (l_2, m_2, u_2) \simeq (l_1 l_2, m_1 m_2, u_1 u_2) \]
3. \((\lambda, \lambda, \lambda) \odot (l_1, m_1, u_1) \approx (\lambda l_1, \lambda m_1, \lambda u_1), \lambda > 0, \lambda \in \mathbb{R}\)

4. \((l_1, m_1, u_1)^{-1} \approx (\frac{1}{u_1}, \frac{1}{m_1}, \frac{1}{l_1})\)

3.8.2 Fuzzy priority theory:

Consider the decision - problem discussed in 3.8 (a). Now assume that we estimate the ratio \(w_i/w_j\) by a fuzzy number \(\tilde{r}_{ij}\). Here we distinguish between \(r = (l, m, u)\) where \(l \leq m \leq u\) does not necessarily hold. We donote \(\tilde{r}_{ij}\) again by \((l_{ij}, m_{ij}, u_{ij})\), where \(l_{ij}, m_{ij}\), and \(u_{ij}\) are again the lower, modal and upper value of \(r_{ij}\), respectively. Thus, we are given a fuzzy matrix \(\tilde{R} = (\tilde{r}_{jk})\) and we want to obtain fuzzy, estimates for the weights \(w_1, w_2, \ldots, w_n\) from the matrix. Proceeding as before we finally obtain the linear system of equations (A) again however, the \(\tilde{x}_i (l_i, m_i, u_i)\) and \(\tilde{y}_{jk} = (l_{ijk}, m_{ijk}, u_{ijk}) = l_n \tilde{r}_{jk}\) are fuzzy numbers now. Keeping in mind the rules for addition and subtraction of fuzzy numbers given in 3.8 (a), we observe that system (A) in its fuzzy version is equivalent to

\[
l_i \sum_{j \neq i}^n \delta_{ij} - \sum_{j \neq i}^n \delta_{ij} u_j = \sum_{j \neq i}^n \sum_{k=1}^{\delta_{ij}} l_{ijk}, \quad i=1,2,\ldots,n. \quad \text{..... (23)}
\]

\[
m_i \sum_{j \neq i}^n \delta_{ij} - \sum_{j \neq i}^n \delta_{ij} m_j = \sum_{j \neq i}^n \sum_{k=1}^{\delta_{ij}} m_{ijk}, \quad i=1,2,\ldots,n. \quad \text{..... (24)}
\]

\[
u_i \sum_{j \neq i}^n \delta_{ij} - \sum_{j \neq i}^n \delta_{ij} l_j = \sum_{j \neq i}^n \sum_{k=1}^{\delta_{ij}} u_{ijk}, \quad i=1,2,\ldots,n. \quad \text{..... (25)}
\]

As \(l_{ijk}\) and \(u_{ijk}\) are lower and upper values of \(l_n \tilde{r}_{jk}\),

\[
l_{ijk} + l_{ijk} = u_{ijk} + u_{ijk} = 0\quad \text{(for i, j=1,\ldots,n, k=1,2,\ldots, \delta_{ij})}
\]
and thus equations (23) and (25) sum up to zero and are linearly dependent. The same is true for (24) also.

Generally, a solution of (23) – (25) is given by,

\[ x_i = (l_i + p_1, m_i + p_2, u_i + p_1), \ i = 1, 2, \ldots, n. \]  \hspace{1cm} (26)

where \( p_1 \) and \( p_2 \) can be chosen arbitrarily.

Now, we have the following:

Remarks:

1. One is not always able to choose the parameters \( p_1 \) and \( p_2 \) such that,

\[ l_i + p_1 \leq m_i + p_2 \leq u_i + p_1, \ i = 1, 2, \ldots, n \]  \hspace{1cm} (27)

However, it is the experience that after taking the exponentials and after normalizing, the resulting fuzzy numbers \( \tilde{\alpha}_i \) are correct again, in the sense that lower value < model value < upper value. However, we have not been able to prove that it is always like this. Thus, one should see (26) as a calculation formula than as a fuzzy number.

2. One may find a double dependency in the linear systems (23) and (25). This is eg. The case, when \( n \) or \( n-1 \) rows in the matrix \( \tilde{R} \) have, apart from the diagonal element, only one other element. Examples of such matrices can be seen in the illustration given in matrices \( R_2 \) and \( R_4 \). In such cases we have three (arbitrary) parameters \( p_1, p_2 \) and \( p_3 \) in equation (26) and the problems is, that, after taking the exponential and normalizing, the parameters \( p_1 \) and \( p_3 \) do not fall out any more. Thus the calculations do not yield a unique solution. However, by putting \( p_1 = p_3 \), the parameters do fall out again and we
have a unique solution. We must emphasize; however that there is no theoretical justification for putting $p_1=p_3$.

A possible way to overcome these difficulties could be the following. It is commonly known that the solution of an undetermined system $Ax=b$ to which the smallest Euclidean norm is given by

$$\tilde{x} = A^T (AA^T)^{-1}b.$$ … (28)

In our case, we have to solve the under determined system:

$$A \tilde{x} = \tilde{y}$$ …. (29)

($\tilde{x}$ and $\tilde{y}$ fuzzy vectors), as given by (23)-(25). Analogous to the non-fuzzy case, a minimum-norm solution would be given by

$$\tilde{x} = A^T (AA^T)^{-1} \tilde{y}.$$ …(30)

Thus $\tilde{x}$ would always be a correct fuzzy number (correct in the sense that lower value < modal value < upper value) and one would obtain a solution for the case mentioned in Remark (2). However, this solution cannot always be a solution to (29), since this would exclude the situation mentioned in Remark (1) (the reason for this fact is that for a triangular fuzzy number $\tilde{y}$ generally

$$A (A^T (AA^T)^{-1} \tilde{y} \neq \tilde{y}, \text{if the matrix A has negative entries.}$$

Taking the exponentials of (26) – and following the fuzzy rules, we obtain

$$\beta_i = \exp(x_i) = (\exp(l_i+p_1), \exp(m_i+p_2), \exp(u_i+p_1), i=1,2…,n$$ …(31)

If we normalize the $\beta_i$; ie if we define $\bar{\alpha}_i$ by
\[ \bar{\alpha}_i = \beta_i \left( \sum_{i=1}^{n} \beta_i \right)^{-1}, \quad i=1,2,\ldots,n \] 

\[ \text{...(32)} \]

It is easy to check that \( \bar{\alpha}_i \) are given by

\[ \bar{\alpha}_i = (r_1 \exp(l_i), r_2 \exp(m_i), r_3 \exp(u_i)) \quad i=1,2,\ldots,n \]

\[ \text{...(33)} \]

where

\[ r_1 = \left( \sum_{i=1}^{n} \exp(u_i) \right)^{-1}, \quad r_2 = \left( \sum_{i=1}^{n} \exp(m_i) \right)^{-1}, \quad r_3 = \left( \sum_{i=1}^{n} \exp(l_i) \right)^{-1} \]

we use \( \bar{\alpha}_i \) as an estimate for \( w_i \)

Thus we have, fuzzy estimates \( \bar{\alpha}_i \) \((i=1,2,\ldots,n)\) for the weights \( w_i \) \((i=1,2,\ldots,n)\) are obtained as follows:

(i) For each pair of factors \( F_i \) and \( F_j \), obtain \( \delta_{ij} (\delta_{ij}=0,1,\ldots) \) fuzzy estimates for the relative significance of \( F_i \) and \( F_j \).

(ii) Solve the linear system (23)-(25) for \( l_i, m_i \) and \( u_i \) \((i=1,2,\ldots,n)\); if the situation mentioned in remark (2) occurs put \( p_1 = p_3 \) or use (30).

(iii) Use \( \bar{\alpha}_i \), as given by (33), as an estimate for \( w_i \). We shall now illustrate the working of this with the help of a hypothetical model:

Suppose that the chancellor of an university has to select a vice-Chancellor based on three candidates recommended by a search committee. Let the candidates be denoted by A,B,C: Now to select the candidate the chancellor wanted the same committee to give their relative preference score for each one of the candidates based on

i. Creativity (C₁)

ii. Creativity in implementations (C₂)

iii. Administrative capabilities (C₃) and
iv. Human maturity (C₄)

Now the first task of the committee is to decide on the relative importance of the decision criteria. Let the matrix R (via pairwise comparison) containing the fuzzy estimates for the relative significance of each pair of factors, is as given in Table.1

Table 1. The matrix R: Pairwise comparison of performance criteria.

<table>
<thead>
<tr>
<th></th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>C₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>C₁</td>
<td>(1,1,1)</td>
<td>(2/3, 1, 3/2)</td>
<td>(2/3, 1, 3/2)</td>
<td>(2/7, 1/3, 2/5)</td>
</tr>
<tr>
<td>C₂</td>
<td>(2/3, 1, 3/2)</td>
<td>(3/2, 2, 5/2)</td>
<td>(5/2, 3, 7/2)</td>
<td>(2/5, 1, 3/2)</td>
</tr>
<tr>
<td>C₃</td>
<td>(3/2, 2, 5/2)</td>
<td>(2, 1, 2/3)</td>
<td>(7/2, 3, 7/2)</td>
<td>(2/3, 2, 5/2)</td>
</tr>
<tr>
<td>C₄</td>
<td>(2, 1, 2/3)</td>
<td>(2, 1, 2/3)</td>
<td>(3/2, 2, 5/2)</td>
<td>(1, 1, 1)</td>
</tr>
</tbody>
</table>
Note (i): Here each committee member gives only a few estimates, i.e., not every cell the matrix has three estimates. Following the rules given in 3.8(b) fuzzy estimates for the weights or priorities of the decision criteria are obtained. They are presented in Table 2.

<table>
<thead>
<tr>
<th>Criterian</th>
<th>Estimated weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>$(0.149, 0.194, 0.256)$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$(0.235, 0.319, 0.431)$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$(0.112, 0.140, 0.180)$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$(0.263, 0.347, 0.451)$</td>
</tr>
</tbody>
</table>

At the second level of the decision procedure, the committee compares the candidates A, B and C under each criteria separately. This results in the matrices $R_1$, $R_2$, $R_3$ and $R_4$ in Table 3.

<p>| Table 3. Matrix $R_1$: Pairwise Comparison of candidates under criterion 1 |
|-----------------|-----|-----|
|                 | A   |     | C   |
| A $(1, 1, 1)$   | $(\frac{2}{3}, 1, \frac{3}{2})$ | $(\frac{2}{3}, 1, \frac{3}{2})$ | $(\frac{2}{3}, 1, \frac{3}{2})$ |
| $(\frac{2}{3}, 1, \frac{3}{2})$ | $(\frac{2}{5}, 1, \frac{2}{3})$ | $(\frac{2}{5}, 1, \frac{2}{3})$ |
| B $(\frac{2}{3}, 1, \frac{3}{2})$ | $(1, 1, 1)$ | $(\frac{2}{5}, 1, \frac{2}{3})$ | $(\frac{2}{5}, 1, \frac{2}{3})$ |</p>
<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(1, 1, 1)</td>
<td>$\left( \frac{5}{2}, 3, \frac{7}{2} \right)$</td>
<td>$\left( \frac{3}{2}, 2, \frac{5}{2} \right)$</td>
</tr>
<tr>
<td>B</td>
<td>$\left( \frac{2}{7}, \frac{1}{3}, \frac{2}{5} \right)$</td>
<td>(1, 1, 1)</td>
<td>-</td>
</tr>
<tr>
<td>C</td>
<td>$\left( \frac{2}{5}, \frac{1}{2}, \frac{2}{3} \right)$</td>
<td>-</td>
<td>(1, 1, 1)</td>
</tr>
</tbody>
</table>

**Matrix R₂: Pairwise Comparison of candidates under criterion 2**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(1, 1, 1)</td>
<td>(\left( \frac{5}{2}, 3, \frac{7}{2} \right))</td>
<td>(\left( \frac{3}{2}, 2, \frac{5}{2} \right))</td>
</tr>
<tr>
<td>B</td>
<td>$\left( \frac{2}{7}, \frac{1}{3}, \frac{2}{5} \right)$</td>
<td>(1, 1, 1)</td>
<td>-</td>
</tr>
<tr>
<td>C</td>
<td>$\left( \frac{2}{5}, \frac{1}{2}, \frac{2}{3} \right)$</td>
<td>-</td>
<td>(1, 1, 1)</td>
</tr>
</tbody>
</table>

**Matrix R₃: Pairwise Comparison of candidates under criterion 3**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(1, 1, 1)</td>
<td>(\left( \frac{5}{2}, 3, \frac{7}{2} \right))</td>
<td>(\left( \frac{5}{2}, 3, \frac{7}{2} \right))</td>
</tr>
<tr>
<td>B</td>
<td>$\left( \frac{5}{2}, 3, \frac{7}{2} \right)$</td>
<td>(1, 1, 1)</td>
<td>-</td>
</tr>
<tr>
<td>C</td>
<td>$\left( \frac{5}{2}, 3, \frac{7}{2} \right)$</td>
<td>-</td>
<td>(1, 1, 1)</td>
</tr>
</tbody>
</table>
\[ \left( \frac{5}{2}, 2, \frac{5}{2} \right) \]

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(1, 1, 1)</td>
<td>-</td>
<td>( \frac{3}{2}, 2, \frac{5}{2} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( \frac{2}{5}, \frac{1}{2}, \frac{2}{3} )</td>
</tr>
<tr>
<td>B</td>
<td>-</td>
<td>(1, 1, 1)</td>
<td>( \frac{2}{2}, 2, \frac{5}{2} )</td>
</tr>
<tr>
<td>C</td>
<td>( \frac{2}{5}, \frac{1}{2}, \frac{2}{3} )</td>
<td>( \frac{2}{5}, \frac{1}{2}, \frac{2}{3} )</td>
<td>(1, 1, 1)</td>
</tr>
</tbody>
</table>

Matrix \( R_4 \): Pairwise Comparison of candidates under criterion 4
As before, these matrices are used to estimate weights, in this case the weights of each candidate under each criterion separately. These estimates are given in Table 4.

**Table 4.**

*Estimates for weights of the candidates under each criterion separately.*

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Candidate A</th>
<th>Candidate B</th>
<th>Candidate C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>$(0.196, 0.289, 0.431)$</td>
<td>$(0.195, 0.265, 0.368)$</td>
<td>$(0.344, 0.449, 0.561)$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$(0.405, 0.546, 0.714)$</td>
<td>$(0.162, 0.182, 0.204)$</td>
<td>$(0.277, 0.273, 0.340)$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$(0.540, 0.579, 0.603)$</td>
<td>$(0.163, 0.217, 0.292)$</td>
<td>$(0.158, 0.205, 0.267)$</td>
</tr>
<tr>
<td>$C_4$</td>
<td>$(0.162, 0.250, 0.344)$</td>
<td>$(0.313, 0.500, 0.763)$</td>
<td>$(0.209, 0.250, 0.305)$</td>
</tr>
</tbody>
</table>

Finally, adding the weights per candidate multiplied by the weights of the corresponding criteria, a final score is obtained for each candidate and presented in Table 5.

**Table 5.**

*Final scores of the Candidates*

<table>
<thead>
<tr>
<th>Final Score</th>
<th>Candidate A.</th>
<th>Candidate B</th>
<th>Candidate C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(0.227, 0.398, 0.705)$</td>
<td>$(0.168, 0.313, 0.579)$</td>
<td>$(0.188, 0.289, 0.504)$</td>
</tr>
</tbody>
</table>

**Fig. 3.6 Membership functions of the final scores.**

---

= A

---

= B

---

= C
In fig 3.6 the membership function of these final score are displayed. They could be interpreted as suitability function that is as a measure of the ability of each alternative to meet the decision criteria. Now, the committee has to evaluate the final scores. It is clear that A is the most preferred candidate.

Now we shall use the fuzzy synthetic extend and use the same problem to the selection of a V.C.

### 3.8.3 Value of the fuzzy synthetic extent

Let \( X = \{x_1, x_2, \ldots, x_n\} \) be an object set, and \( U = \{u_1, u_2, \ldots, u_n\} \) be a goal set. According to the method of extent analysis, we now take each object and perform extent analysis, for each goal respectively. Hence, we can get \( m \) extent analysis values for each object, with the following symbols:

\[
M_{gi}^1, M_{gi}^2, \ldots, M_{gi}^m, \quad i=1, 2, \ldots, n
\]

Where all the \( M_{gi}^j \) (\( j=1, 2, \ldots, m \)) are triangular fuzzy numbers.

**Definition 3.** Let \( M_{gi}^1, M_{gi}^2, \ldots, M_{gi}^m \) be values of extent analysis of the \( i \)th object for \( M \) goals. Then the value of fuzzy synthetic extent with respect to the \( i \)th object is defined as

\[
S_i = \sum_{j=1}^{m} M_{gi}^j \odot \left( \sum_{j=1}^{m} \sum_{j=1}^{m} M_{gi}^j \right)^{-1}
\]

(34)

### 3.8.4 The extent analysis method
The extent analysis method is used to consider to the extent to which an object can satisfy the goal, that is, satisfaction extent. In this method, to “extent” is quantified by using a fuzzy number. On the basis of the fuzzy values for the extent analysis of each object, a fuzzy synthetic degree value can be obtained, which is defined as follows:

Let \( x = \{x_1, x_2, \ldots, x_n\} \) be an object set and \( G = \{g_1, g_2, \ldots, g_m\} \) be a goal set. Now each object is taken an the extent analysis, for each goal, \( g_i \), is performed, respectively. Hence, ‘m’ extend analysis values for each object can be obtained, with the following signs:

\[
\tilde{M}_{gi}^1, \tilde{M}_{gi}^2, \ldots, \tilde{M}_{gi}^m, \quad i=1, 2, \ldots, n
\] (35)

where, all the \( \tilde{M}_{gi}^j \) \((j=1, 2, \ldots, m)\) are triangular fuzzy numbers.

Now the method is as follows:

**Step 1:** The value of fuzzy synthetic extent with respect to the \( i \)th object is defined as:

\[
\bar{S}_i = \sum_{j=1}^{m} \tilde{M}_{gi}^j \otimes \left( \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{M}_{gi}^j \right)^{-1}
\] (36)

Here \( \bar{S} \) is defined as the fuzzy synthetic extent value and \( \otimes \) is defined as the fuzzy operation multiplication.

To obtain \( \sum_{j=1}^{m} \tilde{M}_{gi}^j \) perform the fuzzy addition of \( m \) extent analysis values for a particular matrix such that

\[
\sum_{j=1}^{m} \tilde{M}_{gi}^j = \left[ \sum_{j=1}^{m} l_j, \sum_{j=1}^{m} m_j, \sum_{j=1}^{m} u_j \right]
\] (37)

and to obtain, \( \left[ \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{M}_{gi}^j \right]^{-1} \), perform the fuzzy addition operation on \( \tilde{M}_{gi}^j \)

\((j=1, 2, \ldots, m)\) values such that

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{M}_{gi}^j, = \left[ \sum_{i=1}^{m} l_i, \sum_{i=1}^{m} m_i, \sum_{i=1}^{m} u_i \right]
\] (38)

Now compute the inverse of the vector in Equation (9) as such that:
\[
\left( \sum_{i=1}^{n} \sum_{j=1}^{m} \widehat{M}_{ij} \right)^{-1} = \left( \frac{1}{\sum_{i=1}^{n} u_i}, \frac{1}{\sum_{i=1}^{n} m_i}, \frac{1}{\sum_{i=1}^{n} l_i} \right)
\]

(39)

\[
V(\widehat{M}_2 \geq \widehat{M}_1) = \frac{\min(u_{\widehat{M}_1}(x), u_{\widehat{M}_2}(y))}{\min(v_{\widehat{M}_1}(x), v_{\widehat{M}_2}(y))}
\]

Step 2: The degree of possibility of

\[
\widehat{M}_2 = (l_2, m_2, u_2) \geq \widehat{M}_1 = (l_1, m_1, u_1)
\]

is defined as

\[
V(\widehat{M}_2 \geq \widehat{M}_1) = \frac{1}{\sum_{i=1}^{n} u_i}
\]

and this can equivalently be expressed as follows

\[
V(\widehat{M}_2 \geq \widehat{M}_1) = 1 \quad \text{if } M_2 \geq M_1
\]

\[
= \begin{cases} 
1, & \text{if } M_2 \geq M_1 \\
0, & \text{if } l_1 \geq u_2 \\
\frac{l_1 - l_2}{(m_2 - u_2) - (m_1 - l_1)}, & \text{otherwise} 
\end{cases}
\]

where, \( d \) is the ordinate of the highest intersection point \( D \) between \( \mu_{M_1} \) and \( \mu_{M_2} \) in fig (3.4).

To compare \( M_1 \) and \( M_2 \), the values of \( V(M_1 \geq M_2) \) and \( V(M_2 \geq M_1) \) are needed.
Step 3: Three degree of possibility for a convex fuzzy number to be greater than k convex fuzzy numbers \( \tilde{M}_i \) (i=1,2,…,k) can be defined by

\[
V(\tilde{M} \geq \tilde{M}_1, \tilde{M}_2, \ldots, \tilde{M}_k) = V[\tilde{M} \geq \tilde{M}_1] \cap (\tilde{M} \geq \tilde{M}_2) \cap \ldots \cap (\tilde{M} \geq \tilde{M}_k)]
\]

\[
= \min V(\tilde{M} \geq \tilde{M}_i), \quad i = 1,2,\ldots,k
\]

Assume that:

\[
d'(A_i) = \min V(S_i \geq S_k), \quad k = 1,2,\ldots,n; \quad k \neq i
\]

Then the weight vector is given by,

\[
W^l = [d'(A_1), d'(A_2), \ldots, d'(A_n)]^T
\]

where, \( A_i \) (i=1,2,…,n) are n elements.

Step 4: Via normalization, the normalized weight vectors are

\[
W = [d(A_1), d(A_2), \ldots, d(A_n)]^T
\]

where, W is a non fuzzy number.

3.8.5 Presentation method of fuzzy numbers for the pairwise comparison scale.

The first task of the fuzzy AHP method is to decide on the relative importance of each pair of factors in the same hierarchy. By using triangular fuzzy evaluation matrix \( A=(a_{ij})_{n \times m} \) is constructed. For example, essential or strong importance of element \( i \) over element \( j \) under a certain criterion: then \( a_{ij} = (l, 5, u) \) when \( l \) and \( u \) represent a fuzzy degree of judgement. The greater \( u- l \), the fuzzier the degree; when \( u- l = 0 \), the judgement is a non fuzzy number. This stays the same to scale 5 under general meaning.
If strong importance of element j over element l holds, then the pairwise comparison scale can be represented by the fuzzy number

\[ a_{ij}^{-1} = \left( \frac{1}{u}, \frac{1}{m}, \frac{1}{l} \right) \]

### 3.8.6 Calculation of priority vectors of the fuzzy AHP

Let \( A = (a_{ij})_{nxm} \) be a fuzzy pairwise comparison matrix, where \( a_{ij} = (l_{ij}, m_{ij}, u_{ij}) \), which are satisfied with

\[ l_{ij} = \frac{1}{l_{ij}}, \quad m_{ij} = \frac{1}{m_{ij}}, \quad u_{ij} = \frac{1}{u_{ij}} \]

To obtain the estimates for the vectors of weights under consideration we need to consider a principle of comparison for fuzzy numbers. In fact, two questions may arise.

1) What is the fuzzy value of the least or greatest number from a family of fuzzy numbers?

2) Which is the greatest or least among several fuzzy numbers?

The answer to the first is given below in Definition 4. However the answer to the second question requires efforts. We must evaluate the degree of possibility for \( x \in R \) fuzzily restricted to belong to \( M \), to be greater than \( y \in R \) fuzzily restricted to belong to \( M \). Thus, we give the definition as follows.

**Definition 4:** The degree of possibility of \( M_1 \geq M_2 \) is defined as

\[ V(M_1 \geq M_2) = \sup_{x \geq y} \left[ \min(\mu_{M_1}(x), \mu_{M_2}(y)) \right] \]

\[ \ldots (44) \]
When a pair \((x, y)\) exists such that \(x \geq y\) and \(\mu_{M_1}(x) = \mu_{M_2}(y) = 1\), then we have
\[
V(M_1 \geq M_2) = 1
\]

Since \(M_1\) and \(M_2\) are convex fuzzy numbers we have that
\[
V(M_1 \geq M_2) = 1 \text{ iff } M_1 \geq M_2,
\]
\[
V(M_2 \geq M_1) = \text{hgt} (M_1 \cap M_2) = \mu_{M_1}(d) \quad \text{\ldots (45)}
\]

where \(d\) is the ordinate of the highest intersection point \(D\) between \(\mu_{M_1}\) and \(\mu_{M_2}\) (given in fig 3.5)

When \(M_1 = (l_1, m_1, u_1)\) and \(M_2 = (l_2, m_2, u_2)\), the ordinate of \(D\) is given by equation (46)
\[
V(M_2 \geq M_1) = \text{hgt} (M_1 \cap M_2) = \frac{l_1 - u_2}{(m_2 - u_2) - (m_1 - l_1)} \quad \text{\ldots (46)}
\]

To compare \(M_1\) and \(M_2\), we need both the values of \(V(M_1 \geq M_2)\) and \(V(M_2 \geq M_1)\)

![Fig 3.5](image)

**Definition 3**
The degree possibility for a convex fuzzy number to be greater than k convex fuzzy numbers \( M_i \) (i=1,2,...,k) can be defined by

\[
V(M \geq M_1, M_2, \ldots, M_k) = V[M \geq M_1] \text{ and } (M \geq M_2) \text{ and} \ldots \text{ and } (M \geq M_k)
\]

\[
= \min V[M \geq M_i], \ i=1, \ 2\ldots \ k \quad \ldots \ (47)
\]

Table - 3.2

The matrix \( R \), pairwise comparison of performance criteria

<table>
<thead>
<tr>
<th></th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>(1,1,1)</td>
<td>( \left( \frac{2}{3}, 1, \frac{3}{2} \right) )</td>
<td>( \left( \frac{2}{3}, 1, \frac{3}{2} \right) )</td>
<td>( \left( \frac{2}{7}, \frac{1}{3}, \frac{2}{5} \right) )</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>( \left( \frac{3}{2}, 2, \frac{5}{2} \right) )</td>
<td>(1,1,1)</td>
<td>( \left( \frac{5}{2}, 3, \frac{7}{2} \right) )</td>
<td>( \left( \frac{2}{3}, 1, \frac{3}{2} \right) )</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>( \left( \frac{2}{3}, 1, \frac{3}{2} \right) )</td>
<td>( \left( \frac{2}{7}, \frac{1}{3}, \frac{2}{5} \right) )</td>
<td>(1,1,1)</td>
<td>( \left( \frac{2}{5}, 1, \frac{2}{3} \right) )</td>
</tr>
<tr>
<td>( C_4 )</td>
<td>( \left( \frac{5}{2}, 3, \frac{7}{2} \right) )</td>
<td>( \left( \frac{2}{3}, 1, \frac{3}{2} \right) )</td>
<td>( \left( \frac{2}{3}, 1, \frac{3}{2} \right) )</td>
<td>(1,1,1)</td>
</tr>
</tbody>
</table>
Assume that
\[ d^i(A_i) = \min V(S_i \geq S_k), \quad \text{for } k = 1, 2, \ldots, n; \quad k \neq i \quad \ldots (48) \]

Then the weight vector is given by
\[ w^i = [d^1(A_1), d^1(A_2), \ldots, d^1(A_n)]^T \quad \ldots (49) \]

where \( A_i(i=1,2\ldots n) \) are \( n \) elements via normalization, we get the normalized weight vectors
\[ W = (d(A_1), d(A_2), \ldots, d(A_n))^T \quad \ldots (50) \]

where \( W \) is a nonfuzzy number

### 3.9 APPLICATION OF FUZZY AHP IN GROUP DECISIONS

Suppose that in an university a post of a Professor is vacant, and three serious candidates remain in the screening. We call them as \( A_1, A_2, A_3 \). A committee was convened to decide which applicant is best qualified for the job. The committee has three members and they have identified the following as the decision criteria.

1) Mathematical creativity (\( C_1 \))

2) Creativity implementations (\( C_2 \))

3) Administrative capabilities (\( C_3 \))

4) Human maturity (\( C_4 \))

**Step 1:**

Via pairwise comparison, the fuzzy evaluation matrix \( R \), which is relevant to the objective is constructed and presented in Table 3.2

Taking the average value we obtain Table 3.3.
Table 3.3

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$W_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>(1,1,1)</td>
<td>(0.86,1.17,1.56)</td>
<td>(0.67,1.15)</td>
<td>(0.33,0.39,0.49)</td>
<td>0.13</td>
</tr>
<tr>
<td>$C_2$</td>
<td>(0.64,0.85,1.16)</td>
<td>(1,1,1)</td>
<td>(2.5,3,3.5)</td>
<td>(0.98,1.33,1.83)</td>
<td>0.41</td>
</tr>
<tr>
<td>$C_3$</td>
<td>(0.87,1.149)</td>
<td>(0.29,0.33,0.40)</td>
<td>(1,1,1)</td>
<td>(0.4,0.5,0.67)</td>
<td>0.03</td>
</tr>
<tr>
<td>$C_4$</td>
<td>(2.04,2.56,3.03)</td>
<td>(0.55,0.75,1.05)</td>
<td>(1.49,2.25)</td>
<td>(1,1,1)</td>
<td>0.43</td>
</tr>
</tbody>
</table>

Now, by applying formula (3) we have

\[ S_1 = (2.86, 3.56, 4.55) \odot \left( \frac{1}{23.18}, \frac{1}{18.88}, \frac{1}{15.59} \right) \]

\[ = (0.12, 0.19, 0.29) \]

\[ S_2 = (5.09, 6.18, 7.49) \odot \left( \frac{1}{23.18}, \frac{1}{18.88}, \frac{1}{15.59} \right) \]

\[ = (0.22, 0.32, 0.48) \]

\[ S_3 = (2.56, 2.83, 3.56) \odot \left( \frac{1}{23.18}, \frac{1}{18.88}, \frac{1}{15.59} \right) \]

\[ = (0.11, 0.15, 0.23) \]

\[ S_4 = (5.08, 6.31, 7.58) \odot \left( \frac{1}{23.18}, \frac{1}{18.88}, \frac{2}{15.59} \right) \]

\[ = (0.21, 0.33, 0.49) \]

using formula (16) and (17)
\[
V(S_1 \geq S_2) = \frac{0.22 - 0.29}{(0.19 - 0.29) - (0.32 - 0.22)} = 0.35
\]

\[
V(S_1 \geq S_3) = 1
\]

\[
V(S_1 \geq S_4) = \frac{0.21 - 0.29}{(0.19 - 0.29) - (0.33 - 0.21)} = 0.32
\]

\[
V(S_2 \geq S_1) = 1 , \ V(S_2 \geq S_3) = 1
\]

\[
V(S_2 \geq S_4) = \frac{0.21 - 0.48}{(0.32 - 0.48) - (0.33 - 0.21)} = 0.96
\]

\[
V(S_3 \geq S_1) = 0.73
\]

\[
V(S_3 \geq S_2) = 0.06
\]

\[
V(S_3 \geq S_4) = 0.10
\]

\[
V(S_4 \geq S_1) = 1
\]

\[
V(S_4 \geq S_2) = 1
\]

\[
V(S_4 \geq S_3) = 1
\]

Finally, by using formula (12) we obtain

\[
d^1(C_1) = V(S_1 \geq S_2, S_3, S_4) = \min (0.35, 1, 0.32)
\]

\[
= 0.32,
\]

\[
d^1(C_2) = V(S_2 \geq S_1, S_3, S_4)
\]

\[
= \min (1, 1, 0.96) = 0.96,
\]

\[
d^1(C_3) = V(S_3 \geq S_1, S_2, S_4)
\]

\[
= \min (0.73, 0.06, 0.10) = 0.06,
\]

\[
d^1(C_4) = V(S_4 \geq S_1, S_2, S_3)
\]
\[ \text{min}(1, 1, 1) = 1 \]

Hence \( W^1 = (0.32, 0.96, 0.06, 1)^T \)

Now via normalization, we obtain the weight vector with respect to the decision criteria \( C_1, C_2, C_3 \) and \( C_4 \).

\[ W = (0.13, 0.41, 0.03, 0.43)^T \]

**Step – 2.**

At the second level of the decision procedure, the committee compares candidates \( A_1, A_2 \) and \( A_3 \) under each of the criteria separately. This results in the matrices \( R_1, R_2, R_3 \) and \( R_4 \) which are shown in Tables 3.3a’, 33b’, 3.3c’, 3.3d’.

**Table 3.3a**

**The Matrix \( R_1 \).**

<table>
<thead>
<tr>
<th>( C_1 )</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>(1,1,1)</td>
<td>( \left( \frac{2}{3}, 1, \frac{3}{2} \right) )</td>
<td>( \left( \frac{2}{3}, 1, \frac{3}{2} \right) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \left( \frac{2}{3}, 1, \frac{3}{2} \right) )</td>
<td>( \left( \frac{2}{3}, 1, \frac{3}{2} \right) )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( \left( \frac{2}{3}, 1, \frac{3}{2} \right) )</td>
<td>(1,1,1)</td>
<td>( \left( \frac{2}{5}, \frac{1}{2}, \frac{2}{3} \right) )</td>
</tr>
<tr>
<td></td>
<td>( \left( \frac{2}{3}, 1, \frac{3}{2} \right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( A_3 )</td>
<td>( \left( \frac{2}{3}, 1, \frac{3}{2} \right) )</td>
<td>( \left( \frac{3}{2}, 2, \frac{5}{2} \right) )</td>
<td>(1,1,1)</td>
</tr>
<tr>
<td></td>
<td>( \left( \frac{2}{3}, 2, \frac{5}{2} \right) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The Matrix $R_2$.

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$(1,1,1)$</td>
<td>$\left(\frac{5}{2}, 3, \frac{7}{2}\right)$</td>
<td>$\left(\frac{3}{2}, 2, \frac{5}{2}\right)$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$\left(\frac{2}{7}, \frac{1}{3}, \frac{2}{5}\right)$</td>
<td>$(1,1,1)$</td>
<td>--</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$\left(\frac{5}{2}, \frac{1}{2}, \frac{2}{3}\right)$</td>
<td>--</td>
<td>$(1,1,1)$</td>
</tr>
</tbody>
</table>

Table 3.3 c

The Matrix $R_3$.

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$(1,1,1)$</td>
<td>$\left(\frac{5}{2}, 3, \frac{7}{2}\right)$</td>
<td>$\left(\frac{5}{2}, 3, \frac{7}{2}\right)$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$\left(\frac{2}{7}, \frac{1}{3}, \frac{2}{5}\right)$</td>
<td>$\left(\frac{2}{7}, \frac{1}{3}, \frac{2}{5}\right)$</td>
<td>$\left(\frac{2}{3}, 1, \frac{3}{2}\right)$</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$\left(\frac{2}{7}, \frac{1}{3}, \frac{2}{5}\right)$</td>
<td>$\left(\frac{2}{3}, 1, \frac{3}{2}\right)$</td>
<td>$(1,1,1)$</td>
</tr>
</tbody>
</table>

Table 3.3 d

The matrix $R_4$. 

In Table 3.3d, there are two elements such that $l_\mathbf{u}_1 > 0$, and in this case, the elements of the matrix must be taken normalized.

**Table 3.3a’**

<table>
<thead>
<tr>
<th>$C_1$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$W_{C_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(1,1,1)</td>
<td>(0.67,1.1,5)</td>
<td>(0.54,0.75,1.1)</td>
<td>0.28</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.67,1.1,5)</td>
<td>(1,1,1)</td>
<td>(0.4,0.5,0.6)</td>
<td>0.21</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.91, 1.33, 1.85)</td>
<td>(1.5, 2, 2.5)</td>
<td>(1, 1, 1)</td>
<td>0.51</td>
</tr>
</tbody>
</table>

**Table 3.3b’**

<table>
<thead>
<tr>
<th>$C_2$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$W_{C_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.33, 0.33, 0.34)</td>
<td>(0.28, 0.33, 0.34)</td>
<td>(0.25,0.33,0.42)</td>
<td>0.66</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.29, 0.33, 0.4)</td>
<td>(0.33, 0.33, 0.34)</td>
<td>-</td>
<td>0.16</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.24, 0.32, 0.43)</td>
<td>-</td>
<td>(0.33,0.33,0.34)</td>
<td>0.19</td>
</tr>
</tbody>
</table>

**Table 3.3c’**
As before, these matrices are used to estimate weights, in this case the weights of each candidate under each criterion separately. The results are given in Table 3.4.

Table 3.4

<table>
<thead>
<tr>
<th>Criterion</th>
<th>A_1</th>
<th>A_2</th>
<th>A_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>C_1</td>
<td>0.28</td>
<td>0.21</td>
<td>0.51</td>
</tr>
<tr>
<td>C_2</td>
<td>0.66</td>
<td>0.16</td>
<td>0.19</td>
</tr>
<tr>
<td>C_3</td>
<td>0.35</td>
<td>0.33</td>
<td>0.32</td>
</tr>
<tr>
<td>C_4</td>
<td>0.22</td>
<td>0.42</td>
<td>0.36</td>
</tr>
</tbody>
</table>

Finally adding the weights per candidate multiplied by the weights of the corresponding criteria, a final score is obtained for each candidate. Table 3.5 shows these scores.
The ordering relation between candidates is exactly as in Van leerhoven P.J.M. and Pedrycs W.(1983). According to the final score $A_1$ is the preferred candidate.

**Table 3.5**

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Final Scores</td>
<td>0.41</td>
<td>0.28</td>
<td>0.25</td>
</tr>
</tbody>
</table>

### 3.10 COMPARISON OF THE EXTEND ANALYSIS METHOD AND LLSM

According to the complexity of the algorithm we can distinguish between good and bad points of EAM and LLSM. The time complexity and the space complexity is contained in the complexity of the algorithm.

By time complexity, it refers to the time in which the algorithm is completed. We only use the number of times multiplication, which are more or less as a criterion of appraisal. Here consider the time complexity only.

Assume that we give an nxn fuzzy pairwise comparison matrix, by using the EAM and LLSM. The weight vectors with respect to teach element under certain criterion will then be obtained via normalization we get the normalized weight vectors.

Let us count the number of times of multiplication with respect to the two methods, respectively.

Formulas (7), (10) and (14) are major formulas of EAM. In formula (7), to count $S_i(i=1,2,…n)$, we need to use multiplication $6n$ times. In formulas (10), via pairwise comparison, $S_1, S_2,… S_n$, the number of times multiplication is
\[ P_n^2 = n(n-1) \]

Finally, in formula (14), are also need to use multiplication \( n \) times.

Here, the time complexity of EAM is

\[ T_n = 6n+n(n-1)+n=n(n+6) \quad \cdots (51) \]

In the LLSM, the normalized weight vectors are

\[ w_k = \frac{\left( \prod_{j=1}^{n} \prod_{i=1}^{n} a_{kj} \right)^{1/n}}{\sum_{i=1}^{n} \left( \prod_{j=1}^{n} a_{ij} \right)^{1/n}}, \quad k=1,2, \ldots, n \quad \cdots (52) \]

where \( w_k \) is the \( k \)-the component of the weights vector, Evidently, the time complexity of LLSM is

\[ T_n^1 = n(n+1)+n(n+1)+1 \]
\[ = n(n+1)^2+n \quad \cdots (53) \]

Thus

\[ T_n^1 - T_n = n \left( n^2+n-4 \right) \quad \cdots (54) \]

Let \( n=4 \), we obtain \( T_4^1 - T_4 = 64 \)

Evidently, the EAM is better than the LLSM at time complexity.

### 3.11 AHP IN SOLVING G.P

(i) Consistency analysis of individual evaluation
**Hosiery Units (Small, medium and large)**

The participants in each of this group is 50. They are the decision makers. To estimate priorities out of a given pairwise comparison matrix, we followed the suggestion of Saaty (2006, pp77) and used the Eigen vector method for all the three groups (small, medium and Large). To assure a certain quality level of a decision we have to analyze the consistency of the evaluations. For this purpose we calculated the consistency ratio (CR) confirming Saaty (1995, 81), which is defined as the ratio between the consistency of a given evalution matrix (consistency index CI) and the consistency of a random matrix (for Satty’s scale using numerical judgments from \( \frac{1}{9}, \frac{1}{8}, \ldots, 1, 2, \ldots, 9 \) Saaty quotes for a 5-5 reciprocal matrix a random consistency RC of 1.11). The CR of a decision should not exceed a certain level, namely 0.1 for a matrix larger than 4 by 4 (Saaty, 1995, 81). Hence it is decided to include only evaluations which fulfill the condition CR ≤ 0.1.

Confirming Saaty (1980) we can approximate CR via \( \lambda_{max} \):

\[
CI = \frac{\lambda_{max} - n}{n-1} \quad \text{and} \quad CR = \frac{CI}{RC} \leq 0.1
\]

To simplify the calculation of the CR, we use the crisp value \( m_{ij} \). If the CR exceeded the tolerable level of 0.1, we excluded the pairwise comparison matrix of this respondent for further analysis, because this could affect the overall results negatively.

In this respect, it has to be mentioned, that data for this priorities were collected by a simple, spread sheet based computer program with graphical bars. The evaluator received feedback about the priorities and the consistency of their pairwise comparisons immediately and had therefore the possibility to redo them. However, although we
provided this, many evaluations were inconsistent. In total we received only 32 for small units, 36 for medium units and 30 for large units by following the first knock out criterion CR≤ 0.1. We had to now deal with this.

(ii) Aggregation of group decisions

The steps mentioned in (i) refers to the aggregation of the group evaluations. Fuzzy pairwise comparisons can be combined by use of the following algorithm

\[
l_{ij}=\min(l_{ijk}), m_{ij}=\left(\prod_{k=1}^{K} m_{ijk}\right)^{\frac{1}{K}}, u_{ij}=\max(u_{ijk})\text{ where } (l_{ijk}, m_{ijk}, u_{ijk}) \text{ is the fuzzy evaluation of sample members } k(k=1, 2\ldots,K). \text{ However, min and max operations are not appropriate if the sample had a wide range of upper and lower band widths, in otherwise, if evaluations are inhomogeneous. We have to consider that if only one or few decision makers deliver extreme } l_{ijk} \text{ and/or } u_{ijk} \text{ the whole span of fuzzy numbers } (l_{ij}, m_{ij}, u_{ij}) \text{ gets huge. Due to the required number of multiplications and addition operations, the aggregated fuzzy weights can ever exceed the 0-1 or become irrational, which is of course, unsatisfactory. Hence, we decided to use the geometric mean also for } l_{ij} \text{ and } u_{ij} \text{ which is expected to deliver satisfying fuzzy group weightings.}
\]

\[
l_{ij}=\left(\prod_{k=1}^{K} m_{ijk}\right)^{\frac{1}{K}}, m_{ij}=\left(\prod_{k=1}^{K} m_{ijk}\right)^{\frac{1}{K}}, \text{ and } u_{ij}=\left(\prod_{k=1}^{K} u_{ijk}\right)^{\frac{1}{K}}
\]

This approach, though new, is connected with some information loss also. However, this loss should be tolerable, especially if we take into account the main advantage of this approach. In our case, we get the following aggregated fuzzy pairwise comparisons \((l_{ij}, m_{ij}, u_{ij})\):
### Table 3.7

**Pairwise comparison matrix** (Small size firm) (C.R ≤ 0.1, n=32)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(1,1,1)</td>
<td>(0.42, 0.61, 1.20)</td>
<td>(0.32, 0.45, 0.77)</td>
</tr>
<tr>
<td>C</td>
<td>(0.85, 1.74, 2.64)</td>
<td>(1,1,1)</td>
<td>(0.44, 0.74, 1.23)</td>
</tr>
<tr>
<td>D</td>
<td>(1.31, 2.22, 3.10)</td>
<td>(0.80, 1.36, 2.22)</td>
<td>(1,1,1)</td>
</tr>
</tbody>
</table>

### Table 3.8

**Pairwise Comparison Matrix** (Medium size firm) (C.R ≤ 0.1, n=36)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(1,1,1)</td>
<td>(1.07, 1.71, 2.79)</td>
<td>(0.62, 0.88, 1.32)</td>
</tr>
<tr>
<td>C</td>
<td>(0.44, 0.71, 1.12)</td>
<td>(1,1,1)</td>
<td>(0.86, 1.32, 2.00)</td>
</tr>
<tr>
<td>D</td>
<td>(0.25, 0.38, 0.55)</td>
<td>(0.26, 0.35, 0.55)</td>
<td>(1,1,1)</td>
</tr>
</tbody>
</table>

### Table 3.9

**Pairwise Comparison Matrix** (Large size firm) (C.R ≤ 0.1, n=30)

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>A</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>(1,1,1)</td>
<td>(1.31, 2.22, 3.10)</td>
<td>(1.72, 2.35, 3.35)</td>
</tr>
<tr>
<td>A</td>
<td>(0.32, 0.45, 0.77)</td>
<td>(1,1,1)</td>
<td>(0.30, 0.43, 0.58)</td>
</tr>
<tr>
<td>D</td>
<td>(0.36, 0.58, 0.93)</td>
<td>(1.34, 1.26, 1.11)</td>
<td>(1,1,1)</td>
</tr>
</tbody>
</table>
Methods described under 8.4 are used to calculate $\tilde{s}_1$ and $W$.

Accordingly using 7, 8, and 9 we obtain the weights as

$A|C|D \quad W_1 = (0.32, 0.24, 0.16)$ for small units

$A|C|D \quad W_2 = (0.41, 0.28, 0.19)$ for medium units

$C|A|D \quad W_3 = (0.38, 0.23, 0.21)$ for large units

Using these as the weights the weighted goal programming method described in chapter 1 was used to solve the LPP separately for the three groups and the results are presented under.

(a) Small Hosiery units

$X_1 = 6856$ pieces

$X_2 = 22762$ pieces

$X_6 = 986$ pieces

$X_8 = 15142$ pieces

Total = 45746 which is greater than that obtained through Fuzzy.

(b) Medium Hosiery units

$X_1 = 13112$ pieces

$X_2 = 46314$ pieces

$X_6 = 8643$ pieces

$X_8 = 7465$ pieces
Total = 75534 This is also greater than what is got in the other two.

(c) Large Hosiery units

\[ X_2 = 125741 \]
\[ X_3 = 22664 \]
\[ X_7 = 51595 \]

Total = 200000

This is equal to that however the revenue is greater than the total got through the G.P. Thus use of AHP properly with Fuzzy can give the best possible results in G.P.