Appendix

2

Rough Group

In this section, we review the definition of a rough group and prove some results in this context.

Definition 6.7.24. [63] Let \((U, \theta)\) be an approximation space and let \(*\) be a binary operation on \(U\). A subset \(G\) of \(U\) is called a rough group if it satisfies the following properties.

1. \(\forall x, y \in G, x * y \in \theta^-(G)\).
2. Associativity holds in \(\theta^-(G)\).
3. \(\forall x \in G, \exists e \in \theta^-(G)\) such that \(x * e = e * x = e\), \(e\) is called the rough identity.
4. \(\forall x \in G, \exists y \in G\) such that \(x * y = e = y * x\), \(y\) is called the rough inverse.

Definition 6.7.25. [63] A non-empty subset \(H\) of a rough group \(G\) is called its rough subgroup if it is a rough group itself with respect to the same operation on \(G\).

Definition 6.7.26. [63] A rough subgroup \(N\) of a rough group \(G\) is called a rough invariant subgroup if \(\forall a \in G, a * N = N * a\).

Results of this appendix are published in

Theorem 6.7.27. [63] A necessary and sufficient condition for a rough subgroup \( N \) of \( G \) to be a rough invariant subgroup is that \( \forall a \in G \) and \( n \in N \), \( a \ast n \ast a^{-1} \in N \).

Definition 6.7.28. [63] Let \((U_1, \theta_1)\) and \((U_2, \theta_2)\) be two approximation spaces and \(\ast, \ast'\) be binary operations over \(U_1\) and \(U_2\) respectively. Let \(G_1 \subseteq U_1\) and \(G_2 \subseteq U_2\) be two rough groups.

A mapping \(\phi : \theta_1^{-}(G_1) \rightarrow \theta_2^{-}(G_2)\) satisfying

\[
\phi(x \ast y) = \phi(x) \ast' \phi(y)
\]

\(\forall x, y \in \theta_1^{-}(G_1)\), is called a rough group homomorphism from \(G_1\) to \(G_2\).

Definition 6.7.29. A rough group homomorphism \(\phi\) from a rough group \(G_1\) to a rough group \(G_2\) is called

- a rough epimorphism (or surjective homomorphism) if \(\phi : \theta_1^{-}(G_1) \rightarrow \theta_2^{-}(G_2)\) is onto. That is \(\forall y \in \theta_2^{-}(G_2)\), \(\exists x \in \theta_1^{-}(G_1)\) such that \(\phi(x) = y\)

- a rough embedding (or monomorphism) if \(\phi : \theta_1^{-}(G_1) \rightarrow \theta_2^{-}(G_2)\) is one-one.

- a rough isomorphism if \(\phi : \theta_1^{-}(G_1) \rightarrow \theta_2^{-}(G_2)\) is both one-one and onto.

Definition 6.7.30. A rough group homomorphism is called

- a rough endomorphism if it is a rough homomorphism from the rough group into itself.

- a rough automorphism if it is a rough isomorphism from a rough group into itself.

Theorem 6.7.31. Let \(G\) be a rough group and \(\phi_1\) and \(\phi_2\) be two rough group homomorphisms on \(G\). Then the composition \(\phi_1 \circ \phi_2\) is a rough group homomorphism on \(G\).

Proof. Let \(G\) be a rough group and let \(\phi_1, \phi_2\) be two rough group homomorphisms on \(G\).

Then \(\phi_1, \phi_2 : \theta^{-}(G) \rightarrow \theta^{-}(G)\) such that \(\forall x, y \in \theta^{-}(G)\)

\[
\phi_1(x \ast y) = \phi_1(x) \ast \phi_1(y) \quad \text{and} \quad \phi_2(x \ast y) = \phi_2(x) \ast \phi_2(y)
\]
Now $\forall \ x, y \in \theta^-(G)$

$$(\phi_1 o \phi_2)(x * y) = \phi_1(\phi_2(x * y)) = \phi_1(\phi_2(x) * \phi_2(y)) = (\phi_1 o \phi_2)(x) * (\phi_1 o \phi_2)(y)$$

Therefore, $\phi_1 o \phi_2$ is a rough group homomorphism.

**Theorem 6.7.32.** The rough group endomorphisms of a rough group $G$ form a monoid.

**Proof.** Let $G$ be a rough group and let $\phi_1, \phi_2$ be two rough group homomorphisms on $G$. Then $\phi_1, \phi_2 : \theta^-(G) \to \theta^-(G)$ such that $\forall x, y \in \theta^-(G)$

$$\phi_1(x * y) = \phi_1(x) * \phi_1(y) \quad \text{and} \quad \phi_2(x * y) = \phi_2(x) * \phi_2(y)$$

Now $\forall \ x, y \in \theta^-(G)$

$$(\phi_2 o \phi_1)(x * y) = \phi_2(\phi_1(x * y)) = (\phi_2 o \phi_1)(x) * (\phi_2 o \phi_1)(y)$$

Therefore, $\phi_2 o \phi_1$ is a rough group endomorphism on $G$. Associative property also holds. The identity function $I$ on $\theta^-(G)$ such that $I(x) = x$, is also a rough group endomorphism. Hence the theorem is proved.

**Theorem 6.7.33.** The rough group automorphisms of a rough group $G$ form a group.

**Proof.** Let $G$ be a rough group and let $\phi_1, \phi_2$ be two rough group homomorphisms on $G$.

Then $\phi_1, \phi_2 : \theta^-(G) \to \theta^-(G)$ such that $\forall x, y \in \theta^-(G)$

$$\phi_1(x * y) = \phi_1(x) * \phi_1(y) \quad \text{and} \quad \phi_2(x * y) = \phi_2(x) * \phi_2(y)$$

and $\phi_1, \phi_2$ are one- one and onto.

By theorem (6.7.31), $\phi_1 o \phi_2$ is a rough group homomorphism on $G$. Since $\phi_2$ and $\phi_1$ are one-one and onto, $\phi_1 o \phi_2$ is also one-one and onto. Therefore $\phi_1 o \phi_2$ is a rough group automorphism on $G$.

Associative property also holds. The identity function $I$ on $\theta^-(G)$ such that $I(x) = x$, is also a rough group automorphism. For any rough group automorphism $\phi_1$ on $G$, $\exists$ a rough group automorphism $\phi_2$ on $G$ such that $\phi_1 o \phi_2 = I$. This completes the proof.
Anti-homomorphism of Rough Groups

This section deals with anti-homomorphic properties of rough group. Let \((U_1, \theta_1)\) and 
\((U_2, \theta_2)\) be two approximation spaces and \(*, *'\) be binary operations over \(U_1\) and \(U_2\) respectively. Let \(G_1 \subseteq U_1\) and \(G_2 \subseteq U_2\) be two rough groups.

**Definition 6.7.34.** A mapping \(\phi : \theta_1^{-1}(G_1) \rightarrow \theta_2^{-1}(G_2)\) satisfying

\[
\phi(x * y) = \phi(y) *' \phi(x)
\]

\(\forall x, y \in \theta_1^{-1}(G_1)\), is called a rough group anti-homomorphism from \(G_1\) to \(G_2\).

**Theorem 6.7.35.** Let \(G_1\) and \(G_2\) be two rough groups and \(\phi\) be a rough group anti-epimorphism from \(G_1\) to \(G_2\). If \(*\) satisfies the commutative law, then \(*'\) also satisfies the commutative law.

**Proof.** For \(x', y' \in \theta_2^{-1}(G_2)\), \(\exists x, y \in \theta_1^{-1}(G_1)\) such that \(\phi(x) = x'\) and \(\phi(y) = y'\). Since \(x * y = y * x\) we have

\[
x' *' y' = \phi(x) *' \phi(y) = \phi(y * x) = \phi(x * y) = \phi(y) *' \phi(x) = y' *' x'
\]

Therefore \(*'\) is commutative.

**Theorem 6.7.36.** Let \(\phi\) be a rough group anti-homomorphism from \(G_1\) to \(G_2\). Then

1. If \(e\) is the rough identity of \(G_1\), then \(\phi(e)\) is the rough identity of \(\phi(G_1)\).

2. If \(x \in G_1\), then \(\phi(x^{-1}) = (\phi(x))^{-1}\).

provided \(\phi(\theta_1^{-1}(G_1)) = \theta_2^{-1}(\phi(G_1))\).

**Proof.** For \(x' \in \phi(G_1)\), \(\exists x \in G_1\) such that \(\phi(x) = x'\)

1. Because \(R_1\) is a rough group, \(e \in \theta_1^{-1}(G_1)\) such that \(x * e = x = e * x\). Therefore

\[
x' *' \phi(e) = \phi(x) *' \phi(e) = \phi(e * x) = \phi(e * e) = \phi(e) *' \phi(x) = \phi(e) *' x'
\]

Also \(\phi(e) \in \phi(\theta_1^{-1}(G_1))\). Because \(\phi(\theta_1^{-1}(G_1)) = \theta_2^{-1}(\phi(G_1))\), \(\phi(e) \in \theta_2^{-1}(\phi(G_1))\). Therefore \(\phi(e)\) is the rough identity of \(\phi(G_1)\).
2. Because $G_1$ is a rough group, $x^{-1} \in G_1$ such that $x \ast x^{-1} = e = x^{-1} \ast x$. So we get

$$\phi(x) \ast' \phi(x^{-1}) = \phi(x^{-1} \ast x) = \phi(x \ast x^{-1}) = \phi(x^{-1}) \ast' \phi(x)$$

Thus we get by definition $\phi(x^{-1}) = (\phi(x))^{-1}$.

**Theorem 6.7.37.** Let $\phi$ be a rough group anti- homomorphism from $G_1$ to $G_2$. Then $\phi(G_1)$ is a rough group if $\phi(\theta^-_1(G_1)) = \theta^-_2(\phi(G_1))$.

**Proof.** For $x', y' \in \phi(G_1)$, $\exists x, y \in G_1$ such that $\phi(x) = x'$ and $\phi(y) = y'$.

1. Now $\phi(y \ast x) = \phi(x) \ast' \phi(y) = x' \ast' y'$

   since $\phi(y \ast x) \in \phi(\theta^-_1(G_1))$, we have $\phi(y \ast x) \in \theta^-_2(\phi(G_1))$. That is $x' \ast' y' \in \theta^-_2(\phi(G_1))$

2. Since $e \in \theta^-_1(G_1)$, we have $\phi(e) \in \phi(\theta^-_1(G_1)) = \theta^-_2(\phi(G_1))$. Therefore $\phi(x) \ast' \phi(e) = \phi(e \ast x) = \phi(x \ast e) = \phi(e) \ast' \phi(x)$ $\forall \phi(x) \in \phi(G_1)$.

3. Since $G_1$ is a rough group, $\forall x, y, z \in G_1, x \ast (y \ast z) = (x \ast y) \ast z$

   $$\phi(x \ast (y \ast z)) = \phi((x \ast y) \ast z) = (\phi(x) \ast' \phi(y)) \ast' \phi(z)$$

4. Because $G_1$ is a rough group, for $x \in G_1, \exists x^{-1} \in G_1$ such that $x \ast x^{-1} = e = x^{-1} \ast x$. Thus $\phi(x^{-1}) \in \phi(G_1)$ such that

   $$\phi(x) \ast' \phi(x^{-1}) = \phi(x^{-1} \ast x) = \phi(x \ast x^{-1}) = \phi(x) \ast' \phi(x)$$

   So by definition $\phi(x^{-1}) = (\phi(x))^{-1}$.

Therefore, $\phi(G_1)$ is a rough group.

**Theorem 6.7.38.** Let $H$ be a rough subgroup of $G_1$ and let $\phi$ be a rough group anti- homomorphism from $G_1$ to $G_2$. Then $\phi(H)$ is a rough subgroup of $G_2$ if $\phi(\theta^-_1(H)) = \theta^-_2(\phi(H))$. 
Proof. The proof is similar to that of theorem (6.7.37).

Theorem 6.7.39. Let $H_2$ be a rough subgroup of $G_2$ and let \( \phi \) be a rough group anti-monomorphism from $G_1$ to $G_2$. Then $H_1 = \phi^{-1}(H_2)$ is a rough subgroup of $G_1$ if $\phi(\theta_1^- (H_1)) = \theta_2^- (\phi(H_1))$.

Proof. Since $H_1 = \phi^{-1}(H_2)$, we have $\phi(H_1) = H_2$, and so $\theta_2^- (H_2) = \theta_2^- (\phi(H_1)) = \phi(\theta_1^- (H_1))$

1. \( \forall x, y \in H_1, \) we have $\phi(x), \phi(y) \in H_2$. Since $H_2$ is rough group, we get $\phi(y) \neq \phi(x)$, $x \in \theta_2^- (H_2)$. That is, $\phi(x \ast y) \in \phi(\theta_1^- (H_1))$. Thus we get $x \ast y \in \theta_1^- (H_1)$.

2. \( \forall x \in H_1, \) we have $\phi(x) \in H_2$. Since $H_2$ is a rough group, $\phi(x^{-1}) = (\phi(x))^{-1}$ \( \in H_2 \). That is, $\phi(x^{-1}) \in \phi(H_1)$. Thus $x^{-1} \in H_1$

Therefore $H_1$ is a rough subgroup of $G_1$.

Theorem 6.7.40. Let $N$ be a rough invariant subgroup of $G_1$ and let \( \phi \) be a rough group anti-homomorphism from $G_1$ to $G_2$. Then $\phi(N)$ is a rough invariant subgroup of $G_2$ if $\phi(\theta_1^- (N)) = \theta_2^- (\phi(N))$ and $\phi(G_1) = G_2$.

Proof. The proof is similar to that of theorem (6.7.37).

Theorem 6.7.41. Let $N_2$ be a rough invariant subgroup of $G_2$ and let \( \phi \) be a rough group anti-monomorphism from $G_1$ to $G_2$. Then $N_1 = \phi^{-1}(N_2)$ is a rough invariant subgroup of $G_1$ if $\phi(\theta_1^- (N_1)) = \theta_2^- (\phi(N_1))$ and $\phi(G_1) = G_2$.

Proof. The proof is similar to that of theorem (6.7.39).

Definition 6.7.42. Let $G_1 \subseteq U_1$ and $G_2 \subseteq U_2$ be two rough groups and \( \phi \) be a rough group anti-homomorphism from $G_1$ to $G_2$. Then $\{x/\phi(x) = e_2, \ x \in G_1\}$ where $e_2$ is the rough identity of $G_2$, is called rough group anti-homomorphism kernel, denoted by $^\ast \ker \phi$.

Theorem 6.7.43. Let \( \phi \) be a rough group anti-homomorphism from $G_1$ to $G_2$. Then rough group anti-homomorphism kernel is a rough invariant subgroup of $G_1$.

Proof. \( \forall x, y \in ^\ast \ker \phi, \phi(x) = e_2, \phi(y) = e_2. \)

1. We have $\phi(x \ast y) = \phi(y) \neq \phi(x) = e_2$. Therefore, $x \ast y \in ^\ast \ker \phi \subseteq \theta_1^- (^\ast \ker \phi)$. 

2. Since \( \phi(x^{-1}) = (\phi(x))^{-1} = (e_2)^{-1} = e_2 \), we get \( x^{-1} \in \ker \phi \).
Therefore, \( \ker \phi \) is a rough subgroup of \( G_1 \).

3. \( \forall x \in G \) and \( r \in \ker \phi \), We have \( \phi(x \ast r \ast x^{-1}) = \phi(x^{-1}) \ast \phi(r) \ast \phi(x) = e_2 \).
Therefore, \( x \ast r \ast x^{-1} \in \ker \phi \).

Therefore by theorem (6.7.27), \( \ker \phi \) is a rough invariant subgroup of \( G_1 \).

Theorem 6.7.44. Let \( \phi \) be a rough group anti-epimorphism from \( G_1 \) to \( G_2 \). Then \( \phi \) is a rough group anti-isomorphism if and only if rough group anti-homomorphism kernel is \( \{ e \} \), where \( e \) denotes the rough identity of \( G_1 \).

Proof. It is straight forward.

Theorem 6.7.45. Let \( G \) be a rough group and \( \phi_1 \) and \( \phi_2 \) be two rough group anti-homomorphisms on \( G \). Then the composition \( \phi_1 \circ \phi_2 \) is a rough group homomorphism on \( G \).

Proof. Let \( G \) be a rough group and let \( \phi_1 \), \( \phi_2 \) be two rough group anti-homomorphisms on \( G \).
Then \( \phi_1, \phi_2 : \theta^{-}(G) \rightarrow \theta^{-}(G) \) such that \( \forall x, y \in \theta^{-}(G) \)
\[
\phi_1(x \ast y) = \phi_1(y) \ast \phi_1(x) \quad \text{and} \quad \phi_2(x \ast y) = \phi_2(y) \ast \phi_2(x)
\]
Now \( \forall x, y \in \theta^{-}(G) \)
\[
(\phi_1 \circ \phi_2)(x \ast y) = \phi_1(\phi_2(x \ast y)) = \phi_1(\phi_2(y) \ast \phi_2(x)) = (\phi_1 \circ \phi_2)(x) \ast (\phi_1 \circ \phi_2)(y)
\]
Therefore, \( \phi_1 \circ \phi_2 \) is a rough group homomorphism on \( G \).

Theorem 6.7.46. Let \( G \) be a rough group and \( \phi_1 \) be a rough group anti-homomorphism and \( \phi_2 \) be a rough group homomorphism on \( G \). Then the composition \( \phi_1 \circ \phi_2 \) is a rough group anti-homomorphism on \( G \).

Proof. Let \( G \) be a rough group and let \( \phi_1 \) be a rough group anti-homomorphism on \( G \) and \( \phi_2 \) be a rough group homomorphism on \( G \).
Then \( \phi_1, \phi_2 : \theta^-(G) \to \theta^-(G) \) such that \( \forall x, y \in \theta^-(G) \)

\[
\phi_1(x \ast y) = \phi_1(y) \ast \phi_1(x) \quad \text{and} \quad \phi_2(x \ast y) = \phi_2(x) \ast \phi_2(y)
\]

Now \( \forall x, y \in \theta^-(G) \)

\[
(\phi_1 \circ \phi_2)(x \ast y) = \phi_1(\phi_2(x \ast y)) = \phi_1(\phi_2(x) \ast \phi_2(y)) = (\phi_1 \circ \phi_2)(y) \ast (\phi_1 \circ \phi_2)(x)
\]

Therefore, \( \phi_1 \circ \phi_2 \) is a rough group anti-homomorphism on \( G \).

**Remark.** From the above two theorems we have got that,

- Composition of two rough group anti-homomorphisms is a rough group homomorphism.
- Composition of a rough group homomorphism and a rough group anti-homomorphism is a rough group anti-homomorphism.