5.1 Introduction

Lindley (1958) introduced a distribution as life time model and suggested its applications for studying stress-strength model in reliability. The probability density function of Lindley random variable \( X \), with scale parameter \( \lambda \) is given by

\[
f(x, \lambda) = \frac{\lambda^2}{1 + \lambda} (1 + x) e^{-\lambda x}; x > 0, \lambda > 0
\]  
(5.1.1)
The corresponding cumulative distribution function (c.d.f.) is given by

\[ F(x, \lambda) = 1 - \frac{1 + \lambda + \lambda x}{1 + \lambda} e^{-\lambda x} \] (5.1.2)

It can be seen that this distribution is a mixture of exponential(\(\lambda\)) and gamma(2, \(\lambda\)) distributions. Lindley distribution has drawn much attention in the statistical literature over the great popularity of the well-known exponential distribution. Sankaran (1970) introduced the discrete Poisson-Lindley distribution by combining the Poisson and Lindley distributions. Ghitany, Atieh, Nadarajah (2008) investigated most of the statistical properties of the Lindley distribution. Mahmoudi and Zakerzadeh (2010) proposed an extended version of the compound Poisson distribution which was obtained by compounding the Poisson distribution with the generalized Lindley distribution.

The Lindley distribution is important for studying stress-strength reliability modeling. The stress-strength reliability has been originally proposed by Birnbaum (1956). Birnbaum and McCarty (1958) and Govindarajulu (1968) have discussed the procedure for obtaining the distribution free confidence interval for stress strength reliability and Cheng and Chao (1984) have compared the performances of different methods of constructing the confidence interval of stress strength reliability and proposed a new method for obtaining the s-confidence intervals for the reliability in the stress-strength model. Shanker and Mishra (2013) introduced a two parameter quasi Lindley distribution as a particular case of Lindley distribution. A new generalization of Lindley distribution called Transmuted Quasi Lindley distribution was introduced by Elbatal and Elgarhy (2013).

In this chapter we introduce Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley Distribution and its Applications. First we consider the properties of Extended Lindley distribution. Negative binomial extreme stable Marshall-Olkin Extended Lindley Distribution and its properties are discussed. The quantiles and order statistics are obtained. Record values associated with the new family is also considered. The maximum likelihood estimates of the distribution is obtained by using R programme and is applied to a real data set.

5.2 Extended Lindley Distribution

Consider a particular exponentiation of (5.1.2) to extend the Lindley distribution, for which the distribution function is given by,

\[ F(x) = 1 - \left( \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \]  \hspace{1cm} (5.2.1)

where \( \theta \in \mathbb{R}^- \cup \{0, 1\} \), \( \lambda > 0 \), and \( \beta \geq 0 \).

The extension of the Lindley distribution shall be denoted by extended Lindley (EL) distribution.
EL distribution has several particular cases. For \( \theta = 1 \) and \( \beta = 1 \) the EL distribution is reduced to Lindley distribution and for \( \theta = 0 \) it reduces to the Weibull distribution. Also for \( \beta = 0 \) and \( \theta \in \mathbb{R}^- \), the EL distribution reduces to Pareto distribution given by

\[
F(x) = 1 - \left( \frac{x}{1 + x} \right)^\delta, \quad x > 0, \quad c = 1 + \frac{1}{\lambda} \text{ and } \delta > 0.
\]

(5.2.1) represents the product of the survival functions \((1 - F(x))\) of the Lomax and Weibull distribution respectively, for any \( \beta \) and \( \alpha \in \mathbb{R}^- \) (Murthy, Swartz, and Yuen, 1973). Ghitany et al. (2007) discussed Marshall-Olkin extended Lomax distributions and its application to censored data. Extended Lindley distribution can be seen as a mixture of Lomax and Weibull distribution respectively.

The probability density function of extended Lindley distribution is given by

\[
f(x) = \frac{\lambda(1 + \lambda + \lambda x)^{\theta-1}}{(1 + \lambda)^\theta} \left( \beta(1 + \lambda + \lambda x)(\lambda x)^{\beta-1} - \theta \right) e^{-(\lambda x)^\theta}; \quad x > 0 \quad (5.2.2)
\]

and the corresponding survival function and hazard rate function are given by

\[
\bar{F}(x) = \left( \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\theta}; \quad x > 0, \quad (5.2.3)
\]

and

\[
r(x) = \frac{\beta(1 + \lambda + \lambda x)\lambda x^{\beta-1} - \lambda \theta}{1 + \lambda + \lambda x} \quad (5.2.4)
\]

The first derivative of \( r(x) \) is

\[
r'(x) = \beta(\beta - 1)\lambda x^{\beta-2} + \frac{\lambda^2 \theta}{(1 + \lambda + \lambda x)^2} \quad (5.2.5)
\]

It is obvious that \( r'(x) \leq 0 \), for \( \beta \leq 1 \) and \( \theta \leq 0 \). The function \( r(x) \) is increasing for \( \theta > k \) and decreasing for \( \theta < k \), where \( k = -\beta(\beta - 1)(\lambda x)^{\beta-2}(1 + \lambda + \lambda x)^2 \). For \( \beta > 1 \), \( r(0) = f(0) = \frac{\lambda x}{1+\lambda} \). Therefore at the origin \( r(x) \) varies continuously with the
parameters. This is in contrast with the Weibull and gamma families, where $r(0) = 0$ or $r(0) = \infty$ for both families and hence $r(0)$ is discontinuous in the parameters of such families. For $\beta = 1$, $\lim_{x \to \infty} r(x) = \lambda$, the function $r(x)$ is bounded above by $\lambda$ and continuous in the parameters of the EL distribution.

The $p^{th}$ quantile $x_p$ of the EL distribution, the inverse of the distribution function $F(x_p) = p$ is given by

$$x_p = \left( \frac{\theta}{\lambda^2} \ln(\frac{1 + \lambda + \lambda x_p}{(1 + \lambda)(1 - p)^{\beta}}) \right)^{\frac{1}{\beta}}$$

### 5.3 Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley Distribution

We consider the survival function of extended Lindley distribution $\overline{F}(x) = \left( \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^{\theta} e^{-(\lambda x)^{\beta}}$, $x > 0$. From negative binomial extreme stable family, we introduce Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley distribution. It is denoted by NBESMOEL distribution. The survival function of negative binomial extreme stable Marshall-Olkin extended Lindley distribution is given by

$$\overline{G}(x) = \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left[ \left( 1 - (1 - \alpha) \left( \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^{\theta} e^{-\lambda x)^{\beta}} \right)^{-\gamma} - 1 \right]$$ (5.3.1)

Then the corresponding probability density function is given by

$$g(x) = \frac{(1 - \alpha)^{\gamma} \alpha^\gamma \lambda \left( \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^{\theta - 1} \left[ \beta(1 + \lambda + \lambda x)(\lambda x)^{\beta - 1} - \theta \right] e^{-\lambda x)^{\beta}}}{(1 - \alpha)^{\gamma} \left( 1 - (1 - \alpha) \left( \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^{\theta} e^{-\lambda x)^{\beta}} \right)^{\gamma + 1}}$$ (5.3.2)
for $\alpha < 1$, $\beta, \gamma, \lambda, \theta > 0$ and the cumulative distribution function is given by

$$G(x) = 1 - \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left[ \left( 1 - (1 - \alpha) \left( \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \right)^{-\gamma} - 1 \right]$$

(5.3.3)

Figure 5.1, figure 5.2 and figure 5.3 shows the pdf of NBESMOEL for different combinations of parameter values.

The hazard rate function is given by

$$h(x) = \frac{(1 - \alpha)\gamma F(x) r_F(x)}{(1 - (1 - \alpha) F(x))[1 - (1 - (1 - \alpha) F(x))^\gamma]}$$

$$r(x) = \frac{(1 - \alpha)\gamma \lambda (1 + \lambda + \lambda x)^{\theta-1}}{\left[ (1 + \lambda)^\theta - (1 - \alpha)(1 + \lambda + \lambda x)^\theta e^{-(\lambda x)^\beta} \right] \left[ \beta(1 + \lambda + \lambda x)(\lambda x)^{\beta-1} - \theta e^{-(\lambda x)^\beta} \right] \left[ 1 - \left( 1 - (1 - \alpha) \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \right]}$$

Figure 5.4, figure 5.5 and figure 5.6 show the hazard rate function of NBESMOEL for
Figure 5.2: Hazard rate function of NBESMOEL for various values of parameters.

5.4 Quantiles and Order statistics

The $p^{th}$ quantile function of the distribution is given by

$$x = \left\{ \frac{1}{\lambda^2} \ln \left[ \left( 1 - \frac{\alpha}{((1-p)(1-\alpha^\gamma) + \alpha^\gamma)^\frac{\beta}{\gamma}} \right)^{-1} \left( \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta (1 - \alpha) \right] \right\}^\frac{1}{\beta}$$

Let $X_1, X_2, ..., X_n$ be a random sample taken from the NBESMOEL distribution and $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be the corresponding order statistics. The survival function of NBESMOEL distribution is given by (5.3.1). Then the c.d.f of the minimum order statistic $X_{1:n}$ is given by

$$G_{1:n}(x) = 1 - \left( \tilde{G}(x) \right)^n$$

$$= 1 - \left\{ \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left[ 1 - (1 - \alpha) \left( \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\theta} \right]^{\frac{1}{\gamma}} - 1 \right\}^n$$
The c.d.f of the maximum order statistic $X_{n:n}$ is given by

$$G_{n:n}(x) = [1 - G(x)]^n = \left[ 1 - \left\{ \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left[ 1 - (1 - \alpha) \left( \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \right]^{\gamma} - 1 \right\} \right]^n$$

The probability density function $g_{i:n}(x)$ of the $i^{th}$ order statistics $X_{i:n}$ is given by

$$g_{i:n}(x) = \frac{n!(1 - \alpha)^\gamma \alpha^\gamma}{(i - 1)!(n - i)!(1 - \alpha^\gamma) \left\{ 1 - (1 - \alpha) \left( \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \right\}^{\gamma+1}} \left\{ \frac{\alpha^\gamma}{1 - \alpha^\gamma} \left[ 1 - (1 - \alpha) \left( \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \right]^{\gamma} - 1 \right\}^{i-1} \left( \frac{\alpha^\gamma}{1 - \alpha^\gamma} \right)^{n-i} \left\{ \left[ 1 - (1 - \alpha) \left( \frac{1 + \lambda + \lambda x}{1 + \lambda} \right)^\theta e^{-(\lambda x)^\beta} \right]^{\gamma} - 1 \right\}^{n-i} \frac{(1 + \lambda + \lambda x)^{\beta-1}}{(1 + \lambda)^\beta} \left[ \beta(1 + \lambda + \lambda x)(\lambda x)^{\beta-1} - \theta \right] e^{-(\lambda x)^\beta}$$

### 5.5 Record values

In this section we consider the record statistics of Negative binomial extreme stable Marshall-Olkin extended Lindley distribution with $\gamma = \theta = \beta = 1$ with the pdf given by

$$g(x) = \frac{\alpha \lambda (1 + \lambda)(\lambda + \lambda x)e^{-(\lambda x)}}{[(1 + \lambda) - (1 - \alpha)(1 + \lambda + \lambda x)e^{-(\lambda x)}]^2}; 0 < x < \infty \quad (5.5.1)$$
Using (1.3.1) and (1.3.2), we get the pdf and joint pdf of NBESMOEL\((\alpha, \lambda)\) as

\[
g_R(x) = \frac{\alpha \lambda (1 + \lambda)(\lambda + \lambda x)e^{-(\lambda x)}}{[(1 + \lambda) - (1 - \alpha)(1 + \lambda + \lambda x)e^{-(\lambda x)}]^2}
\]

\[
\frac{1}{(n-1)!} \times \left[-\log \left\{ \frac{\alpha (1 + \lambda + \lambda x)e^{-(\lambda x)}}{(1 + \lambda) - (1 - \alpha)(1 + \lambda + \lambda x)e^{-(\lambda x)}} \right\} \right]^{n-1} \tag{5.5.2}
\]

\[
g_{Rm,Rn}(x,y) = \frac{1}{(n-1)!(n-m-1)!} \left[-\log \left\{ \frac{\alpha (1 + \lambda + \lambda y)e^{-(\lambda y)}}{(1 + \lambda) - (1 - \alpha)(1 + \lambda + \lambda y)e^{-(\lambda y)}} \right\} \right]^{n-m-1}
\]

\[
\left[ \log \left\{ \frac{(1 + \lambda + \lambda x)e^{-(\lambda x)}}{(1 + \lambda) - (1 - \alpha)(1 + \lambda + \lambda x)e^{-(\lambda x)}} \right\} \right]^{n-m-1}
\]

\[
\frac{(1 + \lambda)^2 \alpha \lambda^2 (\lambda + \lambda x)(\lambda + \lambda y)e^{-(\lambda y)}}{[(1 + \lambda) - (1 - \alpha)(1 + \lambda + \lambda x)e^{-(\lambda x)}]^2}
\]

\[
\frac{1}{[(1 + \lambda) - (1 - \alpha)(1 + \lambda + \lambda y)e^{-(\lambda y)}]} \frac{(1 + \lambda + \lambda x)e^{-(\lambda x)}}{1 + \lambda + \lambda x}
\]

### 5.6 Estimation of Parameters

In this section we consider maximum likelihood estimation with respect to a given sample of size \(x_1, x_2, \ldots, x_n\), then the log likelihood function is given by

\[
\log L(\alpha, \beta, \gamma, \lambda, \theta) = n\log(1 - \alpha) + n\log \gamma + n\gamma \log \alpha + n\log \lambda + (\theta - 1)\log(1 + \lambda + \lambda x_i) + n\log \beta + \sum_{i=1}^{n} \log(1 + \lambda + \lambda x_i) + (\beta - 1)\sum_{i=1}^{n} \log(\lambda x_i) - n\log \theta + \lambda \sum_{i=1}^{n} x_i^\beta - n\log(1 - \alpha) - n\theta \log(1 + \lambda) - (\gamma + 1)\sum_{i=1}^{n} \log \left[ 1 - (1 - \alpha) \left( \frac{1 + \lambda + \lambda x_i}{1 + \lambda} \right)^\theta e^{-(\lambda x_i)^\beta} \right]
\]

The partial derivatives of the log likelihood functions with respect to the parameters
are

\[
\frac{\partial \log L}{\partial \alpha} = \frac{-n}{1 - \alpha} + \frac{n\gamma}{\alpha} + \frac{n\gamma\alpha^{\gamma-1}}{1 - \alpha^\gamma} + (\gamma + 1) \sum_{i=1}^{n} \frac{(1 + \lambda + \lambda x_i)^\theta e^{-(\lambda x_i)^\beta}}{1 - (1 - \alpha) \left(1 + \lambda + \lambda x_i\right)^\theta e^{-(\lambda x_i)^\beta}}
\]

\[
\frac{\partial \log L}{\partial \beta} = n\beta + \sum_{i=1}^{n} \log(\lambda x_i) + \lambda^\beta \log \lambda + x_i^\beta \log x_i
\]

\[
+ (\gamma + 1)(1 - \alpha)\beta\lambda^\beta \sum_{i=1}^{n} \frac{(1 + \lambda + \lambda x_i)^\theta e^{-(\lambda x_i)^\beta}}{1 - (1 - \alpha) \left(1 + \lambda + \lambda x_i\right)^\theta e^{-(\lambda x_i)^\beta}}
\]

\[
\frac{\partial \log L}{\partial \gamma} = \frac{n}{\gamma} + n \log \alpha + \frac{n\alpha^\gamma \log \alpha}{1 - \alpha^\gamma} - \sum_{i=1}^{n} \log \left[1 - (1 - \alpha) \left(1 + \lambda + \lambda x_i\right)^\theta e^{-(\lambda x_i)^\beta}\right]
\]

\[
\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + \frac{(\theta - 1)(1 + x_i)}{1 + \lambda + \lambda x_i} + \sum_{i=1}^{n} \frac{1 + x_i}{1 + \lambda + \lambda x_i} + (\beta - 1)n \sum_{i=1}^{n} \frac{x_i}{\lambda x_i}
\]

\[
+ \beta\lambda^{\beta-1} \sum_{i=1}^{n} x_i^\beta - \frac{n\theta}{1 + \lambda} + (\gamma + 1) \sum_{i=1}^{n} \frac{1}{1 - (1 - \alpha) \left(1 + \lambda + \lambda x_i\right)^\theta e^{-(\lambda x_i)^\beta}}
\]

\[
\left[1 - \frac{\left(1 + \lambda + \lambda x_i\right)^\theta e^{-(\lambda x_i)^\beta}}{1 + \lambda}\right]^{\theta-1} \left[1 + \lambda + \lambda x_i\right]^{\theta-1} x_i e^{-(\lambda x_i)^\beta}
\]

\[
\frac{\partial \log L}{\partial \theta} = \log(1 + \lambda + \lambda x_i) - \frac{n}{\theta} - n \log(1 + \lambda)
\]

\[
- (\gamma + 1)(1 - \alpha) \sum_{i=1}^{n} \frac{(1 + \lambda + \lambda x_i)^\theta e^{-(\lambda x_i)^\beta}}{1 - (1 - \alpha) \left(1 + \lambda + \lambda x_i\right)^\theta e^{-(\lambda x_i)^\beta}}
\]

The maximum likelihood estimates can be obtained by solving the equations \( \frac{\partial \log L}{\partial \alpha} = 0 \).
Table 5.1: Estimates, log likelihood and K-S statistic for the data

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameter</th>
<th>Estimates</th>
<th>-log likelihood</th>
<th>K-S statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>NBESMOEL</td>
<td>$\alpha$</td>
<td>2.8352</td>
<td>128.32</td>
<td>0.7624</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>0.1736</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\gamma$</td>
<td>0.8722</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda$</td>
<td>1.5481</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\theta$</td>
<td>0.2915</td>
<td></td>
<td></td>
</tr>
<tr>
<td>EL</td>
<td>$\beta$</td>
<td>0.2044</td>
<td>131.14</td>
<td>0.8531</td>
</tr>
<tr>
<td></td>
<td>$\lambda$</td>
<td>1.3956</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\theta$</td>
<td>0.0119</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$0, \frac{\partial \log L}{\partial \beta} = 0, \frac{\partial \log L}{\partial \gamma} = 0, \frac{\partial \log L}{\partial \lambda} = 0, \frac{\partial \log L}{\partial \theta} = 0$. The equations can be solved using nlm package in R software.

5.7 Data Analysis

In this section we analyze a data set and compare Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley distribution with Extended Lindley distribution. We consider data from Linhart and Zucchini (1986). The following data are failure times of the air conditioning system of an airplane: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

Extended Lindley distribution with parameters $\beta$, $\lambda$ and $\theta$ and Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley distribution with parameters $\alpha$, $\beta$, $\gamma$, $\lambda$ and $\theta$ are fitted to the data. The results are presented in Table 5.1. The QQ plots for the two distributions are shown in Figure 5.7. From Table 5.1, it is seen that the K-S statistic for the NBESMOEL distribution is 0.7624 which is less than that for the EL distribution. Similar is the case with log likelihood values. The Q-Q plot also confirm that the new distribution fits well than the original EL distribution. Hence we conclude that NBESMOEL distribution is a better model for the data set.
5.8 Conclusion

In this chapter we proposed a new distribution namely, Negative Binomial Extreme Stable Marshall-Olkin Extended Lindley distribution. Its properties are obtained. Record values and estimation of parameters are also discussed. We analyze a real data set and compare the goodness of fit to NBESMOEL with EL distribution. We conclude that NBESMOEL distribution is a better fit. The results are given in Table 5.1.

References


