Rough Anti-Fuzzy Ideals

6.1 Introduction

Extensive researches has been carried out to compare the theory of rough set with other theories of uncertainty. Fuzzy set theory is one among them. R. Biswas[9] introduced the idea of anti-fuzzy subgroups which was extended by many researchers. Azam, Mamun, Nasrin[6] studied anti-fuzzy ideal of a ring. Motivated by this, in this chapter, we extend these notions to the rough settings and study roughness of these algebraic structures.

6.2 Rough Anti-fuzzy Subring

In this section, we define rough anti-fuzzy subring and study its homomorphic and anti-homomorphic properties.

1Results of this chapter are published in
1. Rough Anti-Fuzzy Subrings and Their Properties (communicated)
3. Rough Anti-Fuzzy Bi-ideals and Their Properties in Rings (communicated).
4. Rough Anti-Fuzzy Prime and Primary ideals in Rings (communicated).
5. Rough Anti-Fuzzy Semi-Prime ideals in Rings(communicated).
Definition 6.2.1. [5] A fuzzy subset \( \mu \) of a ring \( R \) is called an anti-fuzzy subring of \( R \) if

1. \( \mu(x - y) \leq \mu(x) \lor \mu(y) \)
2. \( \mu(xy) \leq \mu(x) \lor \mu(y) \)

for all \( x, y \in R \).

Definition 6.2.2. A fuzzy subset \( \mu \) of a ring \( R \) is called an upper rough anti-fuzzy subring of \( R \) if \( \theta^- (\mu) \) is an anti-fuzzy subring of \( R \) and a lower rough anti-fuzzy subring of \( R \) if \( \theta^- (\mu) \) is an anti-fuzzy subring of \( R \).

Let \( \mu \) be a fuzzy subset of \( R \) and \( \theta^- (\mu) = (\theta^- (\mu), \theta^- (\mu)) \) a rough fuzzy set. If \( \theta^- (\mu) \) and \( \theta^- (\mu) \) are anti-fuzzy subrings of \( R \), then \( \mu \) is called a rough anti-fuzzy subring.

Example. Consider the ring \( R = (\mathbb{Z}_4, +, .) \) and subring \((S, +, .)\), where \( S = \{0, 2\} \). Define a congruence \( \theta \) on \( \mathbb{Z}_4 \) as \( a \equiv b \mod S \iff a - b \in S \). Define a fuzzy subset \( \mu : R \to [0, 1] \) as, \( \forall x \in R \)

\[
\mu(x) = \begin{cases} 
0.1 & \text{if } x = 0 \\
0.5 & \text{if } x \neq 0
\end{cases}
\]

Then

\[
\theta^- (\mu)(x) = \begin{cases} 
0.5 & \forall x \in R
\end{cases}
\]

and

\[
\theta^- (\mu)(x) = \begin{cases} 
0.1 & \text{if } x = 0, 2 \\
0.5 & \text{if } x = 1, 3
\end{cases}
\]

Clearly \( \mu \) is a rough set. It is obvious that \( \theta^- (\mu) \) is an anti-fuzzy subring.
Clearly $\theta_-(\mu)(x - y) \leq \theta_-(\mu)(x) \vee \theta_-(\mu)(y)$

\[
\begin{array}{c|cccc}
  x:y & 0 & 1 & 2 & 3 \\
  \hline
  0 & 0 & 0 & 0 & 0 \\
  1 & 0 & 1 & 2 & 3 \\
  2 & 0 & 2 & 0 & 2 \\
  3 & 0 & 3 & 2 & 1 \\
\end{array}
\begin{array}{c|cccc}
  \theta_-(\mu)(xy) & 0.1 & 0.1 & 0.1 & 0.1 \\
  0.5 & 0.1 & 0.5 & 0.1 & 0.5 \\
  0.1 & 0.1 & 0.1 & 0.1 & 0.1 \\
  0.5 & 0.1 & 0.5 & 0.1 & 0.5 \\
\end{array}
\]

$\theta_-(\mu)(xy) \leq \theta_-(\mu)(x) \vee \theta_-(\mu)(y)$.

Therefore $\theta_-(\mu)$ is an anti-fuzzy subring. Hence, $\mu$ is a rough anti-fuzzy subring.

**Theorem 6.2.3.** Let $\theta$ be a complete congruence relation on $R$. If $\mu$ is an anti-fuzzy subring of $R$, then $\theta^-(\mu)$ is an anti-fuzzy subring of $R$. 
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Proof. For \( x, y \in R \),

\[
\theta^-(\mu)(x - y) = \bigvee_{z \in [x - y]_\theta} \mu(z)
\]

\[
= \bigvee_{z \in [(x - y)_\theta]_\theta} \mu(z)
\]

\[
= \bigvee_{a \in (x)_\theta, b \in (y)_\theta} \mu(a - b)
\]

\[
\leq \bigvee_{a \in (x)_\theta, b \in (y)_\theta} (\mu(a) \lor \mu(b))
\]

(\because \mu \text{ is an anti-fuzzy subring})

\[
= \bigvee_{a \in (x)_\theta} \mu(a) \lor \bigvee_{b \in (y)_\theta} \mu(b)
\]

\[
= \theta^-(\mu)(x) \lor \theta^-(\mu)(y)
\]

Hence \( \theta^-(\mu)(x - y) \leq \theta^-(\mu)(x) \lor \theta^-(\mu)(y) \).

Also we have,

\[
\theta^-(\mu)(xy) = \bigvee_{z \in [xy]_\theta} \mu(z)
\]

\[
= \bigvee_{a \in (x)_\theta, b \in (y)_\theta} \mu(ab)
\]

\[
\leq \bigvee_{a \in (x)_\theta, b \in (y)_\theta} (\mu(a) \lor \mu(b))
\]

(\because \mu \text{ is an anti-fuzzy subring})

\[
= \bigvee_{a \in (x)_\theta} \mu(a) \lor \bigvee_{b \in (y)_\theta} \mu(b)
\]

\[
= \theta^-(\mu)(x) \lor \theta^-(\mu)(y)
\]

Hence \( \theta^-(\mu)(xy) \leq \theta^-(\mu)(x) \lor \theta^-(\mu)(y) \). Therefore, \( \theta^-(\mu) \) is an anti-fuzzy subring of \( R \).

Remark. The converse of the theorem (6.2.3) does not hold in general.

Example. Consider the ring \( R = (\mathbb{Z}_4, +, \cdot) \) and subring \( (S, +, \cdot) \), where \( S = \{0, 2\} \). Define a congruence \( \theta \) on \( \mathbb{Z}_4 \) as \( a \equiv b \mod S \) iff \( a - b \in S \). Define a fuzzy subset
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$\mu : R \rightarrow [0, 1]$ as

$$\mu(x) = \begin{cases} 
0.5 & \text{if } x=0, 3 \\
0.1 & \text{if } x=1, 2 
\end{cases}$$

\begin{tabular}{c|cccc}
  x : y & 0 & 1 & 2 & 3 \\
  \hline 
  0 & 0 & 3 & 2 & 1 \\
  1 & 1 & 0 & 3 & 2 \\
  2 & 2 & 1 & 0 & 3 \\
  3 & 3 & 2 & 1 & 0 \\
\end{tabular}

\begin{tabular}{c|cccc}
  x : y & 0.5 & 0.1 & 0.1 & 0.5 \\
  \hline 
  0 & 0.5 & 0.5 & 0.1 & 0.1 \\
  1 & 0.1 & 0.5 & 0.5 & 0.1 \\
  2 & 0.1 & 0.1 & 0.5 & 0.5 \\
  3 & 0.5 & 0.1 & 0.1 & 0.5 \\
\end{tabular}

$\mu(x-y)$

Clearly $\mu(x-y) \not\leq \mu(x) \lor \mu(y)$.

Therefore $\mu$ is not an anti-fuzzy subring.

$$\theta^-(\mu)(x) = \begin{cases} 
0.5 & \forall x \in R 
\end{cases}$$

Obviously, $\theta^-(\mu)(x-y) \leq \theta^-(\mu)(x) \lor \theta^-(\mu)(y)$ and

$\theta^-(\mu)(xy) \leq \theta^-(\mu)(x) \lor \theta^-(\mu)(y)$. Therefore $\theta^-(\mu)$ is an anti-fuzzy subring.

**Theorem 6.2.4.** Let $\theta$ be a complete congruence relation on $R$. If $\mu$ is an anti-fuzzy subring of $R$, then $\theta^-(\mu)$ is an anti-fuzzy subring of $R$.

**Proof.** For $x,y \in R$,

$$\theta^-(\mu)(x-y) = \bigwedge_{z \in [x-y]_{\theta}} \mu(z)$$

$$= \bigwedge_{a \in [x]_{\theta}, b \in [y]_{\theta}} \mu(a-b)$$

$$\leq \bigwedge_{a \in [x]_{\theta}, b \in [y]_{\theta}} (\mu(a) \lor \mu(b)) \quad (\because \mu \text{ is an anti-fuzzy subring})$$

$$= \bigwedge_{a \in [x]_{\theta}} \mu(a) \lor \bigwedge_{b \in [y]_{\theta}} \mu(b)$$

$$= \theta^-(\mu)(x) \lor \theta^-(\mu)(y)$$
Hence $\theta_-(\mu)(x - y) \leq \theta_-(\mu)(x) \lor \theta_-(\mu)(y)$. Also we have,

$$\theta_-(\mu)(xy) = \bigwedge_{z \in [xy]_\theta} \mu(z)$$
$$= \bigwedge_{a \in [x]_\theta,b \in [y]_\theta} \mu(ab)$$
$$\leq \bigwedge_{a \in [x]_\theta,b \in [y]_\theta} (\mu(a) \lor \mu(b)) \quad (\because \mu \text{ is an anti-fuzzy subring})$$
$$= \bigwedge_{a \in [x]_\theta} \mu(a) \lor \bigwedge_{b \in [y]_\theta} \mu(b)$$
$$= \theta_-(\mu)(x) \lor \theta_-(\mu)(y)$$

Hence $\theta_-(\mu)(xy) \leq \theta_-(\mu)(x) \lor \theta_-(\mu)(y)$. Therefore, $\theta_-(\mu)$ is an anti-fuzzy subring of $R$.

**Corollary 6.2.5.** Let $\theta$ be a complete congruence relation on $R$ and $\mu$ be an anti-fuzzy subring of $R$, then $\mu$ is a rough anti-fuzzy subring of $R$.

**Proof.** This follows from Theorems (6.2.3) and (6.2.4).

**Definition 6.2.6.** [6] Let $\mu$ be a fuzzy subset of $R$. Then the sets $\mu_t = \{ x \in R \mid \mu(x) \leq t \}$, $\mu_t^* = \{ x \in R \mid \mu(x) < t \}$, where $t \in [0, 1]$ are called respectively, $t$-lower level subset and $t$-strong lower level subset of $\mu$.

**Theorem 6.2.7.** Let $\mu$ be a fuzzy subset of $R$ and $t \in [0, 1]$, then

1. $(\theta^- (\mu))_t^* = \theta_-(\mu_t^*)$
2. $(\theta_-(\mu))_t = \theta^- (\mu_t)$
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Proof. 1. We have

\[ x \in (\theta^{-} (\mu))^s_t \iff \theta^{-} (\mu)(x) < t \]
\[ \iff \bigvee_{a \in [x]_\theta} \mu(a) < t \]
\[ \iff \mu(a) < t \ \forall \ a \in [x]_\theta \]
\[ \iff [x]_\theta \subseteq \mu^s_t \]
\[ \iff x \in \theta_-(\mu^s_t) \]

2. Also we have,

\[ x \in (\theta_-(\mu))^s_t \iff \theta_-(\mu)(x) \leq t \]
\[ \iff \bigwedge_{a \in [x]_\theta} \mu(a) \leq t \]
\[ \iff \exists \ a \in [x]_\theta \text{ such that } \mu(a) \leq t \]
\[ \iff [x]_\theta \cap \mu_t \neq \phi \]
\[ \iff x \in \theta^s_t \]

Remark 6.2.8. [65] Let X and Y be two non-empty sets, \( f : X \rightarrow Y \), \( \mu \) be a fuzzy subset of \( X \). Then \( f_-(\mu) \), the anti-image of \( \mu \) under \( f \) is a fuzzy subset of \( Y \) defined by

\[ f_-(\mu)(y) = \begin{cases} \bigwedge \{ \mu(x) ; f(x) = y \} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases} \]

Theorem 6.2.9. Let \( f \) be a homomorphism (anti-homomorphism) from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be a fuzzy subset of \( R_1 \). Then

1. \( f_-(\theta_1^{-}(\mu)) \supseteq \theta_2^{-}(f_-(\mu)) \). If \( f \) is one to one \( f_-(\theta_1^{-}(\mu)) = \theta_2^{-}(f_-(\mu)) \).

2. \( f_-(\theta_1^{-}(\mu)) = \theta_2^{-}(f_-(\mu)) \).
**Proof.** For \( x \in R_2 \)

\[
\begin{align*}
    f_-(\theta_1^-(\mu))(x) &= \bigwedge_{f(a) = x} \theta_1^-(\mu)(a) = \bigwedge_{f(a) = x} \bigvee_{z \in [a]_{\theta_1}} \mu(z) \\
    &= \bigwedge_{f(a) = x} \bigvee_{a \in [z]_{\theta_1}} \mu(a) \geq \bigvee_{a \in [z]_{\theta_1}} \bigwedge_{f(a) = x} \mu(a) \\
    &= \bigvee_{x \in [f(z)]_{\theta_2}} f_-(\mu)(x) = \bigvee_{f(z) \in [x]_{\theta_2}} f_-(\mu)(f(z)) \\
    &= \theta_2^-(f_-(\mu))(x)
\end{align*}
\]

Therefore, \( f_-(\theta_1^-(\mu)) \supseteq \theta_2^-(f_-(\mu)) \).

If \( f \) is one to one, \( f_-(\theta_1^-(\mu)) = \theta_2^-(f_-(\mu)) \) is clear.

\[
\begin{align*}
    f_-(\theta_1^-(\mu))(x) &= \bigwedge_{f(a) = x} \theta_1^-(\mu)(a) = \bigwedge_{f(a) = x} \bigwedge_{z \in [a]_{\theta_1}} \mu(z) \\
    &= \bigwedge_{f(a) = x} \bigwedge_{a \in [z]_{\theta_1}} \mu(a) = \bigwedge_{a \in [z]_{\theta_1}} \bigwedge_{f(a) = x} \mu(a) \\
    &= \bigwedge_{x \in [f(z)]_{\theta_2}} f_-(\mu)(x) = \bigwedge_{f(z) \in [x]_{\theta_2}} f_-(\mu)(f(z)) \\
    &= \theta_2^-(f_-(\mu))(x)
\end{align*}
\]

Therefore, \( f_-(\theta_1^-(\mu)) = \theta_2^-(f_-(\mu)) \).

**Example.** Consider the onto ring homomorphism \( f : Z_2 \rightarrow \{0\} \). Clearly \( f \) is not one-one. Define a fuzzy set \( \mu : Z_2 \rightarrow [0, 1] \) such that \( \mu(0) = 0 \) and \( \mu(1) = 0.1 \). Define an equivalence relation \( \theta_1 \) on \( Z_2 \) and \( \theta_2 \) on \( \{0\} \) as \( \theta_1 = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \) and \( \theta_2 = \{(0, 0)\} \) respectively.
For $x \in \{0\}$

\[
\begin{align*}
\theta_2^{-1}(f_-(\mu))(x) &= f_-(\theta_1^{-1}(\mu))(0) = \bigwedge_{f(a)=0} \bigvee_{(a,z) \in \theta_1} \mu(z) = \bigwedge_{a=0,1} \bigvee_{(a,z) \in \theta_1} \mu(z) \\
&= \bigwedge_{(0,z) \in \theta_1} \bigvee \mu(z), \bigvee_{(1,z) \in \theta_1} \mu(z) = \bigwedge_{a=0,1} \{\mu(1), \mu(1)\} = \mu(1) = 0.1
\end{align*}
\]

This shows that $f$ is not one-one and $f_-(\theta_1^{-1}(\mu))(x) \neq \theta_2^{-1}(f_-(\mu))(x)$.

**Theorem 6.2.10.** Isomorphic pre-image of a rough anti-fuzzy subring is a rough anti-fuzzy subring.

**Proof.** Let $f$ be an isomorphism from a ring $R_1$ onto a ring $R_2$ and let $\sigma$ be a rough anti-fuzzy subring of $R_2$. Then $\theta_2^{-1}(\sigma)$ and $\theta_2^{-1}(\sigma)$ are anti-fuzzy subrings of $R_2$. For $x, y \in R_1$,

\[
(f^{-1}(\theta_2^{-1}(\sigma)))(x - y) = \theta_2^{-1}(\sigma)f(x - y)
\]

\[
= \theta_2^{-1}(\sigma)(f(x) - f(y)) \quad (\because f \text{ is a homomorphism})
\]

\[
\leq \theta_2^{-1}(\sigma)f(x) \lor \theta_2^{-1}(\sigma)f(y) \quad (\because \theta_2^{-1}(\sigma) \text{ is an anti-fuzzy subring})
\]

\[
= (f^{-1}(\theta_2^{-1}(\sigma)))(x) \lor (f^{-1}(\theta_2^{-1}(\sigma)))(y)
\]

Therefore, $f^{-1}(\theta_2^{-1}(\sigma))(x - y) \leq f^{-1}(\theta_2^{-1}(\sigma))(x) \lor f^{-1}(\theta_2^{-1}(\sigma))(y)$.

Also $(f^{-1}(\theta_2^{-1}(\sigma)))(xy) = \theta_2^{-1}(\sigma)f(xy)$

\[
= \theta_2^{-1}(\sigma)(f(x)f(y)) \quad (\because f \text{ is a homomorphism})
\]

\[
\leq \theta_2^{-1}(\sigma)f(x) \lor \theta_2^{-1}(\sigma)f(y)
\]

\[
\quad (\because \theta_2^{-1}(\sigma) \text{ is an anti-fuzzy subring})
\]

\[
= (f^{-1}(\theta_2^{-1}(\sigma)))(x) \lor (f^{-1}(\theta_2^{-1}(\sigma)))(y)
\]

Therefore $f^{-1}(\theta_2^{-1}(\sigma))(xy) \leq f^{-1}(\theta_2^{-1}(\sigma))(x) \lor f^{-1}(\theta_2^{-1}(\sigma))(y)$. Thus $f^{-1}(\theta_2^{-1}(\sigma))$ is an anti-fuzzy subring of $R_1$. Similarly we can prove that $f^{-1}(\theta_2^{-1}(\sigma))$ is an anti-
fuzzy subring of $R_1$. By remark (5.2.6), $\theta_1^- (f^{-1}(\sigma))$ and $\theta_1^-(f^{-1}(\sigma))$ are anti-fuzzy subrings of $R_1$. Therefore, $f^{-1}(\sigma)$ is a rough anti-fuzzy subring of $R_1$. This proves the theorem.

**Theorem 6.2.11.** Let $f$ be a homomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be an upper rough $f$-invariant anti-fuzzy subring of $R_1$. Then $f(\mu)$ is an upper rough anti-fuzzy subring of $R_2$.

**Proof.** Let $\mu$ be an upper rough anti-fuzzy subring of $R_1$. Then $\theta_1^- (\mu)$ is an anti-fuzzy subring of $R_1$. For $y_1, y_2 \in R_2$, $\exists x_1, x_2 \in R_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

\[
f(\theta_1^- (\mu))(y_1 - y_2) = \bigvee_{t \in f^{-1}(y_1 - y_2)} \theta_1^- (\mu)(t)
\]

\[
= \bigvee_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} \theta_1^- (\mu)(x_1 - x_2) \quad (\because \theta_1^- (\mu) \text{ is } f\text{-invariant})
\]

\[
\leq \bigvee_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} (\theta_1^- (\mu)(x_1) \lor \theta_1^- (\mu)(x_2))
\]

\[
(\because \theta_1^- (\mu) \text{ is an anti-fuzzy subring})
\]

\[
= \bigvee_{x_1 \in f^{-1}(y_1)} \theta_1^- (\mu)(x_1) \lor \bigvee_{x_2 \in f^{-1}(y_2)} \theta_1^- (\mu)(x_2)
\]

\[
= f(\theta_1^- (\mu))(y_1) \lor f(\theta_1^- (\mu))(y_2)
\]

Therefore, $f(\theta_1^- (\mu))(y_1 - y_2) \leq f(\theta_1^- (\mu))y_1 \lor f(\theta_1^- (\mu))y_2$.

Also $f(\theta_1^- (\mu))(y_1y_2) = \bigvee_{t \in f^{-1}(y_1y_2)} \theta_1^- (\mu)(t)$

\[
= \bigvee_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} \theta_1^- (\mu)(x_1x_2) \quad (\because \theta_1^- (\mu) \text{ is } f\text{-invariant})
\]

\[
\leq \bigvee_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} (\theta_1^- (\mu)(x_1) \lor \theta_1^- (\mu)(x_2))
\]

\[
(\because \theta_1^- (\mu) \text{ is an anti-fuzzy subring})
\]

\[
= \bigvee_{x_1 \in f^{-1}(y_1)} \theta_1^- (\mu)(x_1) \lor \bigvee_{x_2 \in f^{-1}(y_2)} \theta_1^- (\mu)(x_2)
\]

\[
= f(\theta_1^- (\mu))(y_1) \lor f(\theta_1^- (\mu))(y_2)
\]

Hence, $f(\theta_1^- (\mu))(y_1y_2) \leq f(\theta_1^- (\mu))y_1 \lor f(\theta_1^- (\mu))y_2$. 


Therefore, \( f(\theta^-_1(\mu)) \) is an anti-fuzzy subring of \( R_2 \). By theorem (5.2.5), \( f(\theta^-_1(\mu)) = \theta^-_2(f(\mu)) \) is an anti-fuzzy subring of \( R_2 \). Hence \( f(\mu) \) is an upper rough anti-fuzzy subring of \( R_2 \), completing the proof.

**Theorem 6.2.12.** Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be a lower rough \( f \)-invariant anti-fuzzy subring of \( R_1 \). Then \( f(\mu) \) is a lower rough anti-fuzzy subring of \( R_2 \).

**Proof.** The proof is similar to that of the theorem (6.2.11).

**Corollary 6.2.13.** Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be a rough \( f \)-invariant anti-fuzzy subring of \( R_1 \). Then \( f(\mu) \) is a rough anti-fuzzy subring of \( R_2 \).

**Proof.** This follows from theorems (6.2.11) and (6.2.12).

**Theorem 6.2.14.** Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be an upper rough \( f \)-invariant anti-fuzzy subring of \( R_1 \). Then \( f(\mu) \) is an upper rough anti-fuzzy subring of \( R_2 \).

**Proof.** Let \( \mu \) be an upper rough anti-fuzzy subring of \( R_1 \). Then \( \theta^-_1(\mu) \) is an anti-fuzzy subring of \( R_1 \). For \( y_1, y_2 \in R_2, \exists x_1, x_2 \in R_1 \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \).

\[
f_-(\theta^-_1(\mu))(y_1 - y_2) = \bigwedge_{t \in f^{-1}(y_1 - y_2)} \theta^-_1(\mu)(t)
\]

\[
= \bigwedge_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} \theta^-_1(\mu)(x_1 - x_2) \quad (\because \theta^-_1(\mu) \text{ is } f \text{-invariant})
\]

\[
\leq \bigwedge_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} \left( \theta^-_1(\mu)(x_1) \lor \theta^-_1(\mu)(x_2) \right) \quad (\because \theta^-_1(\mu) \text{ is an anti-fuzzy subring})
\]

\[
= \bigwedge_{x_1 \in f^{-1}(y_1)} \theta^-_1(\mu)(x_1) \lor \bigwedge_{x_2 \in f^{-1}(y_2)} \theta^-_1(\mu)(x_2)
\]

\[
= f_-(\theta^-_1(\mu))(y_1) \lor f_-(\theta^-_1(\mu))(y_2)
\]
Therefore, \( f_-(\theta_1^- (\mu))(y_1 - y_2) \leq f_-(\theta_1^- (\mu))y_1 \lor f_-(\theta_1^- (\mu))y_2 \).

\[
f_-(\theta_1^- (\mu))(y_1y_2) = \bigwedge_{t \in f^{-1}(y_1y_2)} \theta_1^- (\mu)(t) \\
= \bigwedge_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} \theta_1^- (\mu)(x_1x_2) \quad (\because \theta_1^- (\mu) \text{ is } f\text{-invariant}) \\
\leq \bigwedge_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} (\theta_1^- (\mu)(x_1) \lor \theta_1^- (\mu)(x_2)) \\
= f_-(\theta_1^- (\mu))(y_1) \lor f_-(\theta_1^- (\mu))(y_2)
\]

Therefore, \( f_-(\theta_1^- (\mu))(y_1y_2) \leq f_-(\theta_1^- (\mu))y_1 \lor f_-(\theta_1^- (\mu))y_2 \). Hence, \( f_-(\theta_1^- (\mu)) \) is an anti-fuzzy subring of \( R_2 \). By theorem (6.2.9), \( f_-(\theta_1^- (\mu)) = \theta_2^- (f_-(\mu)) \) is an anti-fuzzy subring of \( R_2 \). Hence \( f_-(\mu) \) is an upper rough anti-fuzzy subring of \( R_2 \). This proves the theorem.

**Theorem 6.2.15.** Let \( f \) be a homomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be a lower rough \( f \)-invariant anti-fuzzy subring of \( R_1 \). Then \( f_-(\mu) \) is a lower rough anti-fuzzy subring of \( R_2 \).

**Proof.** The proof is similar to that of the theorem (6.2.14).

**Corollary 6.2.16.** Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be a rough \( f \)-invariant anti-fuzzy subring of \( R_1 \). Then \( f_-(\mu) \) is a rough anti-fuzzy subring of \( R_2 \).

**Proof.** This follows from theorems (6.2.14) and (6.2.15).

We can see that for homomorphic image, upper approximation need only onto homomorphism where as lower approximation need isomorphism and for homomorphic pre-image, both the approximations need isomorphism. But for homomorphic anti-image, upper approximation need isomorphism where as lower approximation need only onto homomorphism.
The following theorems in anti-homomorphism can be proved in similar way as the corresponding theorems in homomorphism. As before, for anti-homomorphic image, upper approximation need only onto anti-homomorphism where as lower approximation need anti-isomorphism and for anti-homomorphic pre-image, both the approximations need anti-isomorphism. But for anti-homomorphic anti-image, upper approximation need anti-isomorphism where as lower approximation need only onto anti-homomorphism.

**Theorem 6.2.17.** Anti homomorphic image of an upper rough $f$-invariant anti-fuzzy subring is an upper rough anti-fuzzy subring and anti-isomorphic image of a lower rough $f$-invariant anti-fuzzy subring is a lower rough anti-fuzzy subring. Hence anti-isomorphic image of a rough $f$-invariant anti-fuzzy subring is a rough anti-fuzzy subring.

**Theorem 6.2.18.** Anti isomorphic pre-image of a rough anti-fuzzy subring is a rough anti-fuzzy subring.

**Theorem 6.2.19.** Anti isomorphic anti-image of an upper rough $f$-invariant anti-fuzzy subring is an upper rough anti-fuzzy subring and anti-homomorphic anti-image of a lower rough $f$-invariant anti-fuzzy subring is a lower rough anti-fuzzy subring. Hence anti-isomorphic anti-image of a rough $f$-invariant anti-fuzzy subring is a rough anti-fuzzy subring.

### 6.3 Rough Anti-fuzzy Ideal

This section studies homomorphic and anti-homomorphic properties of rough anti-fuzzy ideal.

**Definition 6.3.1.** [6] A fuzzy subset $\mu$ of a ring $R$ is called an anti-fuzzy left (right) ideal of $R$ if

1. $\mu(x - y) \leq \mu(x) \lor \mu(y)$
2. $\mu(xy) \leq \mu(x) \lor \mu(y)$
3. $\mu(xy) \leq \mu(y) \land (\mu(xy) \leq \mu(x))$
for all $x, y \in R$.

**Definition 6.3.2.** A fuzzy subset $\mu$ of a ring $R$ is called an upper rough anti-fuzzy left (right) ideal of $R$ if $\theta^-(\mu)$ is an anti-fuzzy left (right) ideal of $R$ and a lower rough anti-fuzzy left (right) ideal of $R$ if $\theta^-(\mu)$ is an anti-fuzzy left (right) ideal of $R$.

Let $\mu$ be a fuzzy subset of $R$ and $\theta(\mu) = (\theta^-(\mu), \theta^-(\mu))$ a rough fuzzy set. If $\theta^-(\mu)$ and $\theta^-(\mu)$ are anti-fuzzy left (right) ideals of $R$, then $\mu$ is called a rough anti-fuzzy left (right) ideal.

**Example.** Consider the ring $R = (\mathbb{Z}_4, +, \cdot)$ and subring $(S, +, \cdot)$, where $S = \{0, 2\}$. Define a congruence $\theta$ on $\mathbb{Z}_4$ as $a \equiv b \mod S$ if $a - b \in S$. Define a fuzzy subset $\mu : R \to [0, 1]$ as

$$
\mu(x) = \begin{cases} 
0.1 & \text{if } x = 0 \\
0.5 & \text{if } x \neq 0
\end{cases}
$$

Then

$$\theta^-(\mu)(x) = \begin{cases} 
0.5 & \forall x \in R
\end{cases}
$$

and

$$\theta^-(\mu)(x) = \begin{cases} 
0.1 & \text{if } x = 0, 2 \\
0.5 & \text{if } x = 1, 3
\end{cases}
$$

Clearly $\mu$ is a rough fuzzy set. Obviously, $\theta^-(\mu)$ is an anti-fuzzy ideal. We have been proved that $\theta^-(\mu)$ is an anti-fuzzy subring. Also $\theta^-(\mu)(xy) \leq \theta^-(\mu)(x) \wedge \theta^-(\mu)(y)$. Therefore $\theta^-(\mu)$ is an anti-fuzzy ideal. Therefore, $\mu$ is a rough anti-fuzzy ideal.

**Theorem 6.3.3.** Let $\theta$ be a complete congruence relation on $R$. If $\mu$ is an anti-fuzzy left (right) ideal of $R$, then $\theta^-(\mu)$ is an anti-fuzzy left (right) ideal of $R$.

**Proof.** For $x, y \in R$, as in the proof of theorem (6.2.3), $\theta^-(\mu)$ is an anti-fuzzy subring.
Also we have,

\[ \theta^{-}(\mu)(xy) = \bigvee_{z \in [xy]_\theta} \mu(z) = \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu(ab) \leq \bigvee_{b \in [y]_\theta} \mu(b) \ \ (\because \mu \text{ is an anti-fuzzy left ideal}) \]

\[ = \theta^{-}(\mu)(y) \]

Hence \( \theta^{-}(\mu)(xy) \leq \theta^{-}(\mu)(y) \). Therefore, \( \theta^{-}(\mu) \) is an anti-fuzzy left ideal of \( R \). Similarly we can prove the other part also.

**Remark.** The converse of the theorem (6.3.3) does not hold in general.

**Example.** Consider the ring \( R = (\mathbb{Z}/4, +, \cdot) \) and subring \( (S, +, \cdot) \), where \( S = \{0, 2\} \). Define a congruence \( \theta \) on \( \mathbb{Z}/4 \) as \( a \equiv b \text{ mod } S \) iff \( a - b \in S \). Define a fuzzy subset \( \mu : R \to [0, 1] \) as

\[ \mu(x) = \begin{cases} 0.5 & \text{if } x=0, 3 \\ 0.1 & \text{if } x=1, 2 \end{cases} \]

We have been proved that \( \mu \) is not an anti-fuzzy subring. Hence \( \mu \) is not an anti-fuzzy ideal.

\[ \theta^{-}(\mu)(x) = \begin{cases} 0.5 & \forall x \in R \end{cases} \]

Obviously, \( \theta^{-}(\mu)(x-y) \leq \theta^{-}(\mu)(x) \lor \theta^{-}(\mu)(y) \)

Also, \( \theta^{-}(\mu)(xy) \leq \theta^{-}(\mu)(x) \lor \theta^{-}(\mu)(y) \) and \( \theta^{-}(\mu)(xy) \leq \theta^{-}(\mu)(x) \land \theta^{-}(\mu)(y) \).

Therefore \( \theta^{-}(\mu) \) is an anti-fuzzy ideal.

**Theorem 6.3.4.** Let \( \theta \) be a complete congruence relation on \( R \). If \( \mu \) is an anti-fuzzy left (right) ideal of \( R \), then \( \theta^{-}(\mu) \) is an anti-fuzzy left (right) ideal of \( R \).

**Proof.** For \( x,y \in R \), as in the proof of theorem (6.2.4), \( \theta^{-}(\mu) \) is an anti-fuzzy subring.
Also we have,

\[ \theta_-(\mu)(xy) = \bigwedge_{z \in [xy]_\theta} \mu(z) \]
\[ = \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu(ab) \]
\[ \leq \bigwedge_{b \in [y]_\theta} \mu(b) \quad (\because \mu \text{ is an anti-fuzzy left ideal}) \]
\[ = \theta_-(\mu)(y) \]

Hence \( \theta_-(\mu)(xy) \leq \theta_-(\mu)(y) \). Therefore, \( \theta_-(\mu) \) is an anti-fuzzy left ideal of \( R \). Similarly we can prove the other case also, completing the proof.

**Corollary 6.3.5.** Let \( \theta \) be a complete congruence relation on \( R \). If \( \mu \) is an anti-fuzzy left (right) ideal of \( R \), then \( \mu \) is a rough anti-fuzzy left (right) ideal of \( R \).

**Proof.** This follows from Theorems (6.3.3) and (6.3.4).

**Remark.** If \( \theta \) is a complete congruence relation on \( R \) and \( \mu \) is an anti-fuzzy ideal of \( R \), then \( \mu \) is a rough anti-fuzzy ideal of \( R \).

We can see that, for homomorphic image, upper approximation need only onto homomorphism where as lower approximation need isomorphism. But for homomorphic pre-image, both the approximations need isomorphism.

**Theorem 6.3.6.** Isomorphic pre-image of a rough anti-fuzzy left (right) ideal is a rough anti-fuzzy left (right) ideal. Moreover isomorphic pre-image of a rough anti-fuzzy ideal is a rough anti-fuzzy ideal.

**Proof.** Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \sigma \) be a rough anti-fuzzy left ideal of \( R_2 \). Then \( \theta_2^{-}(\sigma) \) and \( \theta_2^{-}(-\sigma) \) are anti-fuzzy left ideals of \( R_2 \). For \( x, y \in R_1 \), as in the proof of theorem (6.2.10), we have

\[ f^{-1}(\theta_2^{-}(\sigma))(x - y) \leq f^{-1}(\theta_2^{-}(\sigma))(x) \lor f^{-1}(\theta_2^{-}(\sigma))(y) \] and
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\[ f^{-1}(\theta_2^-(\sigma))(xy) \leq f^{-1}(\theta_2^-(\sigma))(x) \lor f^{-1}(\theta_2^-(\sigma))(y). \]

Also \[ f^{-1}(\theta_2^-(\sigma))(xy) = \theta_2^-(\sigma)f(xy) \]
\[ = \theta_2^-(\sigma)(f(x)f(y)) \quad (\because f \text{ is a homomorphism}) \]
\[ \leq \theta_2^-(\sigma)f(y) \quad (\because \theta_2^-(\sigma) \text{ is an anti-fuzzy left ideal}) \]
\[ = f^{-1}(\theta_2^-(\sigma))(y) \]

Therefore \[ f^{-1}(\theta_2^-(\sigma))(xy) \leq f^{-1}(\theta_2^-(\sigma))(y). \]

Thus \[ f^{-1}(\theta_2^-(\sigma)) \] is an anti-fuzzy left ideal of \( R_1 \). Similarly we can prove that \[ f^{-1}(\theta_2^-(\sigma)) \]

is an anti-fuzzy left ideal of \( R_1 \). By remark (5.2.6), \( \theta_1^- (f^{-1}(\sigma)) \) and \( \theta_1^- (f^{-1}(\sigma)) \) are anti-fuzzy left ideals of \( R_1 \). Therefore, \( f^{-1}(\sigma) \) is a rough anti-fuzzy left ideal of \( R_1 \).

Similarly we can prove the other case also. Hence the theorem is proved.

**Theorem 6.3.7.** Let \( f \) be a homomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be an upper rough \( f \)-invariant anti-fuzzy left (right) ideal of \( R_1 \). Then \( f(\mu) \) is an upper rough anti-fuzzy left (right) ideal of \( R_2 \). Moreover homomorphic image of an upper rough \( f \)-invariant anti-fuzzy ideal is an upper rough anti-fuzzy ideal.

**Proof.** Let \( \mu \) be an upper rough anti-fuzzy left ideal of \( R_1 \). Then \( \theta_1^- (\mu) \) is an anti-fuzzy left ideal of \( R_1 \). For \( y_1, y_2 \in R_2, \exists x_1, x_2 \in R_1 \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \).

By theorem (6.2.11), \( f(\theta_1^- (\mu)) \) is an anti-fuzzy subring of \( R_2 \). Now

\[
f(\theta_1^- (\mu))(y_1y_2) = \bigvee_{t \in f^{-1}(y_1y_2)} \theta_1^- (\mu)(t)
\]
\[
= \bigvee_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} \theta_1^- (\mu)(x_1x_2) \quad (\because \theta_1^- (\mu) \text{ is } f \text{-invariant})
\]
\[
\leq \bigvee_{x_2 \in f^{-1}(y_2)} \theta_1^- (\mu)(x_2) \quad (\because \theta_1^- (\mu) \text{ is an anti-fuzzy left ideal})
\]
\[
= f(\theta_1^- (\mu))(y_2)
\]

Therefore, \( f(\theta_1^- (\mu))(y_1y_2) \leq f(\theta_1^- (\mu))(y_2) \). Therefore, \( f(\theta_1^- (\mu)) \) is an anti-fuzzy left ideal of \( R_2 \). By theorem (5.2.5), \( f(\theta_1^- (\mu)) = \theta_2^-(f(\mu)) \) is an anti-fuzzy left ideal of \( R_2 \).

Hence \( f(\mu) \) is an upper rough anti-fuzzy left ideal of \( R_2 \). Similarly we can establish the other case also. This proves the theorem.
Theorem 6.3.8. Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be a lower rough \( f \)-invariant anti-fuzzy left (right) ideal of \( R_1 \). Then \( f(\mu) \) is a lower rough anti-fuzzy left (right) ideal of \( R_2 \). Moreover isomorphic image of a lower rough \( f \)-invariant anti-fuzzy ideal is a lower rough anti-fuzzy ideal.

Proof. The proof is similar to that of theorem (6.3.7).

Corollary 6.3.9. Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be a rough \( f \)-invariant anti-fuzzy left (right) ideal of \( R_1 \). Then \( f(\mu) \) is a rough anti-fuzzy left (right) ideal of \( R_2 \). Moreover isomorphic image of a rough \( f \)-invariant anti-fuzzy ideal is a rough anti-fuzzy ideal.

Proof. This follows from theorems (6.3.7) and (6.3.8).

Theorem 6.3.10. Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be an upper rough \( f \)-invariant anti-fuzzy left (right) ideal of \( R_1 \). Then \( f^{-1}(\mu) \) is an upper rough anti-fuzzy left (right) ideal of \( R_2 \). Moreover isomorphic anti-image of an upper rough \( f \)-invariant anti-fuzzy ideal is an upper rough anti-fuzzy ideal.

Proof. Let \( \mu \) be an upper rough anti-fuzzy left ideal of \( R_1 \). For \( y_1, y_2 \in R_2 \), \( \exists x_1, x_2 \in R_1 \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \). By theorem (6.2.14), \( f^{-1}(\mu) \) is an anti-fuzzy subring of \( R_2 \).

Also \( f^{-1}(\mu)(y_1y_2) = \bigwedge_{t \in f^{-1}(y_1y_2)} \theta_1^{-1}(\mu)(t) = \bigwedge_{x_1 \in f^{-1}(y_1), \, x_2 \in f^{-1}(y_2)} \theta_1^{-1}(\mu)(x_1x_2) \quad (\therefore \theta_1^{-1}(\mu) \text{ is } f \text{-invariant}) \leq \bigwedge_{x_2 \in f^{-1}(y_2)} \theta_1^{-1}(\mu)(x_2) \quad (\therefore \theta_1^{-1}(\mu) \text{ is an anti-fuzzy left ideal}) \Rightarrow f^{-1}(\mu)(y_2) \quad \therefore f^{-1}(\mu)(y_1y_2) \leq f^{-1}(\mu)(y_2).

Hence, \( f^{-1}(\mu)(y_1y_2) \leq f^{-1}(\mu)(y_2) \).

Therefore, \( f^{-1}(\mu) \) is an anti-fuzzy left ideal of \( R_2 \). By theorem (6.2.9), \( f^{-1}(\mu) = \theta_2^{-1}(\mu) \) is an anti-fuzzy left ideal of \( R_2 \). Hence \( f^{-1}(\mu) \) is an upper rough anti-fuzzy left ideal of \( R_2 \). Similarly we can establish the other part also. This completes the theorem.
Theorem 6.3.11. Let $f$ be a homomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be a lower rough $f$-invariant anti-fuzzy left (right) ideal of $R_1$. Then $f_-(\mu)$ is a lower rough anti-fuzzy left (right) ideal of $R_2$. Moreover homomorphic anti-image of a lower rough $f$-invariant anti-fuzzy ideal is a lower rough anti-fuzzy ideal.

Proof. The proof is similar to that of the theorem (6.3.10).

Corollary 6.3.12. Let $f$ be an isomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be a rough $f$-invariant anti-fuzzy left (right) ideal of $R_1$. Then $f_-(\mu)$ is a rough anti-fuzzy left (right) ideal of $R_2$. Moreover isomorphic anti-image of a rough $f$-invariant anti-fuzzy ideal is a rough anti-fuzzy ideal.

Proof. This follows from theorems (6.3.10) and (6.3.11).

The following theorems in anti-homomorphism can be proved in similar way as the corresponding theorems with homomorphism. We can see that an anti-homomorphism maps rough anti-fuzzy left ideal into rough anti-fuzzy right ideal and vice versa. As before, for anti-homomorphic image, upper approximation need only onto anti-homomorphism where as lower approximation need anti-isomorphism. But for anti-homomorphic anti-image, upper approximation need anti-isomorphism where as lower approximation need only onto anti-homomorphism.

Theorem 6.3.13. Anti-homomorphic image of an upper rough $f$-invariant anti-fuzzy left (right) ideal is an upper rough anti-fuzzy right (left) ideal. Moreover anti-homomorphic image of an upper rough $f$-invariant anti-fuzzy ideal is an upper rough anti-fuzzy ideal.


Corollary 6.3.15. Anti-isomorphic image of a rough $f$-invariant anti-fuzzy left (right) ideal is a rough anti-fuzzy right (left) ideal. Moreover anti-isomorphic image of a rough $f$-invariant anti-fuzzy ideal is a rough anti-fuzzy ideal.

Theorem 6.3.16. Anti-isomorphic pre-image of a rough anti-fuzzy left (right) ideal is a rough anti-fuzzy right (left) ideal. Moreover anti-isomorphic pre-image of a rough anti-fuzzy ideal is a rough anti-fuzzy ideal.
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Theorem 6.3.17. Anti-isomorphic anti-image of an upper rough $f$-invariant anti-fuzzy left (right) ideal is an upper rough anti-fuzzy right (left) ideal. Moreover anti-isomorphic anti-image of an upper rough $f$-invariant anti-fuzzy ideal is an upper rough anti-fuzzy ideal.

Theorem 6.3.18. Anti-homomorphic anti-image of a lower rough $f$-invariant anti-fuzzy left (right) ideal is a lower rough anti-fuzzy right (left) ideal. Moreover anti-homomorphic anti-image of a lower rough $f$-invariant anti-fuzzy ideal is a lower rough anti-fuzzy ideal.


6.4 Rough Anti-fuzzy Prime Ideal

In this section we define rough anti-fuzzy prime ideal in a ring and study its homomorphic and anti-homomorphic properties.

Definition 6.4.1. [71] An anti-fuzzy ideal $\mu$ of $R$ is called an anti-fuzzy prime ideal if for all $x, y \in R$

$$\mu(xy) = \mu(x) \land \mu(y)$$

Definition 6.4.2. A fuzzy subset $\mu$ of a ring $R$ is called an upper rough anti-fuzzy prime ideal of $R$ if $\theta^-(\mu)$ is an anti-fuzzy prime ideal of $R$ and a lower rough anti-fuzzy prime ideal of $R$ if $\theta_-(\mu)$ is an anti-fuzzy prime ideal of $R$.

Let $\mu$ be a fuzzy subset of $R$ and $\theta(\mu) = (\theta_-(\mu), \theta^-(\mu))$ a rough fuzzy set. If $\theta_-(\mu)$ and $\theta^-(\mu)$ are anti-fuzzy prime ideals of $R$, then $\mu$ is called a rough anti-fuzzy prime ideal.

Example. Consider the ring $R = (Z_4, +, \cdot)$ and subring $(S, +, \cdot)$, where $S = \{0, 2\}$. Define a congruence $\theta$ on $Z_4$ as $a \equiv b \mod S$ iff $a - b \in S$. Define a fuzzy subset $\mu : R \to [0, 1]$ as, for $x \in R$

$$\mu(x) = \begin{cases} 0.1 & \text{if } x=0 \\ 0.5 & \text{if } x \neq 0 \end{cases}$$
Then
\[ \theta^-(\mu)(x) = \begin{cases} 0.5 & \forall x \in R \end{cases} \]

and
\[ \theta_-(\mu)(x) = \begin{cases} 0.1 & \text{if } x = 0, 2 \\ 0.5 & \text{if } x = 1, 3 \end{cases} \]

We have been seen that, \( \theta^-(\mu) \) and \( \theta_-(\mu) \) are anti-fuzzy ideals. Obviously, \( \theta^-(\mu) \) is an anti-fuzzy prime ideal.

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Clearly \( \theta_-(\mu) \) is an anti-fuzzy prime ideal. Therefore, \( \mu \) is a rough anti-fuzzy prime ideal.

**Theorem 6.4.3.** Let \( \theta \) be a complete congruence relation on \( R \). If \( \mu \) is an anti-fuzzy prime ideal of \( R \), then \( \theta^-(\mu) \) is an anti-fuzzy prime ideal of \( R \).

**Proof.** Since \( \mu \) is an anti-fuzzy ideal of \( R \), by theorem (6.3.3), \( \theta^-(\mu) \) is an anti-fuzzy ideal of \( R \). Now for \( x, y \in R \),

\[
\theta^-(\mu)(xy) = \bigvee_{z \in [xy]_{\theta}} \mu(z) \\
= \bigvee_{a \in [x]_{\theta}, b \in [y]_{\theta}} \mu(ab) \\
= \bigvee_{a \in [x]_{\theta}, b \in [y]_{\theta}} (\mu(a) \land \mu(b)) \quad (\because \mu \text{ is an anti-fuzzy prime ideal}) \\
= \bigvee_{a \in [x]_{\theta}} \mu(a) \land \bigvee_{b \in [y]_{\theta}} \mu(b) \\
= \theta^-(\mu)(x) \land \theta^-(\mu)(y)
\]

Therefore, \( \theta^-(\mu) \) is an anti-fuzzy prime ideal of \( R \).
Remark. The converse of the theorem (6.4.3) does not hold in general.

Example. Consider the ring \( R = (\mathbb{Z}_4, +, \cdot) \) and subring \( (S, +, \cdot) \), where \( S = \{0, 2\} \). Define a congruence \( \theta \) on \( \mathbb{Z}_4 \) as \( a \equiv b \mod S \iff a - b \in S \). Define a fuzzy subset \( \mu : R \to [0, 1] \) as

\[
\mu(x) = \begin{cases} 
0.5 & \text{if } x = 0, 3 \\
0.1 & \text{if } x = 1, 2 
\end{cases}
\]

We have been proved that \( \mu \) is not an anti-fuzzy ideal. Hence \( \mu \) is not an anti-fuzzy prime ideal.

\[
\theta^{-}(\mu)(x) = \begin{cases} 
0.5 & \forall x \in R 
\end{cases}
\]

Obviously, \( \theta^{-}(\mu)(x - y) \leq \theta^{-}(\mu)(x) \lor \theta^{-}(\mu)(y) \)

Also, \( \theta^{-}(\mu)(xy) \leq \theta^{-}(\mu)(x) \lor \theta^{-}(\mu)(y) \) and \( \theta^{-}(\mu)(xy) \leq \theta^{-}(\mu)(x) \land \theta^{-}(\mu)(y) \).

Therefore \( \theta^{-}(\mu) \) is an anti-fuzzy ideal. Also \( \theta^{-}(\mu) \) is an anti-fuzzy prime ideal.

Theorem 6.4.4. Let \( \theta \) be a complete congruence relation on \( R \). If \( \mu \) is an anti-fuzzy prime ideal of \( R \), then \( \theta_{-}(\mu) \) is an anti-fuzzy prime ideal of \( R \).

Proof. Since \( \mu \) is an anti-fuzzy ideal of \( R \), by theorem (6.3.4), \( \theta_{-}(\mu) \) is an anti-fuzzy ideal of \( R \). Now for \( x, y \in R \),

\[
\theta_{-}(\mu)(xy) = \bigwedge_{z \in [xy]_{\theta}} \mu(z)
= \bigwedge_{a \in [x]_{\theta}, b \in [y]_{\theta}} \mu(ab)
= \bigwedge_{a \in [x]_{\theta}, b \in [y]_{\theta}} (\mu(a) \land \mu(b)) \quad (\because \mu \text{ is an anti-fuzzy prime ideal})
= \bigwedge_{a \in [x]_{\theta}} \mu(a) \land \bigwedge_{b \in [y]_{\theta}} \mu(b)
= \theta_{-}(\mu)(x) \land \theta_{-}(\mu)(y)
\]

Therefore, \( \theta_{-}(\mu) \) is an anti-fuzzy prime ideal of \( R \).

Corollary 6.4.5. Let \( \theta \) be a complete congruence relation on \( R \). If \( \mu \) is an anti-fuzzy prime ideal of \( R \), then \( \theta(\mu) \) is an anti-fuzzy prime ideal of \( R \).

Proof. This follows from theorems (6.4.3) and (6.4.4).
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Theorem 6.4.6. Isomorphic pre-image of a rough anti-fuzzy prime ideal is a rough anti-fuzzy prime ideal.

Proof. Let $\sigma$ be a rough anti-fuzzy prime ideal of $R_2$. Then $\theta_2^{-1}(\sigma)$ and $\theta_2^{-1}(\sigma)$ are anti-fuzzy prime ideals of $R_2$. Hence by theorem (6.3.6), $f^{-1}(\theta_2^{-1}(\sigma))$ and $f^{-1}(\theta_2^{-1}(\sigma))$ are anti-fuzzy ideals of $R_1$.

For $x, y \in R_1$,

\[ f^{-1}(\theta_2^{-1}(\sigma))(xy) = \theta_2^{-1}(\sigma)f(xy) \\
= \theta_2^{-1}(\sigma)[f(x)f(y)] \quad (\because f \text{ is a homomorphism}) \\
= \theta_2^{-1}(\sigma)f(x) \land \theta_2^{-1}(\sigma)f(y) \quad (\because \theta_2^{-1}(\sigma) \text{ is an anti-fuzzy prime ideal}) \\
= f^{-1}(\theta_2^{-1}(\sigma))(x) \land f^{-1}(\theta_2^{-1}(\sigma))(y) \]

Similarly we can prove that $f^{-1}(\theta_2^{-1}(\sigma))(xy) = f^{-1}(\theta_2^{-1}(\sigma))(x) \land f^{-1}(\theta_2^{-1}(\sigma))(y)$. Thus $f^{-1}(\theta_2^{-1}(\sigma))$ and $f^{-1}(\theta_2^{-1}(\sigma))$ are anti-fuzzy prime ideals of $R_1$. By remark (5.2.6), we get $\theta_1^{-1}(f^{-1}(\sigma))$ and $\theta_1^{-1}(f^{-1}(\sigma))$ are anti-fuzzy prime ideals of $R_1$. Hence $f^{-1}(\sigma)$ is a rough anti-fuzzy prime ideal of $R_1$.

Theorem 6.4.7. Let $f$ be a homomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be an upper rough $f$-invariant anti-fuzzy prime ideal of $R_1$. Then $f(\mu)$ is an upper rough anti-fuzzy prime ideal of $R_2$.

Proof. Let $\mu$ be an upper rough $f$-invariant anti-fuzzy prime ideal of $R_1$. Then $\theta_1^{-1}(\mu)$ is an anti-fuzzy prime ideal of $R_1$. Hence by theorem (6.3.7), $f(\theta_1^{-1}(\mu))$ is an anti-fuzzy
ideal of $R_2$. For $y_1, y_2 \in R_2$, there exist $x_1, x_2 \in R_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

\[
f(\theta^-_1(\mu))(y_1 y_2) = \bigvee_{t \in f^{-1}(y_1 y_2)} \theta^-_1(\mu)(t)
\]
\[
= \bigvee_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} \theta^-_1(\mu)(x_1 x_2) \quad (\because \theta^-_1(\mu) \text{ is } f\text{-invariant})
\]
\[
= \bigvee_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} (\theta^-_1(\mu)(x_1) \land \theta^-_1(\mu)(x_2))
\]
\[
\quad \quad (\because \theta^-_1(\mu) \text{ is an anti-fuzzy prime ideal})
\]
\[
= \bigvee_{x_1 \in f^{-1}(y_1)} \theta^-_1(\mu)(x_1) \land \bigvee_{x_2 \in f^{-1}(y_2)} \theta^-_1(\mu)(x_2)
\]
\[
= f(\theta^-_1(\mu))(y_1) \land f(\theta^-_1(\mu))(y_2)
\]

Therefore, $f(\theta^-_1(\mu))$ is an anti-fuzzy prime ideal of $R_2$. By theorem (5.2.5), $\theta^-_2(f(\mu))$ is an anti-fuzzy prime ideal of $R_2$. This completes the theorem.

**Theorem 6.4.8.** Let $f$ be an isomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be a lower rough $f$-invariant anti-fuzzy prime ideal of $R_1$. Then $f(\mu)$ is a lower rough anti-fuzzy prime ideal of $R_2$.

**Proof.** The proof is similar to that of theorem (6.4.7).

**Corollary 6.4.9.** Let $f$ be an isomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be a rough $f$-invariant anti-fuzzy prime ideal of $R_1$. Then $f(\mu)$ is a rough anti-fuzzy prime ideal of $R_2$.

**Proof.** This follows from theorems (6.4.7) and (6.4.8).

**Theorem 6.4.10.** Let $f$ be an isomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be an upper rough $f$-invariant anti-fuzzy prime ideal of $R_1$. Then $f(\mu)$ is an upper rough anti-fuzzy prime ideal of $R_2$.

**Proof.** Let $\mu$ be an upper rough $f$-invariant anti-fuzzy prime ideal of $R_1$. Then $\theta^-_1(\mu)$ is an anti-fuzzy prime ideal of $R_1$. Hence by theorem (6.3.10), $f(\theta^-_1(\mu))$ is an anti-fuzzy ideal of $R_2$. For $y_1, y_2 \in R_2$, there exist $x_1, x_2 \in R_1$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Therefore, $f(\theta^-_1(\mu))$ is an anti-fuzzy prime ideal of $R_2$. By theorem (5.2.5), $\theta^-_2(f(\mu))$ is an anti-fuzzy prime ideal of $R_2$. This completes the theorem.
6.4. ROUGH ANTI-FUZZY PRIME IDEAL

\[ f_-(\theta^{-}_1(\mu))(y_1 y_2) = \bigwedge_{t \in f^{-1}(y_1 y_2)} \theta^{-}_1(\mu)(t) \]
\[ = \bigwedge_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} \theta^{-}_1(\mu)(x_1 x_2) \quad (\because \theta^{-}_1(\mu) \text{ is } f\text{-invariant}) \]
\[ = \bigwedge_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} (\theta^{-}_1(\mu)(x_1) \land \theta^{-}_1(\mu)(x_2)) \]
\[ = f_-(\theta^{-}_1(\mu))(y_1) \land f_-(\theta^{-}_1(\mu))(y_2) \]

Therefore, \( f_-(\theta^{-}_1(\mu)) \) is an anti-fuzzy prime ideal of \( R_2 \). By theorem (6.2.9), \( \theta^{-}_2(f_-(\mu)) \) is an anti-fuzzy prime ideal of \( R_2 \). This proves the theorem.

**Theorem 6.4.11.** Let \( f \) be a homomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be a lower rough \( f \)-invariant anti-fuzzy prime ideal of \( R_1 \). Then \( f_-(\mu) \) is a lower rough anti-fuzzy prime ideal of \( R_2 \).

**Proof.** The proof is similar to that of theorem (6.4.10).

**Corollary 6.4.12.** Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be a rough \( f \)-invariant anti-fuzzy prime ideal of \( R_1 \). Then \( f_-(\mu) \) is a rough anti-fuzzy prime ideal of \( R_2 \).

**Proof.** This follows from theorems (6.4.10) and (6.4.11).

The following theorems with anti homomorphism can be proved in a similar way as the corresponding theorems with homomorphism.

**Theorem 6.4.13.** Anti-homomorphic image of an upper rough \( f \)-invariant anti-fuzzy prime ideal is an upper rough anti-fuzzy prime ideal.

**Theorem 6.4.14.** Anti-isomorphic image of a lower rough \( f \)-invariant anti-fuzzy prime ideal is a lower rough anti-fuzzy prime ideal.
**Corollary 6.4.15.** Anti-isomorphic image of a rough \( f \)-invariant anti-fuzzy prime ideal is a rough anti-fuzzy prime ideal.

**Theorem 6.4.16.** Anti-isomorphic anti-image of an upper rough \( f \)-invariant anti-fuzzy prime ideal is an upper rough anti-fuzzy prime ideal.

**Theorem 6.4.17.** Anti-homomorphic anti-image of a lower rough \( f \)-invariant anti-fuzzy prime ideal is a lower rough anti-fuzzy prime ideal.

**Corollary 6.4.18.** Anti-isomorphic anti-image of a rough \( f \)-invariant anti-fuzzy prime ideal is a rough anti-fuzzy prime ideal.

**Theorem 6.4.19.** Anti-isomorphic pre-image of a rough anti-fuzzy prime ideal is a rough anti-fuzzy prime ideal.

### 6.5 Rough Anti-fuzzy Primary Ideal

In this section, we define rough anti-fuzzy primary ideal and study its homomorphic and anti-homomorphic properties.

**Definition 6.5.1.** [71] An anti-fuzzy ideal \( \mu \) of \( R \) is called an anti-fuzzy primary ideal if for all \( x, y \in R \)

\[
\mu(xy) = \mu(x) \wedge \mu(y^n), \text{ for some positive integer } n.
\]

**Definition 6.5.2.** A fuzzy subset \( \mu \) of a ring \( R \) is called an upper rough anti-fuzzy primary ideal of \( R \) if \( \theta^-(\mu) \) is an anti-fuzzy primary ideal of \( R \) and a lower rough anti-fuzzy primary ideal of \( R \) if \( \theta_-(\mu) \) is an anti-fuzzy primary ideal of \( R \).

Let \( \mu \) be a fuzzy subset of \( R \) and \( \theta(\mu) = (\theta_-(\mu), \theta^-(\mu)) \) a rough fuzzy set. If \( \theta_-(\mu) \) and \( \theta^-(\mu) \) are anti-fuzzy primary ideals of \( R \), then \( \mu \) is called a rough anti-fuzzy primary ideal.

**Example.** Consider the ring \( R = (\mathbb{Z}_4, +, \cdot) \) and subring \( (S, +, \cdot) \), where \( S = \{0, 2\} \).

Define a congruence \( \theta \) on \( \mathbb{Z}_4 \) as \( a \equiv b \mod S \) iff \( a - b \in S \). Define a fuzzy subset \( \mu : R \to [0, 1] \) as,

\[
\mu(x) = \begin{cases} 
0 & \text{if } x=0 \\
0.5 & \text{if } x=2 \\
1 & \text{if } x =1, 3
\end{cases}
\]
\[ \theta^{-}(\mu)(x) = \begin{cases} 0.5 & \text{if } x = 0, 2 \\ 1 & \text{if } x = 1, 3 \end{cases} \]

and

\[ \theta_{-}(\mu)(x) = \begin{cases} 0 & \text{if } x = 0, 2 \\ 1 & \text{if } x = 1, 3 \end{cases} \]

Clearly \( \mu \) is a rough set.

\[ \theta^{-}(\mu)(x - y) \]

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\[ \theta^{-}(\mu)(xy) \]

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Clearly \( \theta^{-}(\mu)(x - y) \leq \theta^{-}(\mu)(x) \lor \theta^{-}(\mu)(y) \)

\( \theta^{-}(\mu)(xy) \leq \theta^{-}(\mu)(x) \lor \theta^{-}(\mu)(y) \) and \( \theta^{-}(\mu)(xy) \leq \theta^{-}(\mu)(x) \land \theta^{-}(\mu)(y) \).

Therefore \( \theta^{-}(\mu) \) is an anti-fuzzy ideal.

\[ \theta_{-}(\mu)(x - y) \]

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Clearly $\theta_-(\mu)(x - y) \leq \theta_-(\mu)(x) \lor \theta_-(\mu)(y) \;$
\[\theta_-(\mu)(xy) \leq \theta_-(\mu)(x) \lor \theta_-(\mu)(y) \quad \text{and} \quad \theta_-(\mu)(xy) \leq \theta_-(\mu)(x) \land \theta_-(\mu)(y).\]

Therefore $\theta_-(\mu)$ is an anti-fuzzy ideal. Also $\theta^-(\mu)$ and $\theta_-(\mu)$ are anti-fuzzy primary ideals. Therefore, $\mu$ is a rough anti-fuzzy primary ideal.

**Theorem 6.5.3.** Let $\theta$ be a complete congruence relation on $R$. If $\mu$ is an anti-fuzzy primary ideal of $R$, then $\theta^-(\mu)$ is an anti-fuzzy primary ideal of $R$.

**Proof.** Since $\mu$ is an anti-fuzzy ideal of $R$, by theorem (6.3.3), $\theta^-(\mu)$ is an anti-fuzzy ideal of $R$. Now for $x, y \in R$,

\[
\theta^-(\mu)(xy) = \bigvee_{z \in [xy]|\sigma} \mu(z) = \bigvee_{a \in [x]|\sigma, b \in [y]|\sigma} \mu(ab) = \bigvee_{a \in [x]|\sigma, b \in [y]|\sigma} (\mu(a) \land \mu(b^n)) \quad \text{for some positive integer } n
\]

$\therefore \mu$ is an anti-fuzzy primary ideal

\[
= \bigvee_{a \in [x]|\sigma} \mu(a) \land \bigvee_{b \in [y]|\sigma} \mu(b^n) = \theta^-(\mu)(x) \land \theta^-(\mu)(y^n)
\]

Therefore, $\theta^-(\mu)$ is an anti-fuzzy primary ideal of $R$.

**Remark.** The converse of the theorem (6.5.3) does not hold in general.

**Theorem 6.5.4.** Let $\theta$ be a complete congruence relation on $R$. If $\mu$ is an anti-fuzzy primary ideal of $R$, then $\theta_-(\mu)$ is an anti-fuzzy primary ideal of $R$.

**Proof.** Since $\mu$ is an anti-fuzzy ideal of $R$, by theorem (6.3.4), $\theta_-(\mu)$ is an anti-fuzzy
ideal of $R$. Now for $x, y \in R$,

$$
\theta_-(\mu)(xy) = \bigwedge_{z \in [xy]_\sigma} \mu(z) \\
= \bigwedge_{a \in [x]_\sigma, b \in [y]_\sigma} \mu(ab) \\
= \bigwedge_{a \in [x]_\sigma, b \in [y]_\sigma} (\mu(a) \land \mu(b^n)) \quad \text{for some positive integer } n \\
= \theta_-(\mu)(x) \land \theta_-(\mu)(y^n)
$$

Therefore, $\theta_-(\mu)$ is an anti-fuzzy primary ideal of $R$.

**Corollary 6.5.5.** Let $\theta$ be a complete congruence relation on $R$. If $\mu$ is an anti-fuzzy primary ideal of $R$, then $\theta(\mu)$ is an anti-fuzzy primary ideal of $R$.

**Proof.** This follows from theorems (6.5.3) and (6.5.4).

**Theorem 6.5.6.** Isomorphic pre-image of a rough anti-fuzzy primary ideal is a rough anti-fuzzy primary ideal.

**Proof.** Let $f$ be an isomorphism from a ring $R_1$ onto a ring $R_2$ and let $\sigma$ be a rough anti-fuzzy primary ideal of $R_2$. Then $\theta_2^- (\sigma)$ and $\theta_2^- (\sigma)$ are anti-fuzzy primary ideals of $R_2$. Hence by theorem (6.3.6), $f^{-1}(\theta_2^- (\sigma))$ and $f^{-1}(\theta_2^- (\sigma))$ are anti-fuzzy ideals of $R_1$. For $x, y \in R_1$,

$$
f^{-1}(\theta_2^- (\sigma))(xy) = \theta_2^- (\sigma)f(xy) \\
= \theta_2^- (\sigma)(f(x)f(y)) \quad (\because f \text{ is a homomorphism}) \\
= \theta_2^- (\sigma)f(x) \land \theta_2^- (\sigma)(f(y))^n \quad \text{for some positive integer } n \\
\quad (\because \theta_2^- (\sigma) \text{ is an anti-fuzzy primary ideal}) \\
= \theta_2^- (\sigma)f(x) \land \theta_2^- (\sigma)(f(y^n)) \\
= f^{-1}(\theta_2^- (\sigma))(x) \land f^{-1}(\theta_2^- (\sigma))(y^n)
$$
Similarly we can prove that \( f^{-1}(\theta_{2-}(\sigma))(xy) = f^{-1}(\theta_{2-}(\sigma))(x) \wedge f^{-1}(\theta_{2-}(\sigma))(y^n) \). Thus \( f^{-1}(\theta_{2+}(\sigma)) \) and \( f^{-1}(\theta_{2-}(\sigma)) \) are anti-fuzzy primary ideals of \( R_1 \). By remark (5.2.6), we get \( \theta_1^{-1}(f^{-1}(\sigma)) \) and \( \theta_1^{-1}(f^{-1}(\sigma)) \) are anti-fuzzy primary ideals of \( R_1 \). Hence \( f^{-1}(\sigma) \) is a rough anti-fuzzy primary ideal of \( R_1 \).

**Theorem 6.5.7.** Let \( f \) be a homomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be an upper rough \( f \)-invariant anti-fuzzy primary ideal of \( R_1 \). Then \( f(\mu) \) is an upper rough anti-fuzzy primary ideal of \( R_2 \).

**Proof.** Let \( \mu \) be an upper rough \( f \)-invariant anti-fuzzy primary ideal of \( R_1 \). Then \( \theta_1^{-1}(\mu) \) is an anti-fuzzy primary ideal of \( R_1 \). Hence by theorem (6.3.7), \( f(\theta_1^{-1}(\mu)) \) is an anti-fuzzy ideal of \( R_2 \). For \( y_1, y_2 \in R_2 \), \( \exists x_1, x_2 \in R_1 \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \).

\[
\begin{align*}
 f(\theta_1^{-1}(\mu))(y_1y_2) &= \bigvee_{t \in f^{-1}(y_1y_2)} \theta_1^{-1}(\mu)(t) \\
 &= \bigvee_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} \theta_1^{-1}(\mu)(x_1x_2) \quad (\because \theta_1^{-1}(\mu) \text{ is } f \text{-invariant}) \\
 &= \bigvee_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} (\theta_1^{-1}(\mu)(x_1) \wedge \theta_1^{-1}(\mu)((x_2)^n)) \\
 &= \bigvee_{x_1 \in f^{-1}(y_1)} \theta_1^{-1}(\mu)(x_1) \wedge \bigvee_{x_2 \in f^{-1}(y_2)} \theta_1^{-1}(\mu)((x_2)^n) \\
 &= f(\theta_1^{-1}(\mu))(y_1) \wedge f(\theta_1^{-1}(\mu))(y_2^n)
\end{align*}
\]

Therefore, \( f(\theta_1^{-1}(\mu)) \) is an anti-fuzzy primary ideal of \( R_2 \). By theorem (5.2.5), \( \theta_2^{-1}(f(\mu)) \) is an anti-fuzzy primary ideal of \( R_2 \). Hence the theorem is proved.

**Theorem 6.5.8.** Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be a lower rough \( f \)-invariant anti-fuzzy primary ideal of \( R_3 \). Then \( f(\mu) \) is a lower rough anti-fuzzy primary ideal of \( R_2 \).

**Proof.** The proof is similar to that of theorem (6.5.7).

**Corollary 6.5.9.** Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be a rough \( f \)-invariant anti-fuzzy primary ideal of \( R_1 \). Then \( f(\mu) \) is a rough anti-fuzzy primary ideal of \( R_2 \).
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Proof. This follows from theorems (6.5.7) and (6.5.8).

**Theorem 6.5.10.** Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be an upper rough \( f \)-invariant anti-fuzzy primary ideal of \( R_1 \). Then \( f_-(\mu) \) is an upper rough anti-fuzzy primary ideal of \( R_2 \).

Proof. Let \( \mu \) be an upper rough \( f \)-invariant anti-fuzzy primary ideal of \( R_1 \). Then \( \theta_1^-(\mu) \) is an anti-fuzzy primary ideal of \( R_1 \). Hence by theorem (6.3.10), \( f_-(\theta_1^-(\mu)) \) is an anti-fuzzy ideal of \( R_2 \). For \( y_1, y_2 \in R_2 \), \( \exists x_1, x_2 \in R_1 \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \).

\[
f_-(\theta_1^-(\mu))(y_1 y_2) = \bigwedge_{t \in f^{-1}(y_1 y_2)} \theta_1^-(\mu)(t)
\]

\[
= \bigwedge_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} \theta_1^-(\mu)(x_1 x_2) \quad (\because \theta_1^-(\mu) \text{ is } f \text{-invariant})
\]

\[
= \bigwedge_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} (\theta_1^-(\mu)(x_1) \land \theta_1^-(\mu)((x_2)^n))
\]

\[
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The following theorems with anti homomorphism can be proved in a similar way as the corresponding theorems with homomorphism.

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**Theorem 6.5.19.** Anti-isomorphic pre-image of a rough anti-fuzzy primary ideal is a rough anti-fuzzy primary ideal.

### 6.6 Rough Anti-fuzzy Semi-Prime Ideal

In this section we define rough anti-fuzzy semi-prime ideal and study its homomorphic and anti-homomorphic properties.

**Definition 6.6.1.** [6] An anti-fuzzy ideal \( \mu \) of \( R \) is called an anti-fuzzy semi-prime ideal if for all \( x \in R \)

\[
\mu(x^2) = \mu(x)
\]

**Definition 6.6.2.** A fuzzy subset \( \mu \) of a ring \( R \) is called an upper rough anti-fuzzy semi-prime ideal of \( R \) if \( \theta^-(\mu) \) is an anti-fuzzy semi-prime ideal of \( R \) and a lower rough anti-fuzzy semi-prime ideal of \( R \) if \( \theta_-\mu \) is an anti-fuzzy semi-prime ideal of \( R \).

Let \( \mu \) be a fuzzy subset of \( R \) and \( \theta(\mu) = (\theta_-\mu, \theta^-\mu) \) a rough fuzzy set. If \( \theta_-\mu \)
and $\theta^-(\mu)$ are anti-fuzzy semi-prime ideals of $R$, then $\mu$ is called a rough anti-fuzzy semi-prime ideal.

**Example.** Consider the ring $R = (\mathbb{Z}_6, +, \cdot)$ and subring $(S, +, \cdot)$, where $S = \{0, 2, 4\}$.
Define a congruence $\theta$ on $\mathbb{Z}_6$ as $a \equiv b \mod S$ iff $a - b \in S$. Define a fuzzy subset $\mu : R \to [0, 1]$ as, for $0 < a < b < 1$

$$
\begin{align*}
\mu(x) &= \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x = 2, 4 \\
3 & \text{if } x = 1, 3, 5 
\end{cases} \\
\theta^-(\mu)(x) &= \begin{cases} 
0 & \text{if } x = 0, 2, 4 \\
1 & \text{if } x = 1, 3, 5 
\end{cases} \\
\theta_-(\mu)(x) &= \begin{cases} 
0 & \text{if } x = 0, 2, 4 \\
1 & \text{if } x = 1, 3, 5 
\end{cases}
\end{align*}
$$

and

Clearly $\mu$ is a rough set.
Clearly $\theta^{-}(\mu)(x - y) \leq \theta^{-}(\mu)(x) \lor \theta^{-}(\mu)(y)$.

$\theta^{-}(\mu)(xy) \leq \theta^{-}(\mu)(x) \lor \theta^{-}(\mu)(y)$ and $\theta^{-}(\mu)(xy) \leq \theta^{-}(\mu)(x) \land \theta^{-}(\mu)(y)$.

Therefore $\theta^{-}(\mu)$ is an anti-fuzzy ideal.

Clearly $\theta^{-}(\mu)(x - y) \leq \theta^{-}(\mu)(x) \lor \theta^{-}(\mu)(y)$.

$\theta^{-}(\mu)(xy) \leq \theta^{-}(\mu)(x) \lor \theta^{-}(\mu)(y)$ and $\theta^{-}(\mu)(xy) \leq \theta^{-}(\mu)(x) \land \theta^{-}(\mu)(y)$.

Therefore $\theta_{-}(\mu)$ is an anti-fuzzy ideal. Therefore $\mu$ is a rough anti-fuzzy ideal.

Also $\theta^{-}(\mu)(x) = \theta^{-}(\mu)(x^2)$ . Clearly $\theta^{-}(\mu)$ is an anti-fuzzy semi-prime ideal. Hence
\[\mu\] is an upper rough anti-fuzzy semi-prime ideal.

Again \(\theta_-(\mu)(x) = \theta_-(\mu)(x^2)\). Clearly \(\theta_-(\mu)\) is an anti-fuzzy semi-prime ideal. Hence 
\(\mu\) is a lower rough anti-fuzzy semi-prime ideal. Therefore \(\mu\) is a rough anti-fuzzy semi-prime ideal.

**Theorem 6.6.3.** Let \(\theta\) be a complete congruence relation on \(R\). If \(\mu\) is an anti-fuzzy semi-prime ideal of \(R\), then \(\theta^-(\mu)\) is an anti-fuzzy semi-prime ideal of \(R\).

**Proof.** Since \(\mu\) is an anti-fuzzy ideal of \(R\), by theorem (6.3.3), \(\theta^-(\mu)\) is an anti-fuzzy ideal of \(R\). Now for \(x \in R\),

\[
\theta^-(\mu)(x^2) = \bigvee_{z \in [x^2]_\theta} \mu(z) \\
= \bigvee_{a \in [x]_\theta} \mu(a^2) \\
= \bigvee_{a \in [x]_\theta} \mu(a) \quad (\because \mu \text{ is an anti-fuzzy semi-prime ideal}) \\
= \theta^-(\mu)(x)
\]

Therefore, \(\theta^-(\mu)\) is an anti-fuzzy semi-prime ideal of \(R\).

**Theorem 6.6.4.** Let \(\theta\) be a complete congruence relation on \(R\). If \(\mu\) is an anti-fuzzy semi-prime ideal of \(R\), then \(\theta_-(\mu)\) is an anti-fuzzy semi-prime ideal of \(R\).

**Proof.** Since \(\mu\) is an anti-fuzzy ideal of \(R\), by theorem (6.3.4), \(\theta_-(\mu)\) is an anti-fuzzy ideal of \(R\). Now for \(x \in R\),

\[
\theta_-(\mu)(x^2) = \bigwedge_{z \in [x^2]_\theta} \mu(z) \\
= \bigwedge_{a \in [x]_\theta} \mu(a^2) \\
= \bigwedge_{a \in [x]_\theta} \mu(a) \quad (\because \mu \text{ is an anti-fuzzy semi-prime ideal}) \\
= \theta_-(\mu)(x)
\]

Therefore, \(\theta_-(\mu)\) is an anti-fuzzy semi-prime ideal of \(R\).
CHAPTER 6. ROUGH ANTI-FUZZY IDEALS

**Corollary 6.6.5.** Let $\theta$ be a complete congruence relation on $R$. If $\mu$ is an anti-fuzzy semi-prime ideal of $R$, then $\theta(\mu)$ is an anti-fuzzy semi-prime ideal of $R$.

**Proof.** This follows from theorems (6.6.3) and (6.6.4).

**Theorem 6.6.6.** Isomorphic pre-image of a rough anti-fuzzy semi-prime ideal is a rough anti-fuzzy semi-prime ideal.

**Proof.** Let $f$ be an isomorphism from a ring $R_1$ onto a ring $R_2$ and let $\sigma$ be a rough anti-fuzzy semi-prime ideal of $R_2$. Then $\theta_2^-(\sigma)$ and $\theta_2^-(\sigma)$ are anti-fuzzy semi-prime ideals of $R_2$. Hence by theorem (6.3.6), $f^{-1}(\theta_2^-(\sigma))$ and $f^{-1}(\theta_2^-(\sigma))$ are anti-fuzzy ideals of $R_2$. For $x \in R_1$,

\[
  f^{-1}(\theta_2^-(\sigma))(x^2) = \theta_2^-(\sigma)f(x^2) \\
  = \theta_2^-(\sigma)[f(x)]^2 \\
  = \theta_2^-(\sigma)f(x) \\
  = f^{-1}(\theta_2^-(\sigma))(x)
\]

Thus $f^{-1}(\theta_2^-(\sigma))$ is an anti-fuzzy semi-prime ideal of $R_1$. Similarly we get, $f^{-1}(\theta_2^-(\sigma))$ is an anti-fuzzy semi-prime ideal of $R_1$. By remark (5.2.6), we get $\theta_1^-(f^{-1}(\sigma))$ and $\theta_1^-(f^{-1}(\sigma))$ are anti-fuzzy semi-prime ideals of $R_1$. This proves the theorem.

**Theorem 6.6.7.** Let $f$ be a homomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be an upper rough $f$-invariant anti-fuzzy semi-prime ideal of $R_1$. Then $f(\mu)$ is an upper rough anti-fuzzy semi-prime ideal of $R_2$.

**Proof.** Let $\mu$ be an upper rough $f$-invariant anti-fuzzy semi-prime ideal of $R_1$. Then $\theta_1^-(\mu)$ is an anti-fuzzy semi-prime ideal of $R_1$. Hence by theorem (6.3.7), $f(\theta_1^-(\mu))$ is
an anti-fuzzy ideal of $R_2$. For $y_1 \in R_2$, $\exists x_1 \in R_1$ such that $f(x_1) = y_1$

\[
f(\theta^-_1(\mu))(y^2_1) = \bigvee_{t \in f^{-1}(y^2_1)} \theta^-_1(\mu)(t)
\]

\[
= \bigvee_{x_1 \in f^{-1}(y_1)} \theta^-_1(\mu)(x^2_1) \quad (\because \theta^-_1(\mu) \text{ is } f \text{-invariant})
\]

\[
= \bigvee_{x_1 \in f^{-1}(y_1)} \theta^-_1(\mu)(x_1) \quad (\because \theta^-_1(\mu) \text{ is an anti-fuzzy semi-prime ideal})
\]

\[
= f(\theta^-_1(\mu))(y_1)
\]

Therefore, $f(\theta^-_1(\mu))$ is an anti-fuzzy semi-prime ideal of $R_2$. By theorem (5.2.5), $\theta^-_2(f(\mu))$ is an anti-fuzzy semi-prime ideal of $R_2$, as required.

**Theorem 6.6.8.** Let $f$ be an isomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be a lower rough $f$-invariant anti-fuzzy semi-prime ideal of $R_1$. Then $f(\mu)$ is a lower rough anti-fuzzy semi-prime ideal of $R_2$.

*Proof.* The proof is similar to that of theorem (6.6.7).

**Corollary 6.6.9.** Let $f$ be an isomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be a rough $f$-invariant anti-fuzzy semi-prime ideal of $R_1$. Then $f(\mu)$ is a rough anti-fuzzy semi-prime ideal of $R_2$.

*Proof.* This follows from theorems (6.6.7) and (6.6.8).

**Theorem 6.6.10.** Let $f$ be an isomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be an upper rough $f$-invariant anti-fuzzy semi-prime ideal of $R_1$. Then $f_-(\mu)$ is an upper rough anti-fuzzy semi-prime ideal of $R_2$.

*Proof.* Let $\mu$ be an upper rough $f$-invariant anti-fuzzy semi-prime ideal of $R_1$. Then $\theta^-_1(\mu)$ is an anti-fuzzy semi-prime ideal of $R_1$. Hence by theorem (6.3.10), $f_-(\theta^-_1(\mu))$
is an anti-fuzzy ideal of $R_2$. For $y_1 \in R_2$, $\exists x_1 \in R_1$ such that $f(x_1) = y_1$.

\[
f^{-1}(\theta^{-1}_1(\mu))(y_1^2) = \bigwedge_{t \in f^{-1}(y_1^2)} \theta^{-1}_1(\mu)(t)
\]

\[
= \bigwedge_{x_1 \in f^{-1}(y_1)} \theta^{-1}_1(\mu)(x_1^2) \quad (\because \theta^{-1}_1(\mu) \text{ is } f\text{-invariant})
\]

\[
= \bigwedge_{x_1 \in f^{-1}(y_1)} \theta^{-1}_1(\mu)(x_1) \quad (\because \theta^{-1}_1(\mu) \text{ is an anti-fuzzy semi-prime ideal})
\]

\[
= f^{-1}(\theta^{-1}_1(\mu))(y_1)
\]

Therefore, $f^{-1}(\theta^{-1}_1(\mu))$ is an anti-fuzzy semi-prime ideal of $R_2$. By theorem (6.2.9), $\theta^{-1}_2(f^{-1}(\mu))$ is an anti-fuzzy semi-prime ideal of $R_2$. This completes the theorem.

**Theorem 6.6.11.** Let $f$ be a homomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be a lower rough $f$-invariant anti-fuzzy semi-prime ideal of $R_1$. Then $f_{-}(\mu)$ is a lower rough anti-fuzzy semi-prime ideal of $R_2$.

**Proof.** The proof is similar to that of theorem (6.6.10).

**Corollary 6.6.12.** Let $f$ be an isomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be a rough $f$-invariant anti-fuzzy semi-prime ideal of $R_1$. Then $f_{-}(\mu)$ is a rough anti-fuzzy semi-prime ideal of $R_2$.

**Proof.** This follows from theorems (6.6.10) and (6.6.11).

The following theorems with anti homomorphism can be proved in a similar way as the corresponding theorems with homomorphism.

**Theorem 6.6.13.** Anti-homomorphic image of an upper rough $f$-invariant anti-fuzzy semi-prime ideal is an upper rough anti-fuzzy semi-prime ideal.


**Corollary 6.6.15.** Anti-isomorphic image of a rough $f$-invariant anti-fuzzy semi-prime ideal is a rough anti-fuzzy semi-prime ideal.

**Theorem 6.6.16.** Anti-isomorphic pre-image of a rough anti-fuzzy semi-prime ideal is a rough anti-fuzzy semi-prime ideal.
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**Theorem 6.6.17.** Anti-isomorphic anti-image of an upper rough \( f \)-invariant anti-fuzzy semi-prime ideal is an upper rough anti-fuzzy semi-prime ideal.

**Theorem 6.6.18.** Anti-homomorphic anti-image of a lower rough \( f \)-invariant anti-fuzzy semi-prime ideal is a lower rough anti-fuzzy semi-prime ideal.

**Corollary 6.6.19.** Anti-isomorphic anti-image of a rough \( f \)-invariant anti-fuzzy semi-prime ideal is a rough anti-fuzzy semi-prime ideal.

### 6.7 Rough Anti-fuzzy Bi-ideal

In this section we define rough anti-fuzzy bi-ideal and study its homomorphic and anti-homomorphic properties.

**Definition 6.7.1.** [18] A fuzzy subset \( \mu \) of a ring \( R \) is called an anti-fuzzy bi-ideal of \( R \) if

1. \( \mu(x - y) \leq \mu(x) \lor \mu(y) \)
2. \( \mu(xy) \leq \mu(x) \lor \mu(y) \)
3. \( \mu(xyz) \leq \mu(x) \lor \mu(z) \)

for all \( x, y, z \in R \).

**Definition 6.7.2.** A fuzzy subset \( \mu \) of a ring \( R \) is called an upper rough anti-fuzzy bi-ideal of \( R \) if \( \theta^-(\mu) \) is an anti-fuzzy bi-ideal of \( R \) and a lower rough anti-fuzzy bi-ideal of \( R \) if \( \theta_-(\mu) \) is an anti-fuzzy bi-ideal of \( R \).

Let \( \mu \) be a fuzzy subset of \( R \) and \( \theta(\mu) = (\theta_-(\mu), \theta^-(\mu)) \) a rough fuzzy set. If \( \theta_-(\mu) \) and \( \theta^-(\mu) \) are anti-fuzzy bi-ideals of \( R \), then \( \mu \) is called a rough anti-fuzzy bi-ideal.

**Example.** Let \( R \) be the ring of all \( 2 \times 2 \) matrices over \( Z_4 \) with respect to the matrix addition and multiplication and subring \( S = \{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \} \). Let \( \mu \) be a
fuzzy subset of $R$ defined as, for $0 < p < q < 1$

$$\mu \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \begin{cases} 0 & \text{if } a=b=c=d=0 \\ p & \text{if } a \text{ is a non-zero even integer and } b=c=d=0 \\ q & \text{if } a \text{ is a non-zero odd integer and } b=c=d=0 \\ 1 & \text{if otherwise} \end{cases}$$

Define a congruence $\theta$ on $R$ as $r \equiv r' \mod S$ iff $r - r' \in S$.

$$\theta^-(\mu) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = 1 \quad \forall a, b, c, d$$

and

$$\theta^-_{\mu} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \begin{cases} 0 & \text{if } a=b=c=d=0 \\ p & \text{if } a \text{ is a non-zero even integer and } b=c=d=0 \\ q & \text{if } a \text{ is a non-zero odd integer and } b=c=d=0 \\ 1 & \text{if otherwise} \end{cases}$$

Clearly $\mu$ is a rough set and $\theta^-(\mu)$ and $\theta^-_{\mu}$ are anti-fuzzy bi-ideals. Hence $\mu$ is a rough anti-fuzzy bi-ideal.

**Theorem 6.7.3.** Let $\theta$ be a complete congruence relation on $R$. If $\mu$ is an anti-fuzzy bi-ideal of $R$, then $\theta^-(\mu)$ is an anti-fuzzy bi-ideal of $R$.

**Proof.** For $x, y, z \in R$, as in the proof of theorem (6.2.3), $\theta^-(\mu)$ is an anti-fuzzy subring of $R$.
That is, $\theta^-(\mu)(x - y) \leq \theta^-(\mu)(x) \lor \theta^-(\mu)(y)$
and \( \theta^-(\mu)(xy) \leq \theta^-(\mu)(x) \lor \theta^-(\mu)(y) \).

Also \( \theta^-(\mu)(xyz) = \bigvee_{t \in [xyz]_\theta} \mu(t) \)
\[
= \bigvee_{a \in [x]_\theta, b \in [y]_\theta, c \in [z]_\theta} \mu(abc)
\leq \bigvee_{a \in [x]_\theta, c \in [z]_\theta} \mu(a) \lor \mu(c) \quad (\because \mu \text{ is an anti-fuzzy bi-ideal})
\]
\[
= \bigvee_{a \in [x]_\theta} \mu(a) \lor \bigvee_{c \in [z]_\theta} \mu(c)
\]
\[
= \theta^-(\mu)(x) \lor \theta^-(\mu)(z)
\]

Hence \( \theta^-(\mu)(xyz) \leq \theta^-(\mu)(x) \lor \theta^-(\mu)(z) \)

Therefore, \( \theta^-(\mu) \) is an anti-fuzzy bi-ideal of \( R \).

**Theorem 6.7.4.** Let \( \theta \) be a complete congruence relation on \( R \). If \( \mu \) is an anti-fuzzy bi-ideal of \( R \), then \( \theta^-(\mu) \) is an anti-fuzzy bi-ideal of \( R \).

**Proof.** For \( x, y, z \in R \), by theorem (6.2.4), \( \theta^-(\mu) \) is an anti-fuzzy subring. That is, \( \theta^-(\mu)(x - y) \leq \theta^-(\mu)(x) \lor \theta^-(\mu)(y) \)

and \( \theta^-(\mu)(xy) \leq \theta^-(\mu)(x) \lor \theta^-(\mu)(y) \).

Also we have, \( \theta^-(\mu)(xyz) = \bigwedge_{t \in [xyz]_\theta} \mu(t) \)
\[
= \bigwedge_{a \in [x]_\theta, b \in [y]_\theta, c \in [z]_\theta} \mu(abc)
\leq \bigwedge_{a \in [x]_\theta, c \in [z]_\theta} \mu(a) \lor \mu(c) \quad (\because \mu \text{ is an anti-fuzzy bi-ideal})
\]
\[
= \bigwedge_{a \in [x]_\theta} \mu(a) \lor \bigwedge_{c \in [z]_\theta} \mu(c)
\]
\[
= \theta^-(\mu)(x) \lor \theta^-(\mu)(z)
\]

Hence \( \theta^-(\mu)(xy) \leq \theta^-(\mu)(x) \lor \theta^-(\mu)(z) \). Therefore, \( \theta^-(\mu) \) is an anti-fuzzy bi-ideal of \( R \).

**Corollary 6.7.5.** Let \( \theta \) be a complete congruence relation on \( R \) and \( \mu \) be an anti-fuzzy bi-ideal of \( R \), then \( \mu \) is a rough anti-fuzzy bi-ideal of \( R \).
Proof. This follows from Theorems (6.7.3) and (6.7.4).

**Theorem 6.7.6.** [6] Let \( \mu \) be a fuzzy subset of \( R \). Then \( \mu \) is an anti-fuzzy ideal if and only if \( \mu_t \) and \( \mu^*_t \) are, if they are nonempty, ideals of \( R \) for every \( t \in [0, 1] \).

**Theorem 6.7.7.** [19] Let \( \mu \) be a fuzzy subset of \( R \). Then \( \mu \) is an anti-fuzzy bi-ideal if and only if \( \mu_t \) and \( \mu^*_s \) are, if they are nonempty, bi-ideals of \( R \) for every \( t \in [0, 1] \).

**Theorem 6.7.8.** Every rough anti-fuzzy left (right) ideal of a ring \( R \) is a rough anti-fuzzy bi-ideal.

Proof. Let \( \mu \) be a rough anti-fuzzy left ideal of \( R \). Then \( \theta^-(\mu) \) and \( \theta_-(\mu) \) are anti-fuzzy left ideals of \( R \). This implies, for all \( x, y, z \in R \)

\[
\begin{align*}
\theta^-(\mu)(x - y) &\leq \theta^-(\mu)(x) \lor \theta^-(\mu)(y), \\
\theta^-(\mu)(xy) &\leq \theta^-(\mu)(x) \lor \theta^-(\mu)(y) \\
\text{and } \theta^-(\mu)(xy) &\leq \theta^-(\mu)(y)
\end{align*}
\]

Now, \( \theta^-(\mu)(xyz) = \theta^-(\mu)(x(yz)) \)

\[
\begin{align*}
&\leq \theta^-(\mu)(yz) \\
&\leq \theta^-(\mu)(z) \\
&\leq \theta^-(\mu)(x) \lor \theta^-(\mu)(z) \quad \text{for } x, y, z \in R
\end{align*}
\]

Therefore, \( \theta^-(\mu) \) is an anti-fuzzy bi-ideal of \( R \). Hence \( \mu \) is an upper rough anti-fuzzy bi-ideal of \( R \). Similarly, \( \theta_-(\mu) \) is an anti-fuzzy bi-ideal of \( R \). Hence \( \mu \) is a lower rough anti-fuzzy bi-ideal of \( R \). Therefore, \( \mu \) is a rough anti-fuzzy bi-ideal of \( R \). Similarly, we can prove the other statement also.

**Remark.** The converse of the above theorem is not always true. The following theorem gives a condition under which the converse becomes true.

**Theorem 6.7.9.** Every rough anti-fuzzy bi-ideal of \( R \) is a rough anti-fuzzy left (right) ideal, if every bi-ideal of a ring \( R \) is a left (right) ideal of \( R \).

Proof. Let \( \mu \) be a rough anti-fuzzy bi-ideal of \( R \). Then \( \theta^-(\mu) \) and \( \theta_-(\mu) \) are anti-fuzzy bi-ideals of \( R \). Then by theorem (6.7.7), \( (\theta^-(\mu))_t \), \( (\theta_-(\mu))_t \) and \( (\theta^-(\mu))_t^* \), \( (\theta_-(\mu))_t^* \) are
bi-ideals of \( R \). Therefore, by assumption, \((\theta^- (\mu))_l, (\theta^- (\mu))_r\) and \((\theta^- (\mu))_l^*, (\theta^- (\mu))_r^*\) are left (right) ideals of \( R \). Hence by theorem (6.7.6), \( \theta^- (\mu) \) and \( \theta_- (\mu) \) are anti-fuzzy left (right) ideals of \( R \). Therefore, \( \mu \) is a rough anti-fuzzy left (right) ideal.

**Theorem 6.7.10.** Isomorphic pre-image of a rough anti-fuzzy bi-ideal is a rough anti-fuzzy bi-ideal.

**Proof.** Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \sigma \) be a rough anti-fuzzy bi-ideal of \( R_2 \). Then \( \theta^-_2 (\sigma) \) and \( \theta_-^2 (\sigma) \) are anti-fuzzy bi-ideals of \( R_2 \). For \( x, y, z \in R \), by theorem (6.2.10), \( f^{-1}(\theta^-_2 (\sigma)) \) is an anti-fuzzy subring.

Also \( f^{-1}(\theta^-_2 (\sigma))(xyz) = \theta^-_2 (\sigma)(f(xyz)) \)
\[ = \theta^-_2 (\sigma)(f(x)f(y)f(z)) \quad (\because f \text{ is a homomorphism}) \]
\[ \leq \theta^-_2 (\sigma)f(x) \lor \theta^-_2 (\sigma)f(z) \quad (\because \theta^-_2 (\sigma) \text{ is an anti-fuzzy bi-ideal}) \]
\[ = f^{-1}(\theta^-_2 (\sigma))(x) \lor f^{-1}(\theta^-_2 (\sigma))(z) \]

Therefore, \( f^{-1}(\theta^-_2 (\sigma)) \) is an anti-fuzzy bi-ideal. Similarly we can show that \( f^{-1}(\theta_-^2 (\sigma)) \) is an anti-fuzzy bi-ideal. By remark (5.2.6), \( \theta^-_1 (f^{-1}(\sigma)) \) and \( \theta_-^1 (f^{-1}(\sigma)) \) are anti-fuzzy bi-ideals, completing the proof.

**Theorem 6.7.11.** Homomorphic image of an upper rough \( f \)-invariant anti-fuzzy bi-ideal is an upper rough anti-fuzzy bi-ideal.

**Proof.** Let \( f \) be a homomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be an upper rough \( f \)-invariant anti-fuzzy bi-ideal of \( R_1 \). Then \( \theta^-_1 (\mu) \) is an anti-fuzzy bi-ideal of \( R_1 \). For \( y_1, y_2, y_3 \in R_2, \exists x_1, x_2, x_3 \in R_1 \) such that \( f(x_1) = y_1, f(x_2) = y_2 \) and \( f(x_3) = y_3 \).
By theorem (6.2.11), \( f(\theta_1^- (\mu)) \) is an anti-fuzzy subring.

Also we have
\[
\begin{align*}
\theta_1^- (\mu) (y_1 y_2 y_3) &= \bigvee_{t \in f^{-1}(y_1 y_2 y_3)} \theta_1^- (\mu) (t) \\
&= \bigvee_{x_1 \in f^{-1}(y_1), \ x_2 \in f^{-1}(y_2), \ x_3 \in f^{-1}(y_3)} \theta_1^- (\mu) (x_1 x_2 x_3) \\
&\quad \quad \quad \quad \quad \quad \quad (\because \theta_1^- (\mu) \text{ is } f\text{-invariant}) \\
&\leq \bigvee_{x_1 \in f^{-1}(y_1)} \theta_1^- (\mu) (x_1) \lor \bigvee_{x_3 \in f^{-1}(y_3)} \theta_1^- (\mu) (x_3) \\
&\quad \quad \quad \quad \quad \quad \quad (\because \theta_1^- (\mu) \text{ is an anti-fuzzy bi-ideal}) \\
&= f(\theta_1^- (\mu)) (y_1) \lor f(\theta_1^- (\mu)) (y_3)
\end{align*}
\]

Therefore, \( f(\theta_1^- (\mu)) \) is an anti-fuzzy bi-ideal. By theorem (5.2.5), \( \theta^- \ (f(\mu)) \) is an anti-fuzzy bi-ideal, this proves the theorem.

**Theorem 6.7.12.** Isomorphic image of a lower rough \( f\)-invariant anti-fuzzy bi-ideal is a lower rough anti-fuzzy bi-ideal.

**Proof.** The proof is similar to that of theorem (6.7.11).

**Corollary 6.7.13.** Isomorphic image of a rough \( f\)-invariant anti-fuzzy bi-ideal is a rough anti-fuzzy bi-ideal.

**Proof.** This follows from (6.7.11) and (6.7.12).

**Theorem 6.7.14.** Let \( f \) be an isomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be an upper rough \( f\)-invariant anti-fuzzy bi-ideal of \( R_1 \). Then \( f_-(\mu) \) is an upper rough anti-fuzzy bi-ideal of \( R_2 \).

**Proof.** Let \( \mu \) be an upper rough \( f\)-invariant anti-fuzzy bi-ideal of \( R_1 \). Then \( \theta_1^- (\mu) \) is an anti-fuzzy bi-ideal of \( R_1 \). Hence by theorem (6.3.10), \( f_-(\mu) \) is an upper rough anti-fuzzy ideal of \( R_2 \). Hence by theorem (6.7.8), \( f_-(\mu) \) is an upper rough anti-fuzzy bi-ideal of \( R_2 \). Hence the theorem is proved.

**Theorem 6.7.15.** Let \( f \) be a homomorphism from a ring \( R_1 \) onto a ring \( R_2 \) and let \( \mu \) be a lower rough \( f\)-invariant anti-fuzzy bi-ideal of \( R_1 \). Then \( f_-(\mu) \) is a lower rough anti-fuzzy bi-ideal of \( R_2 \).
Proof. The proof is similar to that of theorem (6.7.14).

Corollary 6.7.16. Let $f$ be an isomorphism from a ring $R_1$ onto a ring $R_2$ and let $\mu$ be a rough $f$-invariant anti-fuzzy bi-ideal of $R_1$. Then $f_-(\mu)$ is a rough anti-fuzzy bi-ideal of $R_2$.

Proof. This follows from theorems (6.7.14) and (6.7.15).

The following theorems in anti-homomorphisms can be proved in similar way as the corresponding theorems with homomorphism.

Theorem 6.7.17. Anti-homomorphic image of an upper rough $f$-invariant anti-fuzzy bi-ideal is an upper rough anti-fuzzy bi-ideal.


Theorem 6.7.21. Anti-homomorphic anti-image of a lower rough $f$-invariant anti-fuzzy bi-ideal is a lower rough anti-fuzzy bi-ideal.


Theorem 6.7.23. Anti-isomorphic pre-image of a rough anti-fuzzy bi-ideal is a rough anti-fuzzy bi-ideal.