SECTION - A

STATICAL PROBLEMS
Elastic plates of variable modulus of rigidity play an important role in engineering structures; aircraft industry, shipbuilding industry are two important fields where such plates are widely used. Since the flexural rigidity $D$ of the plate depends upon the Young's modulus $E$ and the thickness $h$ of the plate, variation in $D$ may result from the variation of either $E$ or $h$. But in considering the variation in thickness $h$, one has to see that there is no abrupt variation in $h$, otherwise the differential equation of equilibrium may not remain valid in such a case. But for a general variation in $D$, the governing differential equation is a fourth order partial differential equation with variable coefficients and it is difficult to obtain its general solution. As a result, either we have to restrict variation in $D$ to get a complete solution or have to attempt for approximate solution by using approximate methods such as Ritz method, Galerkin method, perturbation method etc. In this chapter, the first alternative is opted.

Two different types of plate problems are discussed here. The first deals with the nonsymmetrical bending of a circular plate whereas the second is an investigation on the critical load for buckling of a rectangular plate. The effects of nonhomogeneity have been demonstrated by numerical examples in both the problems.
§1. NONSYMMETRICAL BENDING OF CIRCULAR PLATE WITH
AXISYMMETRICALLY VARYING FLEXURAL RIGIDITY

Abstract: Expressions for the deflection and moments have been obtained for a circular plate with axisymmetrically varying flexural rigidity under different types of nonsymmetrical load conditions. The effects of varying flexural rigidity have been shown in graphs and tables.

1.1 Introduction

The problems of symmetrical bending of nonhomogeneous circular plates have been solved by many investigators [21, 25, 10, 12] under different boundary conditions. Some problems of symmetrical bending of circular plates of non-uniform thickness or circular plates of variable flexural rigidity have also been discussed in Timoshenko and Woinowsky-Krieger [95]. But for nonsymmetrical bending of circular plates with a variable modulus of rigidity, no such attempt for the solution has been noticed by the author. Thus it is the object of this paper to find the deflection and moments in an annular circular plate with a variable modulus of rigidity under nonsymmetrical load conditions. The nonhomogeneity in this problem arises due to the variation of $D$ with radial distance. For the sake of simplicity and to get a complete solution, we have assumed in our problem that $D$ varies as the square of

the radial distance from the centre of the plate. With this assumption the governing differential equation becomes comparatively easy to handle and hence the solution becomes possible. Finally in order to study the effect of axisymmetrically varying flexural rigidity on deflection and moments, numerical evaluation of deflection and moments have been made for different values of the parameters involved.

1.2 Formulation of the problem and its general solution

Let the centre of the annular circular plate of inner radius \( b \) and outer radius \( a \) be taken as the pole of the polar co-ordinates \((r, \theta)\). Then, in terms of the non-dimensional variable \( \xi = r/a \), the differential equation of equilibrium of classical Kirchoff plate theory may be written as (see e.g. \([95]\), p 173 and p 282),

\[
\nabla^2 (D \nabla^2 w) - (1 - \nu) \left[ \frac{\partial^2 D}{\partial \xi^2} \left\{ \frac{1}{\xi^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{\xi} \frac{\partial w}{\partial \xi} \right\} \right]
+ \frac{1}{\xi} \frac{\partial D}{\partial \xi} \frac{\partial^2 w}{\partial \theta^2} = qa^4,
\]

where

\[
v^2 = \frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} \left( \frac{1}{\xi^2} \frac{\partial}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial^2}{\partial \theta^2}.
\]
In (1), the deflection \( w = w(\xi, \theta) \) and \( q = q(\xi, \theta) \) is the normal load acting on the plate at \( (\xi, \theta) \), \( D \) is the flexural rigidity of the plate given by

\[
D = Eh^3 / \left[ 12(1 - v^2) \right]
\]

and is supposed to be a function of \( \xi \) alone, \( h \) is the thickness of the plate.

If \( D \) satisfies a power law variation

\[
D = D' \xi^n,
\]

(2)

\( n \) being any real number, then the equation (1) becomes

\[
P' \left[ \left( \frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} + \frac{1}{\xi^2} \frac{\partial^2}{\partial \theta^2} \right) (\xi^n \frac{\partial^2 w}{\partial \xi^2} + \xi^{n-1} \frac{\partial w}{\partial \xi} + \xi^{n-2} \frac{\partial^2 w}{\partial \theta^2}) \right]
- n(1 - v) \left\{ (n-1)\xi \frac{\partial^2 w}{\partial \theta^2} + (n-1)\xi \frac{\partial w}{\partial \theta} + \xi^{n-2} \frac{\partial^2 w}{\partial \theta^2} \right\}
= qa .
\]

(3)

As equation (3) takes a relatively simple form and admits of a complete solution for \( n = 2 \), here we shall solve the problem for \( n = 2 \) only.

Let \( w_0(\xi, \theta) \) be the particular solution of (3) and \( w_1(\xi, \theta) \) be the solution of the equation (3) with right side zero,
then we have \( w = w_0 + w_1 \).

For \( w_1 \), we shall assume

\[
\begin{align*}
  w_1(\xi, \theta) &= R_0(\xi) + \sum_{m=1}^{\infty} R_m(\xi) \cos m\theta \\
  &= \sum_{m=1}^{\infty} R'_m(\xi) \sin m\theta.
\end{align*}
\]

The function \( R_0(\xi) \) corresponds to the symmetrical bending of the plate. Substitution of (4) into the differential equation for \( w_1 \) with \( n = 2 \) leads to the system of equations

\[
\begin{align*}
  \xi^2 \frac{d^2 V_m}{d\xi^2} + \frac{d}{d\xi} \left[ \xi^2 \frac{d}{d\xi} - \left( m^2 + 2(1-\nu) \right) \right] V_m &= 0, \ m = 0, 1, 2, \ldots \tag{5}
\end{align*}
\]

where

\[
V_m = \xi^2 \frac{d^2 R_m}{d\xi^2} + \frac{d}{d\xi} \left( \xi^2 \frac{d}{d\xi} - m^2 R_m \right), \ m = 0, 1, 2, \ldots
\]

or \( V_m \) having similar expression with \( R_m \) replaced by \( R'_m \).

From (5) and (6) we may easily obtain

\[
R_0 = A_0 \ln \xi + B_0 + C_0 \xi + D_0 \xi^2,
\]

\[
R_m = A_m \xi^m + B_m \xi^{-m} + C_m \xi^2 + D_m \xi^m, \ (m \geq 1)
\]

(7)
where \( s_{ml}, s_{m2} = \pm \left[m^2 + 2(1 - \nu)\right]^{\frac{1}{2}} \).

Similar expressions can be written for the function \( R^{'m} \). Substituting \( R^m \) and \( R^{'m} \) in (4), we obtain \( w^{'m} \). When \( q \) is specified, \( w^o \) can be found and hence \( w \) can be known. The constants \( A_m, B_m, C_m, D_m \) are to be determined from the boundary conditions. Knowing \( w \), we may also calculate the moments \( M_r, M_t, M_{rt} \) from the relations, (see e.g. [95], p 283)

\[
M_r = -\frac{D}{a^2} \left[ \frac{\partial^2 w}{\partial \xi^2} + \nu \left( \frac{1}{\xi} \frac{\partial w}{\partial \xi} + \frac{1}{\xi^2} \frac{\partial^2 w}{\partial \eta^2} \right) \right].
\]

\[
M_t = -\frac{D}{a^2} \left( \frac{1}{\xi} \frac{\partial w}{\partial \xi} + \frac{1}{\xi^2} \frac{\partial^2 w}{\partial \eta^2} + \nu \frac{\partial^2 w}{\partial \eta^2} \right),
\]

\[
M_{rt} = (1 - \nu) \frac{D}{a^2} \left( \frac{1}{\xi} \frac{\partial^2 w}{\partial \xi \partial \eta} - \frac{1}{\xi^2} \frac{\partial w}{\partial \eta} \right).
\]

1.3 Boundary conditions

We shall consider two cases of practical interest.

Case I. Inner boundary \( \xi = n \) (\( n = \frac{b}{a} \)) is clamped while the outer boundary \( \xi = 1 \) is free with no loads being applied. Thus

\[
w = \frac{\partial w}{\partial \xi} = 0 \text{ at } \xi = n \]

(11)
and \( M_r = 0 \) at \( \xi = 1 \),

\[
S = - \frac{D'}{a^3} \left[ \frac{d}{d\xi}[\frac{2}{\xi^3} \frac{\partial w}{\partial \xi} + \frac{\partial^3 w}{\partial \xi^3}] + 3\xi \frac{\partial w}{\partial \xi} - 2(1 - \nu) \frac{\partial^2 w}{\partial \xi^2} \right]
\]

\[+ (2\nu - 1) \frac{\partial w}{\partial \xi} \right] - \frac{1}{2a} \frac{\partial}{\partial \theta} M_{rt} = 0 \quad \text{at} \quad \xi = 1 .
\]

Here \( S \) is the resultant internal transverse normal shear force (see [95], p 284).

Case II. The circular ring plate is clamped along the inner edge \( \xi = \eta \) and loaded by a concentrated force \( Q \) at the outer boundary.

In this case, we have

\[
v = \frac{\partial w}{\partial \xi} = 0 \quad \text{at} \quad \xi = \eta .
\]

\[
(13)
\]

For the outer boundary, which is loaded only in one point, the conditions are (Timoshenko and Woinowsky-Krieger [95])

\[ M_r = 0 \]

and

\[
S = \frac{Q}{\pi a} \left[ \frac{1}{2} + \sum_{m=1}^{\infty} \cos m\theta \right] \quad \text{at} \quad \xi = 1 .
\]

\[ (14) \]
1.4 A particular example in Case I

As a particular case, let us consider a distribution of load which actually is the distribution in some practical cases

$q = k \xi \cos \theta$, \hspace{1cm} (15)

where $k$ is a constant.

If (15) is assumed, we find from (4) that the only non-zero $R_q$ is $R_\varphi$ and the deflection $w$ expressed in terms of the non-dimensional radius $\xi$ of the plate becomes

$w = \left[ A_1 \xi + \frac{B_1}{\xi} + C_1 \xi S_{11} + D_1 \xi S_{12} + L_\xi^3 \right] \cos \theta$, \hspace{1cm} (16)

where $L = \frac{ka^4}{16B' \left(3 + \nu\right)}$.

The moments $M_r$, $M_t$, $M_{rt}$ are then obtained from (8), (9) and (10) as

$M_r = -\frac{D'}{a^2} \left[ 2(3 + \nu) L_\xi^3 + \frac{2B_1}{\xi} (1 - \nu) + C_1 (s_{11} - 1)(s_{11} + \nu) \xi S_{11} \right. 
\left. + D_1 (s_{12} - 1)(s_{12} + \nu) \xi S_{12} \right] \cos \theta$, \hspace{1cm} (17)
\[ M_t = -\frac{D'}{a^2} \left[ 2(1+3\nu)L \xi^3 - \frac{2B_1}{\xi} (1+\nu) + C_1(s_{11}-1)(1+\nu s_{11}) \xi s_{11} \right. \\
+ D_1(s_{12}-1)(1+\nu s_{12}) \xi s_{12} \left. \right] \cos \theta , \] (18)
and
\[ M_{rt} = -(1-\nu) \frac{D'}{a^2} \left[ 2L \xi^3 - \frac{2B_1}{\xi} + C_1(s_{11}-1) \xi s_{11} \right. \\
+ D_1(s_{12}-1) \xi s_{12} \left. \right] \sin \theta . \] (19)

The boundary conditions (11) and (12) determine the arbitrary constants \( A_1, B_1, C_1 \) and \( D_1 \) in (16)-(19) from the set of linear equations
\[ G X = F , \]
where
\[ G = (G_{ij}), \quad i,j = 1,2,3,4 \]
\[ X = (A_1, B_1, C_1, D_1) , \]
\[ F = (F_1, F_2, F_3, F_4) , \]
\[ G_{11} = \eta, \quad G_{12} = 1/\eta, \quad G_{13} = \eta s_{11}, \quad G_{14} = s_{12}, \]
\[ G_{21} = 1, \quad G_{22} = -1/\eta, \quad G_{23} = s_{11} s_{11}^{-1}, \quad G_{24} = s_{12} s_{12}^{-1}. \]
\[ \varepsilon_{31} = 0, \quad \varepsilon_{32} = 2(1-\nu), \quad \varepsilon_{33} = (s_{11}-1)(s_{11}+\nu), \]
\[ \varepsilon_{34} = (s_{12}-1)(s_{12}+\nu), \]
\[ \varepsilon_{41} = 0, \quad \varepsilon_{42} = 6(1-\nu), \quad \varepsilon_{43} = s_{11}(s_{11}^2 + 3\nu - 4) + 3(1-\nu), \]
\[ \varepsilon_{44} = s_{12}(s_{12}^2 + 3\nu - 4) + 3(1-\nu), \]
\[ F_1 = -\eta^3 L, \quad F_2 = -3\eta^2 L, \quad F_3 = -2(3+\nu)L, \quad F_4 = -6(3+\nu)L. \]

Here \( X' \) represents transpose of the matrix \( X \).

1.5 Solution of the problem in Case II

If instead of a distributed load, only a concentrated force load \( Q \) is applied at a point \( A \) on the boundary of the plate with centre at the point \( O \) then measuring \( \theta \) from the line \( QA \), we may express \( w \) for the inner part of the plate in the form of series

\[ w = R_0(\xi) + \sum_{m=1}^{\infty} R_m(\xi) \cos m\theta, \tag{20} \]

where

\[ R_0(\xi) = A_0 \ln \xi + B_0 + C_0 \xi^{s_{01}} + D_0 \xi^{s_{02}} \]

and

\[ R_m(\xi) = A_m \xi^m + B_m \xi^{-m} + C_m \xi^{s_{m1}} + D_m \xi^{s_{m2}}, \quad m \geq 1. \]
The constants appearing in $R_m$ ($m \geq 0$) are to be determined from the boundary conditions (13) and (14). Applying these boundary conditions we obtain a set of four linear equations for four constants $A_m, B_m, C_m, D_m$ ($m \geq 0$) which may be solved easily. From (8) and (20), the radial moment $M_r$ is given by

$$M_r = -\frac{D'}{a^2} \left[ (-1+\nu)A_0 + s_{01} C_0 (s_{01} - 1+\nu) \xi - 1 \right]$$

$$+ s_{02} D_0 (s_{02} - 1+\nu) \xi + \sum_{m=1}^{\infty} m(n-1)(1-\nu)A_m \xi^m$$

$$+ m(m+1)(1-\nu)B_m \xi^m + \left[ s_{m1} (s_{m1} - 1) + \nu(s_{m1} - m^2) \right] C_m \xi^m$$

$$+ \left[ s_{m2} (s_{m2} - 1) + \nu(s_{m2} - m^2) \right] C_m \xi^m \cos \theta.$$  \hspace{1cm} (21)

The deflections and moments in the associated homogeneous case in which $D = D' = \text{constant}$ may be obtained from the general solution given to Timoshenko and Krieger [95].

1.6 Numerical results

To study the effect of axisymmetrically varying flexural rigidity as stipulated in (2) where $n = 2$, on the deflection and moments we have considered two different sizes of the plate viz., $\eta = 0.25, \eta = 0.5$ and have computed $w$ and $M_r$ for different value of $\xi$ and $\theta$. For a comparative study, these values are shown in
graphs and tables. For Case I with the surface load of the particular form (15), we have computed variations of

\[ \bar{w} = \frac{w}{L} \]

and

\[ \bar{M}_r = \frac{M_r a^2}{(D l)} \]

with radial distance \( \xi \). These are shown in Fig. 1, Fig. 2 and Fig. 3. The broken lines in each of these figures represent corresponding results in the associated homogeneous case. The values of \( \bar{w} \) and \( \bar{M}_r \) for fixed \( \xi \) but for different \( \vartheta \) are shown in Table I and II respectively along with the values \( \bar{w}^* \) and \( \bar{M}_r^* \) for the homogeneous case. The behaviours of deflection and moment are quite clear from the graphs and tables. For case II, the variations of \( w \) and \( \bar{M}_r = \frac{M_r a^2}{D'} \) are shown in Table III and Table IV respectively. The corresponding results for homogeneous case are given in Table V and Table VI.

From Table IV and Table VI it follows that as in the homogeneous case the largest bending moment \( M_r \) occurs at the inner boundary for \( \vartheta = 0 \).
Table I

Variations of $w$ with $\theta$ for fixed $\xi$ and $\eta$ in Case I

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\eta = 0.25$</th>
<th>$\eta = 0.5$</th>
<th>$\xi = 0.7$</th>
<th>$\eta = 0.25$</th>
<th>$\eta = 0.5$</th>
<th>$\xi = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\bar{w}$</td>
<td>$\bar{w}^*$</td>
<td>$\bar{w}$</td>
<td>$\bar{w}^*$</td>
<td>$\bar{w}$</td>
<td>$\bar{w}^*$</td>
</tr>
<tr>
<td>0</td>
<td>7.062</td>
<td>1.0179</td>
<td>0.3129</td>
<td>0.103</td>
<td>12.04</td>
<td>1.9487</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>4.994</td>
<td>0.7198</td>
<td>0.2213</td>
<td>0.0728</td>
<td>8.517</td>
<td>1.3779</td>
</tr>
<tr>
<td>$3\pi/4$</td>
<td>-4.994</td>
<td>-0.7198</td>
<td>-0.2213</td>
<td>-0.0728</td>
<td>-8.517</td>
<td>-1.3779</td>
</tr>
<tr>
<td>$\pi$</td>
<td>-7.062</td>
<td>-1.0179</td>
<td>-0.3129</td>
<td>-0.103</td>
<td>-12.04</td>
<td>-1.9487</td>
</tr>
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Table-II

Variations of $\bar{M}_T$ with $\theta$ for fixed $\xi$ and $\eta$ in Case I

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\bar{M}_T$</th>
<th>$\bar{M}_T^\eta$</th>
<th>$\bar{M}_T^\eta$</th>
<th>$\bar{M}_T^\eta$</th>
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<tr>
<td>$\theta = 0$</td>
<td>0.7154</td>
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<td>-1.674</td>
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<td>$\pi/4$</td>
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<td>-0.9988</td>
<td>-1.184</td>
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</tr>
<tr>
<td>$3\pi/4$</td>
<td>-0.5059</td>
<td>0.9988</td>
<td>1.184</td>
<td>1.4707</td>
</tr>
<tr>
<td>$\pi$</td>
<td>-0.7154</td>
<td>1.4125</td>
<td>1.674</td>
<td>2.0798</td>
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</table>
### Table-III

Variations of $w$ with $\theta$ for fixed $\xi$ and $\eta$ ($= 0.5$) in Case II

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
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<th>$2\pi/9$</th>
<th>$\pi/3$</th>
<th>$4\pi/9$</th>
<th>$5\pi/9$</th>
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<tbody>
<tr>
<td>$\xi = 0.7$</td>
<td>0.1004</td>
<td>0.0754</td>
<td>0.0316</td>
<td>0.0082</td>
<td>-0.0021</td>
<td>-0.0032</td>
<td>-0.0025</td>
<td>-0.0010</td>
<td>0</td>
<td>0.0002</td>
</tr>
<tr>
<td>$\xi = 1.0$</td>
<td>0.3791</td>
<td>0.2654</td>
<td>0.1191</td>
<td>0.0292</td>
<td>-0.0060</td>
<td>-0.0122</td>
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<td>-0.0034</td>
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### Table-IV

Variations of $\bar{M}_r$ with $\theta$ for fixed $\xi$ and $\eta$ ($= 0.5$) in Case II

<table>
<thead>
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<th>$\theta$</th>
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<th>$2\pi/9$</th>
<th>$\pi/3$</th>
<th>$4\pi/9$</th>
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<tr>
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<td>-2.240</td>
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<td>0.0910</td>
<td>0.1002</td>
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<td>$\xi = 0.7$</td>
<td>-0.7515</td>
<td>-0.5663</td>
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<td>0.0140</td>
<td>0.0090</td>
<td>0.0036</td>
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Table-V

Variations of $w$ with $\theta$ for fixed $\xi$ and $\eta$ (= 0.5) in Case II (Homogeneous case)

<table>
<thead>
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<th>$\theta$</th>
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<th>$\pi/3$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\xi=0.7$</td>
<td>0.0563</td>
<td>0.0440</td>
<td>0.0268</td>
<td>0.0184</td>
<td>0.0164</td>
<td>0.0168</td>
<td>0.0173</td>
<td>0.0176</td>
<td>0.0177</td>
<td>0.0177</td>
</tr>
<tr>
<td>$\xi=1.0$</td>
<td>0.2532</td>
<td>0.1819</td>
<td>0.1010</td>
<td>0.0779</td>
<td>0.0703</td>
<td>0.0714</td>
<td>0.0735</td>
<td>0.0746</td>
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Table-VI

Variations of $\bar{M}_r$ with $\theta$ for fixed $\xi$ and $\eta$ (= 0.5) in Case II (Homogeneous case)

<table>
<thead>
<tr>
<th>$\theta$</th>
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<th>$2\pi/9$</th>
<th>$\pi/3$</th>
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<th>$5\pi/9$</th>
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</thead>
<tbody>
<tr>
<td>$\xi=0.5$</td>
<td>-3.933</td>
<td>-3.004</td>
<td>-1.718</td>
<td>-1.149</td>
<td>-1.106</td>
<td>-1.104</td>
<td>-1.152</td>
<td>-1.173</td>
<td>-1.178</td>
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<tr>
<td>$\xi=0.7$</td>
<td>-1.584</td>
<td>-1.226</td>
<td>-0.8210</td>
<td>-0.5986</td>
<td>-0.5186</td>
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Abstract: In this note, critical load for buckling of a non-homogeneous, rectangular plate of which two opposite edges are simply supported has been found. The plate has been assumed to be acted on by uniform in-plane force over the edge. Levy's method has been followed to find the solution of the problem.

2.1 Introduction

From the structural point of view, knowledge of critical buckling load of elastic plates with variable modulus of rigidity is essential. Many investigators have attempted for the evaluation of such loads under different conditions. Several attempts for the critical load when the thickness varies have been made. Of these, mention may be made of Chehll and Dua [17], Hwang [42] and Whittrick and Ellen [98]. The critical buckling compression as well as critical temperature in a buckled, thin, nonhomogeneous, heated circular plate had been obtained by Chakravorty and Dey [13] where the Young's modulus was assumed to depend on the radial distance.

In the present investigation, our object is to find the critical load for buckling of a nonhomogeneous, rectangular plate
under the action of uniform compressive in-plane force $N$ acting normal to the edges of the plate. The problem has been solved and the critical values of $N$ for different sizes of the plate have been computed numerically and presented in a table.

2.2 Governing equation and solution of the problem

Let us consider a rectangular plate of which two opposite edges are simply supported while the remaining edges are either clamped or simply supported. The plate is loaded by a uniform force distribution $N$ normal to the edges. We choose the middle plane of the plate as the xy plane with x and y-axes parallel to the edges. The governing differential equation of equilibrium for the plate is

$$v^2(4v^2w) - (1 - v) \left[ \frac{d^2D}{dx^2} \frac{d^2w}{dy^2} + 2 \frac{d^2D}{dx^2} \frac{d^2w}{dx^2} \frac{d^2w}{dy^2} \right. $$

$$\left. + \frac{d^2D}{dy^2} \frac{d^2w}{dx^2} \right] + Nv^2w = 0,$$

where

$$v^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

In (1), the deflection $w = w(x,y)$ is measured at $(x,y)$ and $D$ is the variable flexural rigidity of the plate assumed as a linear function of $x$ only,
\[ D = kx + L, \quad (2) \]

\( k \) and \( L \) being constants.

By a choice of axes, (2) may be written as

\[ D = D_0 \left( \frac{x}{a} \right) \quad (3) \]

and the position of the plate may be defined as \( a \leq x \leq 3a, \)
\(-b \leq y \leq b \).

When (3) is used, (1) becomes

\[ \nabla^2 (x \nabla^2 w) + \left( \frac{Na}{D_0} \right) \nabla^2 w = 0. \quad (4) \]

If we introduce nondimensional variables \( \xi \) and \( \eta \) such that

\[ \xi = \frac{x}{a} \quad \text{and} \quad \eta = \frac{y}{b} \quad (5) \]

then the position of the plate is given by \( |\xi - 2| \leq 1, \quad |\eta| \leq 1 \).

Assuming that the edges \( |\eta| = 1 \) are simply supported, we use
Levy's method to solve (4). Thus we write

\[ w = \sum_{m=0}^{\infty} f_m(\xi) \cos s_m \eta, \quad (6) \]

where \( s_m = (2m + 1) \pi / 2 \).

Transforming the equation (4) in terms of \( \xi \) and \( \eta \) from
(5) and substituting \( w \) from (6) we get the system of differential
In order to solve (7) we introduce the transformation

\[ V_m = U_m(Z) \exp\left(-\frac{Z}{2}\right), \]

so that the equation (7) is reduced to a confluent hypergeometric equation

\[ Z \frac{d^2U_m}{dZ^2} + (2 - Z) \frac{dU_m}{dZ} - \left(1 - \frac{p_1^2}{2\beta_m}\right) U_m = 0. \] (10)

The solution of (10) is known (Abramowitz and Stegun [1]) and is given by

\[ U_m(Z) = C_m \varphi_{m1}(Z) + D_m \varphi_{m2}(Z), \quad m = 0, 1, 2, \ldots \]

where \( \varphi_{m1} \) and \( \varphi_{m2} \) are known confluent hypergeometric functions. Hence, from (9) and (8), \( f_m(\xi) \) is given by
\[ f_m(\xi) = A_m \exp(\beta_m \xi) + B_m \exp(-\beta_m \xi) + \frac{\exp(\beta_m \xi)}{2\beta_m} \left[ C_m \psi_m'(\xi) + D_m \psi_m''(\xi) \right] \]  

where

\[ \psi_m(\xi) = 2\beta_m \int_\xi^1 \exp(-2\beta_m \eta) \varphi_m(2\beta_m \eta) \, d\eta \]

\[ i = 1, 2 \]

\[ m = 0, 1, 2, \ldots \]

The arbitrary constants \( A_m, B_m, C_m, D_m \) in (11) are to be determined from the boundary condition on the edges \( |\xi - 2| = 1 \).

In the present problem we assume two types of boundary conditions.

**Case-I.** The edges \( |\xi - 2| = 1 \) are supported.

The equations for determination of \( A_m, B_m, C_m, D_m \) are

\[ f_m(\xi) = 0 = f_m''(\xi) \text{ at } \xi = 1, 3 \]

\[ m = 0, 1, 2, \ldots \]

This leads to the matrix equation

\[ G_m \cdot X_m = 0 \]

where

\[ X_m = (A_m, B_m, C_m, D_m)^T \]

and

\[ G_m = (g_{ij}^m), \, i, j = 1, 2, 3, 4 \] such that

\[ g_{11}^m = \delta_m (i = 1, 2), \quad g_{11}^m = \beta_m (i = 3, 4) \]  

(13a contd.)
\[ e_{12}^m = e_{32}^m / \beta_m = \exp(-2\beta_m), \quad e_{22}^m = e_{42}^m / \beta_m = \exp(-6\beta_m) \]

\[ e_{1,j+2}^m = \psi_{mj}(1)/(2\beta_m), \quad e_{2,j+2}^m = \psi_{mj}(3)/(2\beta_m), \]

\[ e_{3,j+2}^m = \bar{e}_{mj}(1)/(2\beta_m), \quad e_{4,j+2}^m = \bar{e}_{mj}(3)/(2\beta_m), \quad (j = 1, 2) \]

\[ \text{where} \quad \bar{e}_{mj}(\xi) = \beta \psi_{mi}(\xi) + 2 \exp(-2\beta_m \xi) \varphi_{mi}(2\beta_m \xi), \quad i = 1, 2, m = 0, 1, 2, \ldots \]

\textbf{Case II.} The edges \(|\xi - 2| = 1\) are clamped.

The constants \(A_m, B_m, C_m, D_m\) are to be determined from

the equations

\[ f_m(\xi) = 0 = f_m'(\xi) \quad \text{at} \quad \xi = 1, 3, \quad m = 0, 1, 2, \ldots \]

In this case too we obtain the matrix equation (12) with

the elements of \(G_m\) defined by

\[ e_{11}^m = 1, \quad (i = 1, 2, 3, 4) \]

\[ e_{12}^m = -e_{32}^m = \exp(-2\beta_m), \quad e_{22}^m = -e_{42}^m = \exp(-6\beta_m), \]

\[ e_{1,j+2}^m = \psi_{mj}(1)/(2\beta_m), \quad e_{2,j+2}^m = \psi_{mj}(3)/(2\beta_m), \]

\[ \text{(13b contd.)} \]
\[ \varepsilon_{3,j+2} = \frac{\zeta_j(1)}{2\beta_m}, \quad \varepsilon_{4,j+2} = \frac{\zeta_j(2)}{2\beta_m}, \quad (j = 1,2) \quad (13b) \]

where

\[ \zeta_j(\xi) = \varphi_m(\xi) + 2 \exp(-2\beta_m\xi) \int_{\xi}^{\infty} \varphi_m(2\beta_m\tau) \, d\tau, \quad \xi = 1,2 \]

\[ m = 0,1,2,\ldots \]

In both cases, the smallest value of \( N \) for which one of \( |G_m| \) given by (13a) or (13b), vanishes, will correspond to the critical value of the compressive load \( N \).

Thus the equation for determination of \( N \) is

\[ |G_m| = 0, \quad (14) \]

which leads to

\[ \varphi_m(2\beta_m)\varphi_m(2\beta_m) - \varphi_m(2\beta_m)\varphi_m(2\beta_m) = 0 \quad (15a) \]

for Case I and

\[ \exp(-2\beta_m) \left[ W_{m1} R_{m2}(1) - W_{m2} R_{m1}(1) \right] - \left[ \frac{W_{m1}}{S_{m2} + R_{m2}(1)} \right] = 0 \quad (15b) \]

for Case II where

\[ F_{m1} = 2\beta_m \int_{1}^{3} \exp(-2\beta_m\xi) \int \varphi_m(2\beta_m\xi) \, d\xi \, d\xi, \]
\[ W_{m_i} = F_{m_i} + S_{m_i}, \quad S_{m_i} = R_{m_i}(3) - R_{m_i}(1), \]

\[ R_{m_i}(\xi) = \exp(-2\beta_{m_i} \xi) \int_{\xi}^{\infty} \varphi_{m_i}(2\beta_{m_i} \xi) \, d\xi, \quad i = 1, 2, \]

\[ m = 0, 1, 2, \ldots \]

2.3 **Numerical calculation**

The critical value of the compressive load \( N \) for buckling is to be determined from (14). Since, each \( |C_m| \) involves confluent hypergeometric functions, numerical procedures have been followed to determine \( N \) and it has been observed that critical value of \( N \) corresponds to the vanishing of \( |C_0| \) irrespective of the size of the plate, as in the case of buckling of a homogeneous rectangular plate. The Table VII gives at least some informations for the magnitudes of the critical loads when all the edges are simply supported or two simply supported and two clamped under material nonhomogeneity of the medium as stipulated by (2) or (3). The Table VII also shows the corresponding critical load values for the associative homogeneous medium and for different sizes of the plate.
Table VII
Variations of $N'$ with size of the plate

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<tr>
<th>Value of $N'$ = $N_{cr} \frac{D_a}{a^2}$</th>
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<td>clamped ends</td>
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<th>Homogeneous</th>
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<th>Homogeneous</th>
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