In this chapter our aim is to investigate problems of torsional oscillation of a semi-infinite circular cylinder — the material of which has been taken as nonhomogeneous and the oscillation of the cylinder is supposed to be the result of an impulsive twist at the boundary of it. In literature this kind of problem has been the subject of many investigators, but as we have devoted ourselves to study the effect of nonhomogeneity on several elasticity problems, we like to study the effect on this kind of problem also.
§1. TORSIONAL OSCILLATION OF A SEMI-INFINITE NONHOMOGENEOUS CIRCULAR CYLINDER*

Abstract: This paper deals with the torsional oscillation of a semi-infinite, nonhomogeneous circular cylinder under the action of an impulsive twist applied at the end of the cylinder. Laplace transform technique is adopted to find the solution and the expressions for the stresses have been given.

1.1 Introduction

A good number of investigators had solved the problem of torsional oscillation of a circular cylinder. Of them, mention may be made of Kolsky [52] and Davies [24], who assumed the displacement component as a simple harmonic function of the axial co-ordinate $z$ and time $t$. The problem of torsional oscillation of an isotropic, semi-infinite, circular cylinder acted on by an impulsive twist at the end was investigated by Mitra [66]. Chatterjee [15] extended his problem for the case of a transversely isotropic material. Further extension was made by Banerjee [8], who considered a cylinder of nonhomogeneous, transversely isotropic material where the elastic constants were assumed to vary exponentially with the length of the cylinder. Mandal [64] considered the case where both the Young's modulus and density obey the same square law variation in the axial coordinate.

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In our present investigation our aim is to study the problem discussed by Mandal [64] with the modification that impulsive torsional twist is applied at the end of the semi-infinite circular cylinder and the twist is steadily maintained thereafter.

The material of the cylinder is assumed as nonhomogeneous in the sense that both the density and the Young's modulus have arbitrary power law variations with the axial coordinate. Laplace transform has been used to solve the problem and in the inversion of the transformed functions we have followed the same procedure as has been followed in §2 Chapter II.

1.2 Formulation of the problem

We consider a semi-infinite circular cylinder of radius $b$ whose curved surface is stress-free. We take $z$-axis along the axis of the semi-infinite cylinder $z \geq 0$ and use cylindrical co-ordinates $(r, \theta, z)$ to determine positions in the cylinder. At the end $z = 0$, an impulsive twist is applied and is steadily maintained.

From the symmetry of the body and from the nature of the applied force, it is expected that the components $u_r$ and $u_z$ of the displacement vector $(u_r, u_\theta, u_z)$ vanish, while $u_\theta$ will be independent of $\theta$. Thus,

$$u_r = u_z = 0, \quad u_\theta = u_\theta(r, z, t).$$

(1)
In view of (1), the stress components become

\[ \sigma_\theta = \mu \left( \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \]
\[ \sigma_z = \mu \frac{\partial u_\theta}{\partial z}. \]  

\[ (2) \]

In our discussion, we shall suppose that the rigidity modulus \( \mu \) and the density \( \rho \) are not constants but are functions of the axial distance \( z \) such that

\[ \mu = \mu_0 (1 + kz)^m, \quad \rho = \rho_0 (1 + kz)^n, \]  

\[ (3) \]

\( \mu_0, \rho_0, k, m, n \) all being constants.

Inserting (3) into (2) we find that

\[ \sigma_\theta = \mu_0 (1 + kz)^{m-1} \left[ \frac{\partial V}{\partial r} - \frac{V}{r} \right], \]  

\[ (4a) \]

and

\[ \sigma_z = \mu_0 (1 + kz)^{m-2} \left[ (1 + kz) \frac{\partial V}{\partial z} - kV \right], \]  

\[ (4b) \]

where \( V = (1 + kz) u_\theta \).

In view of (4a) and (4b), the only non-vanishing equation of motion becomes
\[ \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} - \frac{V}{r^2} + \frac{\partial^2 V}{\partial z^2} + \frac{k(m-2)}{(1+kz)^2} \left[ (1+kz) \frac{\partial V}{\partial z} - kV \right] \]

\[ = \frac{1}{c_0^2} (1+kz)^{n-m} \frac{\partial^2 V}{\partial t^2} , \quad (5) \]

where \( c_0 = \frac{\mu_0}{\rho_0} \).

The initial and boundary conditions are as follows —

\[ u_\theta = \frac{\partial u_\theta}{\partial t} = 0, \quad t = 0, \quad 0 \leq z < \infty , \]

\[ \sigma_r = 0, \quad r = b, \quad t \geq 0, \quad (6) \]

\[ \sigma_\theta = f(r) H(t), \quad z = 0, \quad 0 \leq r < b , \]

\[ u_\theta/r \text{ is finite at } r = 0, t > 0, \quad u_\theta \text{ is finite when } z \to \infty, t > 0. \]

1.3 Solution of the problem

Let us assume a solution of equation (5) in the form

\[ V = R(r) Z(z,t) . \quad (7) \]

Substituting (7) into (5) we find that

\[ \frac{1}{R} \left[ \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} - \frac{R}{r^2} \right] = -\frac{1}{Z} \left[ \frac{\partial^2 Z}{\partial z^2} + \frac{k(m-2)}{(1+kz)^2} \left( (1+kz) \frac{\partial Z}{\partial z} - kz \right) \right] \]

\[ - \frac{1}{c_0^2} (1+kz)^{n-m} \frac{\partial^2 Z}{\partial t^2} \] 

\[ = -\lambda^2 , \]
X being a constant.

This gives
\[
\frac{d^2R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + (\lambda^2 - \frac{1}{r^2})R = 0
\]  
(8)

and
\[
\frac{k(m-2)}{(1+kz)^2} \left[ (1+kz) \frac{\partial Z}{\partial z} - kz \right] - \frac{1}{c_0^2} \frac{\partial^2 Z}{\partial t^2} + \frac{\partial^2 Z}{\partial z^2} = \lambda^2 Z .
\]  
(9)

The solution of (8) is
\[
R = C_1 \xi + C_2 / \xi \quad \text{when} \quad \lambda = 0
\]
and
\[
R = C_3 J_1(\Delta \xi) + C_4 Y_1(\Delta \xi) \quad \text{when} \quad \lambda > 0,
\]

where \( \xi = r/b \), \( \Delta = \lambda b \) and \( C_1, C_2, C_3, C_4 \) are constants.

Using the fourth boundary condition of (6), we have
\[
C_2 = C_4 = 0
\]
so that \( R = C_1 \xi \) when \( \lambda = 0 \)
and
\[
R = C_3 J_1(\Delta \xi) \quad \text{when} \quad \lambda > 0 .
\]  
(10)

The second boundary condition of (6) will be satisfied if
\[
J_0(\Delta) = 0 .
\]  
(11)
Denoting the roots of the equation (11) by \( \Lambda_n \) \((n = 1, 2, 3, \ldots)\) we may obtain the eigen values \( \lambda_n \) corresponding to \( \Lambda_n \).

Again, using the dimensionless variables

\[ Z^* = \frac{z}{b} \quad \text{and} \quad \tau = c_0 t/b \, , \quad \text{we get from equation (9)} \]

\[
\frac{k_1(m-2)}{(1+k_1 Z^*)^2} \left[ (1+k_1 Z^*) \frac{2Z}{z^*} - k_1 Z \right] - (1+k_1 Z^*)^{n-m} \frac{\partial^2 Z}{\partial \tau^2} + \frac{2Z}{z^*} = \Lambda^2 Z, \quad (12)
\]

where \( k_1 = k b \).

Let an overbar denotes the Laplace transform of a function with respect to \( \tau \). Then, noting the initial condition (6) and taking Laplace transform of (12), we get

\[
\frac{d^2 Z}{d \tau^2} - p^2 (1+k_1 Z^*) \frac{n-m}{Z} + \frac{k_1(m-2)}{1+k_1 Z^*} \frac{d Z}{d \tau} + \frac{k_1^2(m-2)}{(1+k_1 Z^*)^2} = \Lambda^2 \overline{Z} \quad (12a)
\]

It is rather difficult to get exact solution of equation (12a) but it may be reduced to a Bessel equation when \( n = m \). In our discussion we shall be concentrating on the exact solution of the problem assuming \( n = m \).

Using the transformation

\[ \overline{Z} = \overline{\eta}^\alpha \quad \text{with} \quad \eta = 1 + k_1 Z^* \]

and \( \alpha = (3 - m)/2 \),

the equation (12a) may be reduced to a Bessel equation whose
solution, appropriate for the problem, may be written as

\[ Z = B_1 \eta^s K_s(X), \]  

where

\[ s = (m-1)/2, \]  

\[ X = \gamma \eta \]  

and

\[ \gamma = (p^2 + \Lambda^2)/k_1^2. \]

Laplace transform of (7) and (4b) yields

\[ v = D_0 \eta^s K_s(p \eta/k_1) + \sum_{n=1}^{\infty} D_n \eta^s K_s(\gamma_n \eta) J_1(\Lambda_n \xi) \]  

\[ + \sum_{n=1}^{\infty} D_n \eta^s K_{s+1}(\gamma_n \eta) J_1(\Lambda_n \xi) \]  

(15)

and

\[ \sigma_{\theta z}/(\mu_1 k_1) = 2^{-m+1/2} \left[ D_0 \xi(p/k_1) K_{s+1}(p \eta/k_1) \right. \]  

\[ \left. + \sum_{n=1}^{\infty} D_n \gamma_n K_{s+1}(\gamma_n \eta) J_1(\Lambda_n \xi) \right] \]  

(16)

where \( \mu_1 = \mu_0/b. \)

Similar treatment on the third boundary condition of equation (6) yields

\[ \sigma_{\theta z} = f(r)/p = F(\xi)/p \] (say) on \( z = 0. \)  

(17)

Hence, from (16) and (17), we get

\[ \mu_2 \left[ D_0 \xi \mu_1 K_{s+1}(z_0) + \sum_{n=1}^{\infty} D_n \gamma_n K_{s+1}(z_n) J_1(\Lambda_n \xi) \right] = -F(\xi)/p, \]  

(18)
where \( \mu_2 = \mu_1 k_1 \), \( p_1 = p/k_1 \),

\[ z_0 = p_1, \quad z_n = \gamma_n. \]

Using the orthogonality relation of Bessel functions we have from (18),

\[ D_0 = -2T/(\pi \mu_2 p \, d_0) \]

and

\[ D_n = \frac{-2}{\mu_2 p} \int_0^1 J_1(\Lambda_n \xi) F(\xi) \, d\xi \]

where \( d_0 = p_1 K_{s+1}(z_0) \), \( d_n = \gamma_n K_{s+1}(z_n) \) \( (n = 1, 2, 3, \ldots) \)

and \( T = \int_0^1 2\pi \xi^2 F(\xi) \, d\xi. \)

Thus, we have

\[ \frac{\sigma_{\xi\xi}}{2} = \eta^{(m+1)/2} \left[ s_o(\eta, p) + \sum_{n=1}^{\infty} s_n(\eta, p) J_1(\Lambda_n \xi) \right], \quad (19) \]

where

\[ \bar{e}_o(\eta, p) = \frac{K_{s+1}(p_1 \eta)}{p \, K_{s+1}(p_1)}, \]

\[ \bar{e}_n(\eta, p) = \frac{K_{s+1}(\gamma_n \eta)}{p \, K_{s+1}(\gamma_n)}, \quad (n = 1, 2, 3, \ldots) \]
So = \int_0^\infty F(\xi) \, d\xi

and

S_n = \frac{\int_0^\infty J_1(\Lambda_n \xi) F(\xi) \, d\xi}{\left[ J_1(\Lambda_n) \right]^2}.

To find the stresses, we set

\bar{\sigma}_n(\eta, p) = \frac{\bar{\varphi}_n(\eta, p)}{p \, \varphi_n(1, p)} \quad (n = 0, 1, 2, \ldots), \quad (20)

where

\bar{\varphi}_0(\eta, p) = \frac{K_{s+1}(p_1 \eta)}{p_1} \quad (21a)

and

\bar{\varphi}_n(\eta, p) = \frac{K_{s+1}(\gamma_n \eta)}{\gamma_n^{s+1}} \quad (n = 1, 2, \ldots). \quad (21b)

Equation (20) can then be rewritten as

\bar{\sigma}_n(\eta, p) \, \bar{\varphi}_n(1, p) = \bar{\varphi}_n(\eta, p) / p.

Applying convolution theorem for Laplace inversion on the above equation we have

\int_0^\tau s_n(\eta, \xi_1) H(\tau - \xi_1 - 1/k_1) \nu_n(1, \tau - \xi_1) \, d\xi_1

= \int_0^\tau H(\tau' - \eta / k_1) \nu_n(\eta, \tau') \, d\tau',

where \nu_n(\eta, \tau) = H(\tau - \eta / k_1) \psi_n(\eta, \tau),
\[ \psi_n(n, \tau) = \begin{cases} 
\frac{k_1 \sinh\{(s+1) \cosh^{-1} (\tau k_1/n)\}}{(s+1)}, & n = 0 \\
\frac{(\pi/2)^{1/2} (\Lambda_n)^{-s+1/2} j_{s+1/2} j_{s+1/2}(\Lambda_n, y_1)}{(n/k_1)^{s+1}}, & n > 0 
\end{cases} \]

\[ y_1 = (\tau_1^2 - n^2/k_1^2)^{1/2}. \]

Because of the presence of the Heaviside function \( H(\tau - \xi_1 - 1/k_1) \), the left-hand-side of the above equation vanishes when \( \xi_1 > \tau - 1/k_1 = \tau_1 \) (say). Hence, assuming \( k_1 \) to be positive. We may write

\[ \int_0^{\tau_1} s_n(n, \xi_1) \psi_n(1, \tau_1 + 1/k_1 - \xi_1) \, d\xi_1 \]

\[ = H(\tau_1 - \delta) \int_0^{\tau_1 - \delta} \psi_n(n, u + n/k_1) \, du, \quad (22) \]

where

\[ \delta = (n-1)/k_1. \]

Since, the right-hand-side of equation (22) contains \( H(\tau_1 - \delta) \), it follows that there will be no disturbance at \( n \) before the time \( \tau = n/k_1 \) i.e. before \( \tau_1 = \delta \). The disturbance reaches the position \( n \) at time \( n/k_1 \).

The function \( s_n(n, \delta) \) when the disturbance arrives at the position \( n \) may be found to be
\[ s_n(n, \delta) = 1/n^{1/2} \quad (n = 0, 1, 2, \ldots) \]

Thus, we may write

\[ \sigma_{\theta z} = L^{-1} \left[ \sigma_{\theta z} \right] = 2n^{(m+1)/2} \left[ 2\xi S_o \cdot \sigma_o(n, \tau) \right. \]
\[ \left. + \sum_{n=1}^{\infty} S_n \cdot s_n(n, \tau) \cdot j_1(A_n \xi) \right]. \quad (23) \]

Similarly, the radial stress \( \sigma_{r \phi} \) can be obtained as

\[ \sigma_{r \phi} = 2n^{(m+1)/2} \sum_{n=1}^{\infty} t_n(\eta, \tau) \cdot S_n \cdot A_n \cdot j_2(A_n \xi) \quad (24) \]

where

\[ t_n(\eta, p) = \frac{K_s(\gamma_n \eta)}{p \gamma_n K_{s+1}(\gamma_n)} \]

and

\[ t_n(\eta, \delta) = 0. \]

1.4 A particular case

The expressions for the stress components given in (23) and (24) are general in the sense that the function \( F(\xi) \) is arbitrary. In this section we shall assume simplest form of \( F(\xi) \) and take \( F(\xi) = c = \text{const.} \) With such a choice the coefficients \( S_n \) appearing in (23) and (24) have simple forms given by

\[ S_0 = c/3 \]

and

\[ S_n = c \cdot T_n \quad (n = 1, 2, 3, \ldots) \]
where

\[ T_n = \left[ \sum_{k=0}^{\infty} \frac{(-1)^k (A_n/2)^{2k+1}}{k! (k+1)! (2k + 3)} \right]^2 \]

Knowing these coefficients, the stress components \( \sigma_{\theta z} \) and \( \sigma_{\phi z} \) may be known from (23) and (24) respectively.