WELL-POSEDNESS AND SOLUTION METHODS
FOR CERTAIN VARIATIONAL INEQUALITIES

ABSTRACT
SUBMITTED FOR THE AWARD OF THE DEGREE OF
Doctor of Philosophy
In
Mathematics
BY
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UNDER THE SUPERVISION OF
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ALIGARH MUSLIM UNIVERSITY,
ALIGARH, INDIA (202002)
2014
Abstract

The present thesis entitled "Well-Posedness and Solution Methods for Certain Variational Inequalities" is a part of the research work carried out by the author in last four and half years. It studies different kinds of well-posedness of variational-hemivariational inequalities, and various types of iterative methods for finding the solutions of different kinds of variational inequalities.

The present thesis comprises seven chapters and is classified as follows:

Chapter 1 provides a brief introduction of variational inequalities, hierarchical variational inequalities, system of variational inequalities, variational-hemivariational inequalities and split feasibility problems. It consists of five sections. Section 1.1 gives a brief introduction of variational inequalities, formulation of a variational inequality and its geometric interpretation, some known facts about variational inequalities, and iterative schemes for finding the solutions of variational inequalities. For further details on variational inequalities, their generalizations and applications, we refer to [4, 6, 7, 8, 24, 26, 31, 32, 33, 34, 41, 45, 47, 48, 54, 55, 65, 72, 74, 75, 76] and the references therein. In Section 1.2, we give a brief introduction of hierarchical variational inequality problem, that is, a variational inequality problem defined over the set of fixed points of a mapping. We also consider a variational inequality problem defined over the set of intersection of the set of fixed points of a finite family of nonexpansive mappings. For detail study on hierarchical variational inequalities and their applications, we refer to [13, 21, 23, 36, 38, 40, 56, 59, 63, 64, 69, 71; 79, 80, 81, 83, 84, 85, 87] and the references therein. Section 1.3 provides the decomposition of original variational inequality into a system of variational inequalities. It worth to mention that a system of variational inequalities is a very powerful tool to study Nash equilibrium problem for finite number of players. In this section, we consider a system of two variational inequalities which could be useful to study Nash equilibrium problem for two players game. More about the existence and approximate solutions of various systems of variational inequalities, see [1, 2, 3, 5, 20, 44, 70, 77] and the references therein. Section 1.4 introduces the notion of nonconvex superpotential by using the generalized gradient in the sense of Clarke [22]. Due to the lack of convexity, new types of variational expressions known as hemivariational inequality, are obtained. For a comprehensive treatment of the hemivariational inequalities, we refer [67, 68, 73]. In this section, we also give a brief introduction to a variational-hemivariational inequality problem introduced by Motreanu and Radulescu [68]. Last section, (Section 1.5), provides a brief introduction of split feasibility problems (in short, SFP) and a popular algorithm that solves the SFP, that is, the CQ algorithm of Byrne [9, 10] which is found to be a
gradient-projection method (GPM) in convex minimization.

Chapter 2 provides some basic definitions, properties and results which will be used throughout the thesis. This chapter is divided into four sections. Section 2.1 carries out the definitions and results from analysis, in which we discuss the properties of sequences of real numbers. Several results from functional analysis are also given. We give the definition of various kind of mappings and their properties. In Section 2.2, we study metric projection and some of its properties. Section 2.3 deals with some concept form geometry of Banach spaces. In Section 2.4, we gather some definitions, characterizations and few results from nonsmooth analysis.

Chapter 3 deals with several kinds of well-posedness for variational-hemivariational inequalities which includes several classes of variational inequalities as special cases. This chapter consists of four sections. Section 3.1 gives an introduction of well-posedness and well-posedness by perturbations or extended well-posedness, due to Zolezzi [89, 90], for optimization problems. The concept of well-posedness has been extended to other variational problems, such as, variational inequalities [16, 25, 27, 28, 51, 52, 53, 57], saddle point problems [11], Nash equilibrium problems [53, 58, 60, 61, 62, 66], equilibrium problems [29], inclusion problems [50, 49] and fixed point problems [49, 50, 82]. In Section 3.2, we formulate the variational-hemivariational inequality problem and give some of its special cases. In Section 3.3, we introduce the concepts of well-posedness by perturbations for variational-hemivariational inequalities and establish their metric characterizations. We introduce the concept of \( \alpha \)-approximating sequence, strongly and weakly \( \alpha \)-well-posed by perturbations and strongly and weakly generalized \( \alpha \)-well-posed by perturbations for variational-hemivariational inequality problem. Section 3.4 provides the link between the well-posedness by perturbations for a variational-hemivariational inequality problem and the well-posedness by perturbations for the corresponding inclusion problem.

The content of this chapter is published in [CGWa].

In Chapter 4, we consider a general system of nonlinear variational inequalities (in short, GSNVI) in the setting of Banach spaces \( X \). This chapter is divided into three sections. In Section 4.1, we formulate the general system of nonlinear variational inequalities with the help of normalized duality mapping which is considered and studied by Yao et al. [86]. Section 4.2 provides the equivalence between GSNVI and fixed point problem by using a sunny nonexpansive retraction \( \Pi_C : X \to C \), where \( C \) is a nonempty closed convex subset of a real 2-uniformly smooth Banach space \( X \). In Section 4.3, we study the iterative methods for computing the approximate solutions of GSNVI. We also introduce the implicit and explicit algorithms of Mann-type for
solving GSNVI. We show the strong convergence of the sequences generated by the proposed algorithms.

The content of this chapter is accepted in [CGA].

Chapter 5 provides some Mann-type extragradient iterative algorithms with regularization for finding a common element of the solution set $\Lambda$ of a general system of variational inequalities, the solution set $\Gamma$ of a split feasibility problem and the fixed point set $\text{Fix}(S)$ of a strictly pseudo-contractive mapping $S$ in the setting of Hilbert spaces $H$. This chapter consists of two sections. Section 5.1 provides the formulation of the Mann-type extragradient iterative algorithms with regularization for finding a common element of the solution set of GSNVI, the solution set of SFP and the fixed point set of a strictly pseudo-contractive mapping $S$. In Section 5.2, we provide the weak convergence results and their consequences for the proposed algorithms.

The content of this chapter is published in [CGWb].

In Chapter 6, we consider the triple hierarchical variational inequality problem (in short, THVIP). This chapter is divided into four sections. In Section 6.1, we formulate the triple hierarchical variational inequality problem considered and studied by liduka [35, 37]. For the details of several iterative methods for finding the solutions of THVIP, see [12, 14, 15, 17, 35, 37, 39, 42, 43, 46, 78, 88] and the references therein. In Section 6.2, we propose hybrid iterative algorithm for computing a fixed point of a pseudo-contraction mapping and a solution of THVIP in the setting of real Hilbert spaces. We also establish a strong convergence result for the sequences generated by the proposed algorithm. In Section 6.3, we propose multi-step explicit and implicit hybrid extragradient-like methods to compute the approximate solutions of THVIP and present the convergence analysis of the sequences generated by the proposed methods. In Section 6.4, we derive explicit and implicit solution methods for solving a system of hierarchical variational inequalities (SHVI). Under some mild conditions, we prove that the sequences generated by the proposed methods converge strongly to a unique solution of the SHVI.

The content of this chapter is published in [ACG] and communicated in [GCA].

Last chapter, (Chapter 7) deals with the triple hierarchical variational inequality problems (THVIP) defined over the set of solutions of a hierarchical variational inequality problem which is defined over the intersection of the fixed point sets of a finite family of nonexpansive mappings. This chapter consists of two sections. In Section 7.1, we consider the monotone variational inequality with the variational inequality constraint which is defined over the intersection of the fixed point sets of a family of $N$ nonexpansive mappings $T_i : H \rightarrow H$, $i = 1, 2, \ldots, N$, where $N \geq 1$ an integer,
considered and studied by Ceng et al. [15, 88]. In Section 7.2, we propose the hybrid iterative algorithm for computing a common fixed point of a finite family \( \{T_i\}_{i=1}^{N} \) of pseudo-contractive mappings and a solution of THVIP in the setting of a real Hilbert space \( H \). We establish the strong convergence theorem for the sequences generated by the proposed algorithm.

The content of this chapter is published in [ACG].

Finally, the thesis concludes with an extensive bibliography which presents the list of papers, book and articles referred for this research work.
References


THESIS


