8.1 Introduction

Burr (1942) introduced a family of twelve cumulative distribution functions for modeling lifetime data. Burr type III and XII are the two important members of the Burr family. The Burr XII distribution, having logistic and Weibull as special sub-models, is a very popular distribution for modeling lifetime data and survival data. The Burr type XII distribution has been widely used in various fields of sciences, such as in actuarial science, forestry, ecoloxicology, reliability and survival analysis. Shao et al. (2004) studied the models for extended three parameter of Burr type XII distribution and used this distribution to model extreme event with application to flood frequency. The flexibility of Burr type XII distribution has been studied by Rodriguez (1977).

Sandhya et al. (2008) proposed Harris family of discrete distributions. Alice et al.

In this chapter we concentrate on Harris Extended family of distributions. We consider Harris Extended Burr XII and Harris Extended Exponentiated Exponential distribution. Moments, quantiles, entropy, record values and order statistics are derived. The method of maximum likelihood estimates are also obtained. Applications to two sets of real data are carried out to validate the results.

### 8.2 Harris Extended family

Let \( F(x) = F(x; \eta) \) be a baseline cumulative distribution function and \( \bar{F}(x) = 1 - F(x; \eta) \) be the corresponding survival function of a lifetime random variable \( X \), where \( \eta = (\eta_1, \eta_2, ..., \eta_q)' \) is a parameter vector of dimension \( q \). Let \( f(x) = f(x, \eta) \) be the probability density function of \( X \). The survival function of Harris extended family of distribution is
CHAPTER 8. HARRIS EXTENDED BURR XII AND EXPONENTIATED EXPONENTIAL DISTRIBUTION

given by

\[ \tilde{G}(x) = \left[ \frac{\theta(\bar{F}(x))^k}{1 - \theta(\bar{F}(x))^k} \right]^\frac{1}{k}; x > 0. \] (8.2.1)

where \( \bar{\theta} = 1 - \theta, \theta > 0 \) and \( k > 0 \). The new parameters \( \theta > 0 \) and \( k > 0 \) are additional shape parameters to those in \( \eta \) and gives more flexibility.

The Harris extended probability density function is

\[ g(x) = \frac{\theta^\frac{1}{k} f(x)}{[1 - \bar{\theta}(\bar{F}(x))^k]^{\frac{k+1}{k}}}; x > 0. \] (8.2.2)

and the Harris extended failure rate function is

\[ h(x) = \frac{r(x)}{[1 - \bar{\theta}(\bar{F}(x))^k]} \] (8.2.3)

where \( r(x) \) denotes the failure rate function of the baseline distribution.

When \( k = 1 \), the above equations reduces to Marshall-Olkin family of distributions. Therefore Harris extended family of distributions is a generalization of Marshall-Olkin family of distributions.

The Harris Extended probability density function can be expressed as an infinite linear combination of exponentiated base line survival function. Barreto-Souza et al. (2013) discussed general mathematical properties of Marshall-Olkin family of distributions. Similarly using the algebraic equations, we get

\[ g(x) = f(x) \sum_{i=0}^{\infty} w_i(\bar{F}(x))^{ki} \] (8.2.4)

for \( \theta \in (0, 1) \), where \( w_i = w_i(\theta, m) = \theta^\frac{1}{k} \bar{\theta}^i \frac{\Gamma(k^{-1}+i+1)}{\Gamma(k^{-1}+1)} i! \).
For $\theta > 1$

$$g(x) = f(x) \sum_{i=0}^{\infty} v_i(\bar{F}(x))^{ki} \quad (8.2.5)$$

where $v_i = v_i(\theta, k) = (-1)^i \theta^{-1} \sum_{j=i}^{\infty} \binom{j}{i} \frac{\Gamma(k^{-1} + j + 1)}{\Gamma(k^{-1} + 1) j!} i.e,$

$$g(x) = f(x) \begin{cases} 
\sum_{i=0}^{\infty} w_i(\bar{F}(x))^{ki}; & 0 < \theta < 1 \\
\sum_{i=0}^{\infty} v_i(\bar{F}(x))^{ki}; & \theta > 1 
\end{cases}$$

(8.2.4) and (8.2.5) have the same representation except for the coefficients. The HE density function for any $\theta > 0$ can be expressed as the baseline density $f(x)$ multiplied by an infinite power series $\bar{F}(x)$. For more details see Batsidis and Lemonte (2014).

### 8.3 Harris Extended Burr XII distribution

Consider the baseline survival function of Burr XII distribution is $\bar{F}(x) = (1 + x^c)^{-b}, x > 0, c > 0, b > 0$. Then consider the Harris Extended Burr XII distribution denoted as $HEBXII(\theta, k, c, b)$. The survival function is given by

$$\bar{F}(x) = \left[ \frac{\theta (1 + x^c)^{-kb}}{1 - \theta (1 + x^c)^{-kb}} \right]^\frac{1}{k} \quad (8.3.1)$$

The probability density function of HEBXII distribution is given by

$$g(x) = \frac{\theta^\frac{1}{k} b c x^{c-1}(1 + x^c)^{-(b+1)}}{[1 - \theta (1 + x^c)^{-kb}]^{\frac{k+1}{k}}}; x > 0, c > 0, b > 0, \theta > 0, k > 0. \quad (8.3.2)$$

and the corresponding hazard rate function is given by

$$h(x) = \frac{b c x^{c-1}(1 + x^c)^{-1}}{1 - \theta (1 + x^c)^{-kb}} \quad (8.3.3)$$
When \( k=1 \), (8.3.2) becomes Marshall-Olkin Burr XII distribution. When \( \theta = 1, k=1 \) it becomes Burr XII distribution.

Figure 8.1 and figure 8.2 shows the probability density function and hazard rate function of HEBXII distribution with different combinations of parameter values.

The quantile function of the HEBXII distribution is given by

\[
x_p = \left\{ \theta + \theta (1-p)^{-k} \right\}^{\frac{1}{k}} - 1
\]

### 8.3.1 Order Statistics and Record Values

Let \( X_1, X_2, \ldots, X_n \) be a random sample taken from the HEBXII distribution and \( X_{1:n}, X_{2:n}, \ldots, X_{n:n} \) be the corresponding order statistics. The survival function of
HEBXII distribution is given by (8.3.2). The c.d.f. of first order statistic \( X_{1:n} \) is given by

\[
G_{1:n}(x) = 1 - \left\{ \frac{\theta(1 + x^c)^{-kb}}{1 - \theta(1 + x^c)^{-kb}} \right\}^n
= 1 - \theta^n \sum_{i=0}^{\infty} \frac{\Gamma(nk^{-1} + i)}{\Gamma(nk^{-1})i!} (\bar{\theta})^i (1 + x^c)^{-ikb}
\]

The c.d.f. of \( n^{th} \) order statistic \( X_{n:n} \) is given by

\[
G_{n:n}(x) = \left\{ \frac{\theta(1 + x^c)^{-kb}}{1 - \theta(1 + x^c)^{-kb}} \right\}^n
= \sum_{j=0}^{n} \sum_{l=0}^{\infty} (-1)^n \binom{n}{j} \left( \frac{\Gamma(jk^{-1} + l)}{\Gamma(jk^{-1})l!} \right) (\bar{\theta})^l (1 + x^c)^{-lkb}
\]
The probability density function of \( i^{th} \) order statistic \( X_{i:n} \) is given by

\[
g_{i:n}(x; \theta, k, c, b) = \frac{n!}{(i-1)!(n-i)!} \left\{ \frac{\theta \frac{1}{b} c x^{c-1} (1 + x^c)^{-(b+1)}}{[1 - \bar{\theta}(1 + x^c)^{-kb}]^{\frac{1}{b}}} \right\}^{i-1} \left[ \frac{\theta (1 + x^c)^{-kb}}{1 - \bar{\theta}(1 + x^c)^{-kb}} \right]^{\frac{n-i}{k}}
\]

The above probability density function can be written as a mixture of the probability density functions of HEBXII distributed random variables since

\[
g_{i:n}(x; \theta, k, c, b) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{i-1} \left( \begin{array}{c} i - 1 \\ j \end{array} \right) \left( \frac{\Gamma(jk^{-1} + l)}{\Gamma(jk^{-1})} \frac{\Gamma((n-i)k^{-1} + m)}{\Gamma(n-i)k^{-1}m!} \right) \left( \frac{\bar{\theta}^{1 + \frac{n-i}{k}}}{(kl + n - i + 1)} \right) g(x; \theta, k(n-i + j + 1)^{-1}, c, b(kl + n - i + 1))
\]

Now we consider the record statistics of HEBXII distribution. Using (1.3.1), probability density function of \( n^{th} \) record is given by

\[
g_{R:n}(x) = \frac{1}{(n-1)!k^{n-1}} \left\{ \log \left[ \frac{\theta (1 + x^c)^{-kb}}{1 - \bar{\theta}(1 + x^c)^{-kb}} \right] \right\}^{n-1} \theta \frac{1}{b} c x^{c-1} (1 + x^c)^{-(b+1)} \sum_{j=0}^{\infty} \frac{\Gamma(k^{-1} + j + 1)}{\Gamma(k^{-1} + 1)j!} \bar{\theta}^j (1 + x^c)^{-kbj}
\]

### 8.3.2 Entropy

Entropy is a measure of uncertainty regarding a random variable. Rényi entropy is a generalization of Shannon entropy. Rényi entropy of a random variable with probability density function is given by \( I_R(\gamma) = \frac{1}{1-\gamma} \log \int_{0}^{\infty} g^\gamma(x)dx; \gamma > 0, \gamma \neq 1. \)
When $\gamma = 1$ it reduces to Shannon entropy. Now we consider,

$$
\int_0^\infty g^\gamma(x)dx = \left[\sum_{j=0}^\infty \frac{\Gamma(k^{-1} + j + 1)}{\Gamma(k^{-1} + 1)j!} \bar{\theta}^j \bar{\xi}^j b c \right]^\gamma \int_0^\infty x^{(c-1)}(1 + x)^{-(\gamma - 1)(b + kbj + 1)}dx
$$

$$
= \left[\sum_{j=0}^\infty \frac{\Gamma(k^{-1} + j + 1)}{\Gamma(k^{-1} + 1)j!} \bar{\theta}^j \bar{\xi}^j b c \right]^\gamma B \left(\gamma(b + kbj + 1) - \frac{\gamma(c - 1)}{c} - \frac{1}{c}, \frac{\gamma(c - 1)}{c} + \frac{1}{c}\right)
$$

Therefore, the Rényi entropy is given by

$$
I_R(\gamma) = \frac{c}{1 - \gamma} \log \left\{ \sum_{j=0}^\infty \frac{\Gamma(k^{-1} + j + 1)}{\Gamma(k^{-1} + 1)j!} \bar{\theta}^j \bar{\xi}^j b c \right\}^\gamma B \left(\gamma(b + kbj + 1) - \frac{\gamma(c - 1)}{c} - \frac{1}{c}, \frac{\gamma(c - 1)}{c} + \frac{1}{c}\right)
$$

$$
= \frac{c}{1 - \gamma} \left\{ \gamma \log \left[ \sum_{j=0}^\infty \frac{\Gamma(k^{-1} + j + 1)}{\Gamma(k^{-1} + 1)j!} \bar{\theta}^j \bar{\xi}^j b c \right] \right\} + \frac{c}{1 - \gamma} \left\{ \log B \left(\gamma(b + kbj + 1) - \frac{\gamma(c - 1)}{c} - \frac{1}{c}, \frac{\gamma(c - 1)}{c} + \frac{1}{c}\right) \right\}
$$

When $\gamma = 1$, we get Shannon entropy and is given by

$$
S = \log \left[ \sum_{j=0}^\infty \frac{\Gamma(k^{-1} + j + 1)}{\Gamma(k^{-1} + 1)j!} \bar{\theta}^j \bar{\xi}^j b B(b(kj + 1), 1) \right]
$$

### 8.3.3 Moments

Consider the general expression for the moments of Harris Extended distribution in terms of the probability weighted moments of the base line distribution. The probability weighted moments introduced by Greenwood et al. (1979) are expectations of certain functions of a
random variable whose mean exists. It can be defined as

$$
\tau_{p,r} = \int_{-\infty}^{\infty} x^p \left( \bar{F}(x) \right)^r f(x) dx.
$$

From (8.2.4) and for $0 < \theta < 1$, the $s^{th}$ moment of the Harris extended distribution can be expressed as

$$
\mu'_s = \sum_{j=0}^{\infty} w_j \tau_{s,jk}
$$

(8.3.4)

For $\theta > 1$, (8.3.4) holds replacing $w_j$ with $v_j$.

Now we consider the moments of HEBXII distribution. Let $\tau_{s,jk}$ can be expressed as

$$
\tau_{s,jk} = \int_0^{\infty} x^s [\bar{F}(x)]^{jk} f(x) dx
$$

$$
= bc \int_0^{\infty} x^{s+c-1} (1 + x)^{-b(jk+1)-1} dx
$$

$$
= bB \left( b(jk + 1) - \frac{s}{c}, \frac{s}{c} + 1 \right)
$$

Substituting in (8.3.4), we get

$$
\mu'_s = \sum_{j=0}^{\infty} w_j bB \left( b(jk + 1) - \frac{s}{c}, \frac{s}{c} + 1 \right)
$$

Now

$$
\mu'_1 = b \sum_{j=0}^{\infty} w_j B \left( b(jk + 1) - \frac{1}{c}, \frac{1}{c} + 1 \right)
$$

$$
\mu'_2 = b \sum_{j=0}^{\infty} w_j B \left( b(jk + 1) - \frac{2}{c}, \frac{2}{c} + 1 \right)
$$

133
\[ \mu_2 = \left[ b \sum_{j=0}^{\infty} w_j B \left( b(jk + 1) - \frac{2}{c}, \frac{2}{c} + 1 \right) \right] - \left[ b \sum_{j=0}^{\infty} w_j B \left( b(jk + 1) - \frac{1}{c}, \frac{1}{c} + 1 \right) \right]^2 \]

### 8.3.4 Estimation of Parameters

Let \( x_1, x_2, \ldots, x_n \) be a random sample of size \( n \) from HEBXII distribution with parameters \( (\theta, k, \eta) \), where \( \eta = (\eta_1, \eta_2, \ldots, \eta_q)' \) is a parameter vector of dimension \( q \). Here \( \eta = (\eta_1, \eta_2)' = (\eta_1(b), \eta_2(c))' \). Let \( \xi = (\theta, k, \eta)' \) be the parameter vector. The log likelihood function for \( \xi \) based on a given random sample is

\[
\log L(\xi) = \frac{n}{k} \log \theta + \sum_{i=1}^{n} \log \left[ bc(1 + x_i^c)^{-(b+1)}x_i^{c-1} \right] - \frac{k+1}{k} \sum_{i=1}^{n} \log \left[ 1 - \bar{\theta}(1 + x_i^c)^{-bk} \right]
\]

Taking partial derivatives with respect to model parameters, we get

\[
\frac{\partial \log L(\xi)}{\partial \theta} = \frac{n}{k} \theta - \frac{k+1}{k} \sum_{i=1}^{n} \frac{(1 + x_i^c)^{-bk}}{[1 - \theta(1 + x_i^c)^{-bk}]}
\]

\[
\frac{\partial \log L(\xi)}{\partial k} = \frac{-n \log \theta}{k^2} + \frac{1}{k^2} \sum_{i=1}^{n} \log \left[ 1 - \bar{\theta}(1 + x_i^c)^{-bk} \right] + \\frac{\bar{\theta}(k+1)}{k} \sum_{i=1}^{n} \frac{(1 + x_i^c)^{-bk} \log(1 + x_i^c)^{-b}}{[1 - \theta(1 + x_i^c)^{-bk}]}
\]
\[
\frac{\partial \log L(\xi)}{\partial \eta} = \sum_{i=1}^{n} \frac{\partial \log[f_0(x_i; \eta)]}{\partial \eta} + \bar{\theta}(k + 1) \sum_{i=1}^{n} \frac{\bar{F}_0(x_i; \eta)^{k-1}}{1 - \theta \bar{F}_0(x_i; \eta)^k} \frac{\partial \bar{F}_0(x_i; \eta)}{\partial \eta}
\]

Therefore,

\[
\frac{\partial \log L(\xi)}{\partial \eta_1(b)} = \sum_{i=1}^{n} \left[ \frac{1}{b} - \log(1 + x_i^c) \right] + \bar{\theta}(k + 1) \sum_{i=1}^{n} \frac{(1 + x_i^c)^{-b}}{[1 - \theta(1 + x_i^c)^{-bk}]} \log(1 + x_i^c)^{-(1 + x_i^c)^{-b(k-1)}}
\]

\[
\frac{\partial \log L(\xi)}{\partial \eta_1(c)} = \sum_{i=1}^{n} \frac{1}{c} - (b + 1) \frac{x_i^c \log x_i}{1 + x_i^c} + \log x_i - \bar{\theta}b(k + 1) \sum_{i=1}^{n} \frac{(1 + x_i^c)^{-b(k-1)} - (b-1)x_i^c \log x_i}{1 - \theta(1 + x_i^c)^{-bk}}
\]

The minimum likelihood estimator \( \hat{\xi} = (\hat{\theta}, \hat{k}, \hat{\eta}') \) of \( \xi = (\theta, k, \eta)' \) can be obtained by solving the equations \( \frac{\partial \log L(\xi)}{\partial \theta} = 0, \frac{\partial \log L(\xi)}{\partial k} = 0 \) and \( \frac{\partial \log L(\xi)}{\partial \eta} = 0 \). These can be solved using nlm package in R software.

### 8.4 Harris Extended Exponentiated Exponential Distribution

Gupta et al. (1998) introduced exponentiated exponential distribution. The cumulative distribution function of exponentiated exponential is the \( \alpha^{th} \) power of exponential distribution. Consider the base line distribution as exponentiated exponential and its survival function given by \( F(x) = 1 - (1 - e^{-\lambda x})^\alpha \). Here we introduce Harris Extended Exponentiated Exponential Distribution denoted as \( HEEE(\theta, k, \alpha, \lambda) \). Using (8.2.1) and (8.2.2), we get
the survival function and probability density function corresponding to the new distribution and is given by

\[
\bar{G}(x) = \left[ \frac{\theta[1 - (1 - e^{-\lambda x})^\alpha]^k}{1 - \theta[1 - (1 - e^{-\lambda x})^\alpha]^k} \right]^{1/k} \quad (8.4.1)
\]

\[
g(x) = \frac{\theta^{1/k} \alpha \lambda (1 - e^{-\lambda x})^{\alpha - 1} e^{-\lambda x}}{[1 - \theta[1 - (1 - e^{-\lambda x})^\alpha]^k]^{1/k}}; \quad \theta, \alpha, \lambda, k > 0. \quad (8.4.2)
\]

Figure 8.3 shows the probability density function corresponding to various combinations of parameter values.

The failure rate function given by

\[
h(x) = \frac{\alpha \lambda (1 - e^{-\lambda x})^{\alpha - 1} e^{-\lambda x}}{[1 - (1 - e^{-\lambda x})^\alpha] [1 - \theta[1 - (1 - e^{-\lambda x})^\alpha]^k]}
\]
8.4.1 Quantiles and Order statistics

The \( p^{th} \) quantile function of the distribution, the inverse of the distribution function 
\( F(x_p) = p \), is given by

\[
x = -\frac{1}{\lambda} \log \left\{ 1 - \left[ 1 + \frac{1 - p}{[\theta + \hat{\theta}(1 - p)^k]^\frac{1}{k}} \right] \right\}
\]

Let \( X_1, X_2, ..., X_n \) be a random sample taken from the HEEE distribution and 
\( X_{1:n}, X_{2:n}, ..., X_{n:n} \) be the corresponding order statistics. The survival function of 
HEEE distribution is given by (8.4.1). Then the c.d.f. of the first order statistic \( X_{1:n} \) is given by

\[
G_{1:n}(x) = 1 - \left[ \frac{\theta[1 - (1 - e^{-\lambda x})^\alpha]^k}{1 - \theta[1 - (1 - e^{-\lambda x})^\alpha]^k} \right]^\frac{1}{k}
\]

\[
= 1 - \theta^n \hat{\theta}^j \sum_{j=0}^{\infty} \left( \frac{\Gamma(nk^{-1} + j)}{j!nk^{-1}} \right) [1 - (1 - e^{-\lambda x})^\alpha]^{n+kj}
\]

The c.d.f. of the \( n^{th} \) order statistic \( X_{n:n} \) is given by

\[
G_{n:n}(x) = \left\{ 1 - \left[ \frac{\theta[1 - (1 - e^{-\lambda x})^\alpha]^k}{1 - \theta[1 - (1 - e^{-\lambda x})^\alpha]^k} \right]^{\frac{1}{k}} \right\}^n
\]

\[
= \sum_{i=0}^{n} \sum_{j=0}^{\infty} (-1)^i \binom{n}{i} \binom{i+j-1}{i-1} \theta^n \hat{\theta}^j [1 - (1 - e^{-\lambda x})^\alpha]^{i+j}
\]

The probability density function of the \( i^{th} \) order statistic \( X_{i:n} \) is given by
Chapter 8. Harris Extended Burr XII and Exponentiated Exponential Distribution

\[ g_{i:n}(x, \theta, k, \alpha, \lambda) = \frac{n!}{(i-1)!(n-i)!} \frac{\alpha \lambda (1 - e^{-\lambda x})^{\alpha-1} e^{-\lambda x}}{[1 - \bar{\theta}(1 - (1 - e^{-\lambda x})^k)]^{\frac{1}{k}}} \left\{ 1 - \left[ \frac{\theta(1 - (1 - e^{-\lambda x})^k)}{1 - \bar{\theta}(1 - (1 - e^{-\lambda x})^k)} \right]^{\frac{1}{k}} \right\}^{i-1} \frac{\theta^{n-i+j}}{[1 - (1 - e^{-\lambda x})^k]^{(n-i+j)}} 
\]

This can be written as a finite mixture of the probability density functions of HEEE distributed random variables since

\[ g_{i:n}(x, \theta, k, \alpha, \lambda) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=1}^{i-1} (-1)^{i-j} \left( \binom{i-1}{j} \frac{\theta^{i-j}}{[1 - (1 - e^{-\lambda x})^k]^{(n-i+j)}} \right) \]

8.4.2 Record Values

Using (1.3.1), we consider the record statistics of HEEE distribution with pdf given by

\[ g_{R_n}(x) = \frac{1}{(n-1)!} \frac{\alpha \lambda (1 - e^{-\lambda x})^{\alpha-1} e^{-\lambda x}}{[1 - \bar{\theta}(1 - (1 - e^{-\lambda x})^k)]^{\frac{1}{k}}} \left[ \frac{1}{k \log \theta} - \sum_{i=1}^{\infty} \frac{(1 - e^{-\lambda x})^i}{i} - \frac{1}{k} \sum_{i=1}^{\infty} \frac{\bar{\theta}^i(1 - (1 - e^{-\lambda x})^k)^i}{i} \right]^{n-1} \]

8.4.3 Estimation of Parameters

Consider the estimation of unknown parameters by the method of maximum likelihood. Let \( x_1, x_2, \ldots, x_n \) be a random sample of size \( n \) from HEEE distribution with parameters \( (\theta, k, \eta) \). where \( \eta = (\eta_1, \eta_2, \ldots, \eta_q)^t \) ia a parameter vector of dimension \( q \). Here \( \eta = (\eta_1, \eta_2)^t = (\eta_1(\alpha), \eta_\lambda(c))^t \). Let \( \xi = (\theta, k, \eta)^t \) be the parameter vector. The log likelihood
function for $\xi$ based on a given random sample is

$$\log L(\xi) = \frac{n}{k}\log \theta + n\log \alpha + n\log \lambda + (\alpha - 1)\sum_{i=0}^{n}\log(1 - (e^{-\lambda x_i})) -$$

$$\lambda \sum_{i=0}^{n} x_i - \frac{k+1}{k}\sum_{i=0}^{n} \log \left(1 - \tilde{\theta}[1 - (1 - e^{-\lambda x_i})^\alpha]^k\right)$$

The partial derivatives of the log-likelihood function is given by

$$\frac{\partial \log L(\xi)}{\partial \theta} = \frac{n}{k}\theta - \sum_{i=0}^{n} \frac{[1 - (1 - e^{-\lambda x_i})^\alpha]^k}{(1 - \tilde{\theta}[1 - (1 - e^{-\lambda x_i})^\alpha]^k)}$$

$$\frac{\partial \log L(\xi)}{\partial k} = \frac{-n\log \theta}{k^2} - \frac{1}{k^2}\sum_{i=0}^{n} \log \left(1 - \tilde{\theta}[1 - (1 - e^{-\lambda x_i})^\alpha]^k\right) -$$

$$\frac{k + 1}{k}\frac{\tilde{\theta}[1 - (1 - e^{-\lambda x_i})^\alpha]^k\log[1 - (1 - e^{-\lambda x_i})^\alpha]}{(1 - \tilde{\theta}[1 - (1 - e^{-\lambda x_i})^\alpha]^k)}$$

$$\frac{\partial \log L(\xi)}{\partial \eta_1(\alpha)} = \frac{n}{\alpha} + \sum_{i=0}^{n} \log(1 - e^{-\lambda x_i}) - \frac{k+1}{k}\sum_{i=0}^{n} \frac{\tilde{\theta}k[1 - (1 - e^{-\lambda x_i})^\alpha]^{k-1}\log(1 - e^{-\lambda x_i})\log(1 - e^{-\lambda x_i})\alpha}{(1 - \tilde{\theta}[1 - (1 - e^{-\lambda x_i})^\alpha]^k)}$$

$$\frac{\partial \log L(\xi)}{\partial \eta_2(\lambda)} = \frac{n}{\lambda} + (\alpha - 1)\sum_{i=0}^{n} \frac{x_i e^{-\lambda x_i}}{1 - e^{-\lambda x_i}} - \frac{k+1}{k}\sum_{i=0}^{n} \frac{x_i - \frac{k+1}{k}}{(1 - \tilde{\theta}[1 - (1 - e^{-\lambda x_i})^\alpha]^k)}$$

$$\sum_{i=0}^{n} \frac{\tilde{\theta}k[1 - (1 - e^{-\lambda x_i})^\alpha]^{k-1}\alpha x_i(1 - e^{-\lambda x_i})^{\alpha-1}e^{-\lambda x_i}}{(1 - \tilde{\theta}[1 - (1 - e^{-\lambda x_i})^\alpha]^k)}$$

The maximum likelihood estimator $\hat{\xi} = (\hat{\theta}, \hat{k}, \hat{\eta}^\prime)^\prime$ of $\xi = (\theta, k, \eta^\prime)^\prime$ can be obtained by
8.1 Harris Extended Burr XII and Exponentiated Exponential Distribution

solving the equations \( \frac{\partial \log L(\xi)}{\partial \theta} = 0 \), \( \frac{\partial \log L(\xi)}{\partial k} = 0 \) and \( \frac{\partial \log L(\xi)}{\partial \eta} = 0 \). The solutions can be obtained by using nlm package in R software.

8.5 Data Analysis

Here we consider the applications to two data sets. The first data set is used to compare Harris Extended Burr type XII with Burr type XII distribution and the second data is used to compare Harris Extended Exponentiated Exponential distribution with Exponentiated Exponential distribution. To compare the goodness of fit test using information criteria, we use \( AIC = -2 \log L + 2k \), \( BIC = -2 \log L + k \log n \) and the Kolmogrov- Smirnov statistic, where \( k \) is the number of parameters and \( n \) is the sample size.

Data set 1: The data is taken from Choulakian and Stephens (2001) and data represents the exceedances of flood peaks (in \( m^3/s \)) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958-1984, rounded to one decimal place: 1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39, 0.3, 15, 11, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0. To compare the goodness of fit test for HEBXII and BurrXII distribution, we estimate the unknown parameters by the method of maximum likelihood estimation. The maximum likelihood estimates, AIC and BIC values, and K-S statistic are given in Table 8.1. From Table 8.1, it is seen that the smallest AIC and BIC values are obtained for HEBXII distribution. Hence we conclude that HEBXII distribution is a better model for the data set.

Data set 2: The data is taken from Lawless (2003). The data represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test and they are 1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2. The maximum likelihood estimates, AIC and BIC values, and K-S statistic are given in Table
Chapter 8. Harris Extended Burr XII and Exponentiated Exponential Distribution

Table 8.1: Estimates, AIC, BIC and K-S statistic for the data set 1

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameter</th>
<th>Estimates</th>
<th>AIC</th>
<th>BIC</th>
<th>K-S statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>HEBXII</td>
<td>$\theta$</td>
<td>0.0673</td>
<td>618.8546</td>
<td>627.9613</td>
<td>0.5162</td>
</tr>
<tr>
<td></td>
<td>$k$</td>
<td>0.6144</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$b$</td>
<td>0.0999</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>0.1000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>BurrXII</td>
<td>$b$</td>
<td>0.2400</td>
<td>1180.763</td>
<td>1189.869</td>
<td>0.8509</td>
</tr>
<tr>
<td></td>
<td>$c$</td>
<td>143.0157</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8.2: Estimates, AIC, BIC and K-S statistic for the data set 2

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameter</th>
<th>Estimates</th>
<th>AIC</th>
<th>BIC</th>
<th>K-S statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>HEEE</td>
<td>$\theta$</td>
<td>2.264</td>
<td>505.8228</td>
<td>508.655</td>
<td>0.7056</td>
</tr>
<tr>
<td></td>
<td>$k$</td>
<td>0.3661</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\alpha$</td>
<td>0.1629</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda$</td>
<td>0.2147</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EE</td>
<td>$\alpha$</td>
<td>0.5484</td>
<td>1482.743</td>
<td>1485.575</td>
<td>0.8964</td>
</tr>
<tr>
<td></td>
<td>$\lambda$</td>
<td>0.2361</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

8.2. From Table 8.2, it is seen that the smallest AIC and BIC values are obtained for HEEE distribution. Hence we conclude that HEEE distribution is a better model for the data set.

8.6 Conclusion

In this chapter we studied a more generalized version of Marshall-Olkin family namely, the Harris extended family of distributions. In particular we study in detail the Harris extended Burr XII and Harris extended exponentiated exponential distribution. Various properties of these distributions are obtained. Maximum likelihood estimates are derived and applied to two real data sets on exceedances of flood peaks of the Wheaton River and failure times for electronic components in an accelerated life test. The results are given in Table 8.1 and Table 8.2. The study can be extended to various other distributions in a similar manner. The new distributions can serve as competing or complementary models in lifetime
modeling as well as reliability studies.

References


