Chapter 6

Iterative Methods for Triple Hierarchical Variational Inequalities

In this chapter, we consider a triple hierarchical variational inequality problem (THVIP). By combining hybrid steepest-descent method, Mann iteration method and projection method, we present hybrid iterative algorithm for computing a fixed point of a pseudo-contractive mapping and a solution of a triple hierarchical variational inequality in the setting of real Hilbert spaces. Under some conditions, we prove that the sequence generated by the proposed algorithm converges strongly to a fixed point which is also a solution of triple hierarchical variational inequality problem. We also propose multi-step explicit and implicit hybrid extragradient-like methods to compute the approximate solutions of THVIP and present the convergence analysis of the sequences generated by the proposed methods. We derive explicit and implicit solution methods for solving a system of hierarchical variational inequalities (SHVI). Under some mild conditions, it is proven that the sequences generated by the proposed methods converge strongly to a unique solution of the SHVI.
6.1 Introduction and Formulations

In 2009, Iiduka [94, 96] considered a problem which has triple structure, that is, a variational inequality problem defined over the set of solutions of another variational inequality problem which is defined over the set of fixed points of a mapping. Because of the triple structure of the problem, it is called triple hierarchical variational inequality problem. So, a variational inequality problem defined over the set of solutions of a hierarchical variational inequality problem is called a triple hierarchical variational inequality problem (in short, THVIP). Iiduka [94, 96] proposed some iterative methods for computing the approximate solutions of THVIP. The strong convergence of the sequences generated by the proposed methods is studied. Some examples of triple hierarchical variational inequality problems are provided in [96]. Subsequently, Iiduka [98] translated the non-concave utility bandwidth allocation problem with compoundable constraints into a triple hierarchical variational inequality problem. Then, he suggested an iterative method, so called fixed point optimization algorithm, to find the solution of THVIP. The strong convergence of the iterative method is studied. In the last few years, several iterative methods for finding the solutions of THVIP have been proposed and analyzed; for example, see [28, 30, 31, 39, 94, 96, 98, 105, 106, 114, 217, 248] and the references therein. Ceng et al. [30] considered a monotone variational inequality problem defined over the set of solutions of another variational inequality problem which is defined over the intersection of the fixed point sets of N nonexpansive mappings. They proposed two relaxed hybrid steepest-descent algorithms with variable parameters for computing the approximate solutions of these two problems. The strong convergence of these two algorithms is also studied. Recently, Zeng et al. [248] presented strong convergence of relaxed hybrid steepest-descent method under some mild conditions on parametric sequences. The THVIP is further investigated and generalized in [39, 105, 106, 217].

Motivated by the work of Yamada [229] on a hierarchical variational inequality problem
defined over the set of common fixed points of a finite family of nonexpansive mappings, Ceng et al. [31, 248] considered the variational inequality problem with the variational inequality constraint which is defined over the intersection of the fixed point sets of a family of $N$ nonexpansive mappings $T_i : H \to H$, where $N \geq 1$ an integer. Such problem is called a triple hierarchical variational inequality problem for a family of nonexpansive mappings.

**Problem 6.1.1.** Let $H$ be a real Hilbert space. Assume that

(B1) $T : H \to H$ is a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$;

(B2) $A_1 : H \to H$ is an $\alpha$-inverse strongly monotone mapping;

(B3) $A_2 : H \to H$ is a $\beta$-strongly monotone and $L$-Lipschitz continuous mapping;

(B4) $\text{VIP}(\text{Fix}(T), A_1) \neq \emptyset$.

The triple hierarchical variational inequality problem (in short, THVIP) is to find $x^* \in \text{VIP}(\text{Fix}(T), A_1)$ such that

$$\langle A_2 x^*, v - x^* \rangle \geq 0, \; \forall v \in \text{VIP}(\text{Fix}(T), A_1), \quad (6.1.1)$$

that is, to find

$$x^* \in \text{VIP}(\text{VIP}(\text{Fix}(T), A_1), A_2)$$

$$= \{ x^* \in \text{VIP}(\text{Fix}(T), A_1) : \langle A_2 x^*, v - x^* \rangle \geq 0, \; \forall v \in \text{VIP}(\text{Fix}(T), A_1) \}.$$

several examples of THVIP (Problem 6.1.1) can be found in [6].

Iiduka [94, 96] proposed the iterative method to compute the solutions of THVIP (Problem 6.1.1). He established a convergence result for the proposed iterative method. He further studied the convergence of the sequence generated by his iterative method by making some changes in the parametric sequences.
Recently, Ceng et al. [31] extended the algorithm proposed by Iiduka [94, 96] by making some changes in variable parameters. They also established the convergence result for the sequence generated by the proposed algorithm.

Further, Zeng et al. [248, Theorem 3.1] concluded the result given by Ceng et al. [31] by modifying the parametric conditions. They presented an application of THVIP (Problem 6.1.1) to constrained pseudo-inverse problem.

### 6.2 A Hybrid Iterative Algorithm

In this section, we propose the hybrid iterative algorithm for computing a fixed point of a pseudo-contraction mapping and a solution of THVIP (Problem 6.1.1) in the setting of real Hilbert spaces.

**Algorithm 6.2.1.** Suppose that the assumptions (B1)-(B4) in Problem 6.1.1 are satisfied.

**Step 1.** Take \( \mu > 0 \). Put \( C_1 = H \), choose \( x_0 \in H \), \( \lambda_1 \in (0, 2\alpha] \), \( \alpha_1 \in (0, 1) \), \( \beta_1 \in (0, 1) \) arbitrarily, and let \( n := 1 \).

**Step 2.** Given \( x_n \in C_n \), choose \( \lambda_n \in (0, 2\alpha] \), \( \alpha_n \in (0, 1] \) and \( \beta_n \in (0, 1) \) and compute \( x_{n+1} \in C_{n+1} \) as:

\[
\begin{align*}
C_{n+1} := \left\{ z \in C_n : \| \beta_n (I - (I - \alpha_n \mu A_2)T_n)(y_n) \| & \leq 2\beta_n [(x_n - z, (I - (I - \alpha_n \mu A_2)T_n)(y_n)) \\
& \quad - (\alpha_n \mu A_2 T_n(y_n) + \lambda_n A_1(T(y_n), y_n - z)] \right\}, \quad (6.2.1)
\end{align*}
\]

\[
x_{n+1} := P_{C_{n+1}}(x_n), \quad n \geq 0,
\]

where \( T_n := (I - \lambda_n A_1)T \) for all \( n \geq 1 \).

Update \( n := n + 1 \) and go to Step 2.

We now present the criteria for the strong convergence of the sequence generated by the Algorithm 6.2.1.
Theorem 6.2.1. Let \( T : H \to H \) be a \( L \)-Lipschitz continuous pseudo-contractive self-mapping defined on a real Hilbert space \( H \) such that \( \text{Fix}(T) \neq \emptyset \). Assume that \( \{\beta_n\} \subset [a, b] \) for some \( a, b \in (0, \frac{1}{L+1}) \) and \( \{\alpha_n\} \subset (0, 1) \) and \( \{\lambda_n\} \subset (0, 2\alpha) \) such that \( \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \lambda_n = 0 \). Take a fixed number \( \mu \in (0, \frac{2\beta}{L^2}) \). Then, the sequence \( \{x_n\} \) generated by Algorithm 6.2.1 satisfies the following properties:

(a) \( \{x_n\} \) is bounded;

(b) \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \) and \( \lim_{n \to \infty} \|x_n - T(x_n)\| = 0 \);

(c) \( \{x_n\} \) converges strongly to \( P_{\text{Fix}(T)}(x_0) \);

(d) If \( T \) is nonexpansive and \( A_1 \) is injective, then \( P_{\text{Fix}(T)}(x_0) \) is a unique solution of \( \text{THVIP} \) (Problem 6.1.1) provided \( \lim_{n \to \infty} (\|x_n - y_n\| + \alpha_n)/\lambda_n = 0 \).

Proof. We first show that \( P_{\text{Fix}(T)} \) and \( \{x_n\} \) are well defined.

From [150, 253], we note that \( \text{Fix}(T) \) is closed and convex. Indeed, by [253], we can define a mapping \( g : H \to H \) by

\[
g(x) = (2I - T)^{-1}(x), \quad \forall x \in H.
\]

It is clear that \( g \) is a nonexpansive self-mapping such that \( \text{Fix}(T) = \text{Fix}(g) \). Hence, by [153, Proposition 2.1 (iii)], we conclude that \( \text{Fix}(g) = \text{Fix}(T) \) is a closed convex set. This implies that the projection \( P_{\text{Fix}(T)} \) is well defined. It is obvious that \( \{C_n\} \) is closed and convex. Thus, \( \{x_n\} \) is also well defined.

We now show that \( \text{Fix}(T) \subseteq C_n \) for all \( n \geq 0 \). Indeed, taking \( p \in \text{Fix}(T) \), we note that \( (I - T)p = 0 \) and pseudo-contractivity of \( T \) is equivalent to

\[
\langle (I - T)x - (I - T)y, x - y \rangle \geq 0, \quad \forall x, y \in H.
\]  \hspace{1cm} (6.2.2)

By using Lemma 2.1.5 and the inequality (6.2.2), we obtain
\[
\|x_n - p - \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n\|^2
\]
\[
= \|x_n - p\|^2 - \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n\|^2
\]
\[
- 2\beta_n ((I - (I - \alpha_n \mu A_2)T_n)y_n, x_n - p - \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n)
\]
\[
= \|x_n - p\|^2 - \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n\|^2
\]
\[
- 2\beta_n ((I - T)y_n - (I - T)p + \lambda_n A_1(Ty_n), y_n - p)
\]
\[
- 2\beta_n (T_ny_n - (I - \alpha_n \mu A_2)(Ty_n), y_n - p)
\]
\[
- 2\beta_n ((I - (I - \alpha_n \mu A_2)T_n)y_n, x_n - y_n - \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n)
\]
\[
\leq \|x_n - p\|^2 - \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n\|^2
\]
\[
- 2\beta_n (T_ny_n - (I - \alpha_n \mu A_2)T_ny_n + \lambda_n A_1(Ty_n), y_n - p)
\]
\[
- 2\beta_n ((I - (I - \alpha_n \mu A_2)T_n)y_n, x_n - y_n - \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n)
\]
\[
= \|x_n - p\|^2 - \|x_n - y_n + y_n - x_n + \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n\|^2
\]
\[
- 2\beta_n (\alpha_n \mu A_2 T_n y_n + \lambda_n A_1(Ty_n), y_n - p)
\]
\[
- 2\beta_n ((I - (I - \alpha_n \mu A_2)T_n)y_n, x_n - y_n - \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n)
\]
\[
= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n\|^2
\]
\[
- 2(x_n - y_n, y_n - x_n + \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n)
\]
\[
+ 2\beta_n ((I - (I - \alpha_n \mu A_2)T_n)y_n, y_n - x_n + \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n)
\]
\[
- 2\beta_n (\alpha_n \mu A_2 T_n y_n + \lambda_n A_1(Ty_n), y_n - p)
\]
\[
= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n\|^2
\]
\[
- 2(x_n - y_n - \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n, y_n - x_n)
\]
\[
+ \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n)
\]
\[
- 2\beta_n (\alpha_n \mu A_2 T_n y_n + \lambda_n A_1(Ty_n), y_n - p)
\]
\[
\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - x_n + \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n\|^2
\]
\[
+ 2(x_n - y_n - \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n, y_n - x_n)
\]
\[
+ \beta_n(I - (I - \alpha_n \mu A_2)T_n)y_n)
\]
\[
- 2\beta_n (\alpha_n \mu A_2 T_n y_n + \lambda_n A_1(Ty_n), y_n - p)
\]
\[
(6.2.3)
\]
Since $T$ is a $L$-Lipschitz continuous mapping, by Lemma 2.1.11 and Lemma 2.1.12, we have

$$
\| (I - (I - \alpha_n \mu A_2) T_n) x_n - (I - (I - \alpha_n \mu A_2) T_n) y_n \| \\
\leq \| x_n - y_n \| + \| (I - \alpha_n \mu A_2) T_n x_n - (I - \alpha_n \mu A_2) T_n y_n \| \\
\leq \| x_n - y_n \| + (1 - \alpha_n \tau) \| T_n x_n - T_n y_n \| \\
= \| x_n - y_n \| + (1 - \alpha_n \tau) \| (I - \lambda_n A_1) T_n x_n - (I - \lambda_n A_1) T_n y_n \| \\
\leq \| x_n - y_n \| + \| (I - \lambda_n A_1) T_n x_n - (I - \lambda_n A_1) T_n y_n \| \\
\leq \| x_n - y_n \| + \| T x_n - T y_n \| \\
\leq (L + 1) \| x_n - y_n \| ,
$$

(6.2.4)

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu \kappa^2)}$. From (6.2.1), we observe that

$$
x_n - y_n = \beta_n (I - (I - \alpha_n \mu A_2) T_n) x_n .
$$

Hence, by utilizing (6.2.4), we obtain

$$
\langle x_n - y_n - \beta_n (I - (I - \alpha_n \mu A_2) T_n) y_n, y_n - x_n + \beta_n (I - (I - \alpha_n \mu A_2) T_n) y_n \rangle \\
= \beta_n \langle (I - (I - \alpha_n \mu A_2) T_n) x_n - (I - (I - \alpha_n \mu A_2) T_n) y_n, y_n - x_n \\
+ \beta_n (I - (I - \alpha_n \mu A_2) T_n) y_n \rangle \\
\leq \beta_n \| (I - (I - \alpha_n \mu A_2) T_n) x_n - (I - (I - \alpha_n \mu A_2) T_n) y_n \| \\
\| y_n - x_n + \beta_n (I - (I - \alpha_n \mu A_2) T_n) y_n \| \\
\leq \beta_n (L + 1) \| x_n - y_n \| \| y_n - x_n + \beta_n (I - (I - \alpha_n \mu A_2) T_n) y_n \| \\
\leq \frac{\beta_n (L + 1)}{2} \left( \| x_n - y_n \|^2 + \| y_n - x_n + \beta_n (I - (I - \alpha_n \mu A_2) T_n) y_n \|^2 \right) .
$$

(6.2.5)

Combining (6.2.3) and (6.2.5), we get

$$
\| x_n - p - \beta_n (I - (I - \alpha_n \mu A_2) T_n) y_n \|^2 \\
\leq \| x_n - p \|^2 - \| x_n - y_n \|^2 - \| y_n - x_n + \beta_n (I - (I - \alpha_n \mu A_2) T_n) y_n \|^2
$$
\[ + \beta_n(L + 1) \left( \| x_n - y_n \|^2 + \| y_n - x_n + \beta_n(I - (I - \alpha_n \mu A_2) T_n) y_n \|^2 \right) \\
- 2 \beta_n(\alpha_n \mu A_2(T_n y_n) + \lambda_n A_1(T y_n), y_n - p) \\
= \| x_n - p \|^2 + [\beta_n(L + 1) - 1](\| x_n - y_n \|^2 \\
+ \| y_n - x_n + \beta_n(I - (I - \alpha_n \mu A_2) T_n) y_n \|^2) \\
- 2 \beta_n(\alpha_n \mu A_2(T_n y_n) + \lambda_n A_1(T y_n), y_n - p) \\
\leq \| x_n - p \|^2 - 2 \beta_n(\alpha_n \mu A_2(T_n y_n) + \lambda_n A_1(T y_n), y_n - p). \] (6.2.6)

We observe that
\[ \| x_n - p - \beta_n(I - (I - \alpha_n \mu A_2) T_n) y_n \|^2 \\
= \| x_n - p \|^2 - 2 \beta_n\langle x_n - p, (I - (I - \alpha_n \mu A_2) T_n) y_n \rangle \\
+ \| \beta_n(I - (I - \alpha_n \mu A_2) T_n) y_n \|^2. \] (6.2.7)

Therefore, from (6.2.6) and (6.2.7), we have
\[ \| \beta_n(I - (I - \alpha_n \mu A_2) T_n) y_n \|^2 \leq 2 \beta_n[(x_n - p, (I - (I - \alpha_n \mu A_2) T_n) y_n) \\
- \langle \alpha_n \mu A_2(T_n y_n) + \lambda_n A_1(T y_n), y_n - p \rangle], \]
which implies that
\[ p \in C_n, \quad \text{that is, } \quad \text{Fix}(T) \subseteq C_n, \quad \forall n \geq 1. \]

Since \( x_n = P_{C_n}(x_0) \), we have
\[ \langle x_0 - x_n, x_n - y \rangle \geq 0, \quad \forall y \in C_n. \]

Since \( \text{Fix}(T) \subseteq C_n \), we obtain
\[ \langle x_0 - x_n, x_n - u \rangle \geq 0, \quad \forall u \in \text{Fix}(T). \]

Therefore, for all \( u \in \text{Fix}(T) \), we have
\[ 0 \leq \langle x_0 - x_n, x_n - u \rangle \\
= \langle x_0 - x_n, x_n - x_0 + x_0 - u \rangle \]
\[\begin{align*}
&= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - u \rangle \\
&\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - u\|
\end{align*}\]

which implies that
\[\|x_0 - x_n\| \leq \|x_0 - u\|, \quad \forall u \in \text{Fix}(T). \tag{6.2.8}\]

Thus, \(\{x_n\}\) is bounded and so are \(\{y_n\}, \{Ty_n\}, \{T_ny_n\}\). From \(x_n = P_{C_n}(x_0)\) and \(x_{n+1} = P_{C_{n+1}}(x_0) \in C_{n+1} \subseteq C_n\), we have
\[\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0. \tag{6.2.9}\]

Hence,
\[\begin{align*}
0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\
&= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\
&\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|
\end{align*}\]

and therefore,
\[\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.\]

Thus, \(\lim_{n \to \infty} \|x_n - x_0\|\) exists.

From inequality (6.2.9), we obtain
\[\begin{align*}
\|x_{n+1} - x_n\|^2 \\
&= \| (x_{n+1} - x_0) - (x_n - x_0) \|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\
&\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \to 0.
\end{align*}\]

Since \(x_{n+1} \in C_{n+1} \subseteq C_n\), from \(\|x_n - x_{n+1}\| \to 0\), \(\lambda_n \to 0\) and \(\alpha_n \to 0\), it follows that
\[\|\beta_n(I - (I - \alpha_n\mu A_2)T_n)y_n\|^2 \leq 2\beta_n\langle x_n - x_{n+1}, (I - (I - \alpha_n\mu A_2)T_n)y_n \rangle\]
\[-(\alpha_n \mu A_2(T_n y_n) + \lambda_n A_1(T_n y_n), y_n - x_{n+1})\]
\[\leq 2 \beta_n \|x_n - x_{n+1}\| \|I - (I - \alpha_n \mu A_2)T_n\| y_n\| + \|\alpha_n \mu A_2(T_n y_n) + \lambda_n A_1(T_n y_n)\| y_n - x_{n+1}\|
\[\leq 2 \beta_n \|x_n - x_{n+1}\| (\|y_n\| + \|T_n y_n\| + \alpha_n \mu \|A_2(T_n y_n)\|)
\[+ \|\alpha_n \mu \|A_2(T_n y_n)\| + \lambda_n \|A_1(T_n y_n)\|\|y_n - x_{n+1}\|) \rightarrow 0.
\]

We note that $\beta_n \in [a, b]$ for some $a, b \in (0, \frac{1}{2\lambda_1})$, we thus obtain

\[\|y_n - (I - \alpha_n \mu A_2)T_n y_n\| \rightarrow 0.
\]

We also note that

\[\|Ty_n - (I - \alpha_n \mu A_2)T_n y_n\| = \|Ty_n - T_n y_n + \alpha_n \mu A_2(T_n y_n)\|
\[\leq \|Ty_n - (I - \lambda_n A_1)T_n y_n\| + \alpha_n \mu \|A_2(T_n y_n)\|
\[= \lambda_n \|A_1(T_n y_n)\| + \alpha_n \mu \|A_2(T_n y_n)\| \rightarrow 0.
\]

Therefore, we get

\[\|y_n - Ty_n\| \leq \|y_n - (I - \alpha_n \mu A_2)T_n y_n\| + \|Ty_n - (I - \alpha_n \mu A_2)T_n y_n\| \rightarrow 0.
\]

On the other hand, by utilizing Lemma 2.1.11 and Proposition 2.1.12, we deduce that

\[\|x_n - (I - \alpha_n \mu A_2)T_n x_n\|
\[\leq \|x_n - y_n\| + \|y_n - (I - \alpha_n \mu A_2)T_n y_n\|
\[+ \|(I - \alpha_n \mu A_2)T_n y_n - (I - \alpha_n \mu A_2)T_n x_n\|
\[\leq \|x_n - y_n\| + \|y_n - (I - \alpha_n \mu A_2)T_n y_n\|
\[+ (1 - \alpha_n \tau)\|T_n y_n - T_n x_n\|
\[\leq \|x_n - y_n\| + \|y_n - (I - \alpha_n \mu A_2)T_n y_n\|
\[+ \|(I - \lambda_n A_1)Ty_n - (I - \lambda_n A_1)Tx_n\|
\[\leq \|x_n - y_n\| + \|y_n - (I - \alpha_n \mu A_2)T_n y_n\| + \|T(y_n) - T x_n\|\]
\begin{align*}
&\leq \|x_n - y_n\| + \|y_n - (I - \alpha_n \mu A_2) T_n y_n\| + L \|y_n - x_n\|
= &\quad (L + 1)\|x_n - y_n\| + \|y_n - (I - \alpha_n \mu A_2) T_n y_n\|
= &\quad \beta_n (L + 1)\|x_n - (I - \alpha_n \mu A_2) T_n x_n\| + \|y_n - (I - \alpha_n \mu A_2) T_n y_n\|,
\end{align*}
that is,
\|x_n - (I - \alpha_n \mu A_2) T_n x_n\| \leq \frac{1}{1 - \beta_n (L + 1)} \|y_n - (I - \alpha_n \mu A_2) T_n y_n\| \to 0.

We note that
\begin{align*}
\|T x_n - (I - \alpha_n \mu A_2) T_n x_n\| &= \|T x_n - T_n x_n + \alpha_n \mu A_2 (T_n x_n)\|
\leq &\quad \|T x_n - (I - \lambda_n A_1) T x_n\| + \alpha_n \mu \|A_2 (T_n x_n)\|
= &\quad \lambda_n \|A_1 (T x_n)\| + \alpha_n \mu \|A_2 (T_n x_n)\| \to 0.
\end{align*}
Consequently,
\|x_n - T x_n\| \leq \|x_n - (I - \alpha_n \mu A_2) T_n x_n\| + \|T x_n - (I - \alpha_n \mu A_2) T_n x_n\| \to 0. \ (6.2.10)

Relation (6.2.10) and Lemma 2.1.14 guarantee that every weak limit point of the sequence \{x_n\} is a fixed point of \(T\), that is, \(\omega_w(x_n) \subseteq \text{Fix}(T)\). This fact, the inequality (6.2.8) and Lemma 2.1.5 ensure the strong convergence of \{x_n\} to \(P_{\text{Fix}(T)}(x_0)\). Since \(\|x_n - y\| = \|\beta_n (I - (I - \alpha_n \mu A_2) T_n) x_n\| \to 0\), it immediately follows that the sequence \{y_n\} converges strongly to \(P_{\text{Fix}(T)}(x_0)\).

Finally, we prove that whenever \(T\) is nonexpansive and \(A_1\) is injective and \((\|x_n - y_n\| + \alpha_n)/\lambda_n \to 0\) as \(n \to \infty\), \(P_{\text{Fix}(T)}(x_0)\) is the unique solution of Problem 6.1.1.

Indeed, put \(\delta := P_{\text{Fix}(T)}(x_0)\). By condition (B4), we can take an arbitrarily fixed element \(y \in \text{VI}(\text{Fix}(T), A_1)\) and put \(M := \sup \{\|x_n - y\| + \|y_n - y\| : n \geq 1\} < \infty\).

Then, from the condition (B3) and Lemmas 2.1.11 and 2.1.14, it follows that for all \(n \geq 1\),
\begin{align*}
&\|(I - \alpha_n \mu A_2) T_n x_n - y\|^2
= &\quad \|(I - \alpha_n \mu A_2) T_n x_n - (I - \alpha_n \mu A_2) T_n y + (I - \alpha_n \mu A_2) T_n y - y\|^2
\end{align*}
\[ \begin{align*}
&\leq \| (I - \alpha_n A_2)T_n x_n - (I - \alpha_n A_2)T_n y \|^2 \\
&\quad + 2\langle (I - \alpha_n A_2)T_n x_n - y, (I - \alpha_n A_2)T_n y - y \rangle \\
&\leq (1 - \alpha_n \tau)^2 \| T_n x_n - T_n y \|^2 \\
&\quad + 2\langle y - (I - \alpha_n A_2)T_n x_n, \lambda_n A_1(y) + \alpha_n \mu A_2(T_n y) \rangle \\
&\leq \| T_n x_n - T_n y \|^2 + 2\langle y - (I - \alpha_n A_2)T_n x_n, \lambda_n A_1(y) + \alpha_n \mu A_2(T_n y) \rangle \\
&\leq \| x_n - y \|^2 + \| T_n x_n - T_n y \|^2 + 2\langle y - (I - \alpha_n A_1)T_n x_n, \lambda_n A_1(y) + \alpha_n \mu A_2(T_n y) \rangle \\
&\quad + 2\alpha_n \mu (A_2(T_n x_n), \lambda_n A_1(y) + \alpha_n \mu A_2(T_n y)) \\
&\leq \| x_n - y \|^2 + 2\| y - (I - \alpha_n A_1)T_n x_n, \lambda_n A_1(y) + \alpha_n \mu A_2(T_n y) \| \\
&\quad + 2\alpha_n \mu \| A_2(T_n x_n) \| \| \lambda_n A_1(y) + \alpha_n \mu A_2(T_n y) \|
\end{align*} \]

and hence,

\[ \begin{align*}
\| y_n - y \|^2 &= \| (1 - \beta_n)(x_n - y) + \beta_n(I - \alpha_n A_2)T_n x_n - y \|^2 \\
&\leq (1 - \beta_n)\| x_n - y \|^2 + \beta_n\| (I - \alpha_n A_2)T_n x_n - y \|^2 \\
&\leq (1 - \beta_n)\| x_n - y \|^2 + \beta_n\| x_n - y \|^2 \\
&\quad + 2\langle (y - T_n x_n, \lambda_n A_1(y) + \alpha_n \mu A_2(T_n y)) \\
&\quad + \lambda_n \| A_1(T_n x_n) \| + \alpha_n \mu \| A_2(T_n x_n) \| \| \lambda_n A_1(y) + \alpha_n \mu A_2(T_n y) \|, \n\end{align*} \]
\[ = \|x_n - y\|^2 + 2\beta_n \langle y - Tx_n, \lambda_n A_1(y) + \alpha_n \mu A_2(T_n y) \rangle \\
+ (\lambda_n \|A_1(Tx_n)\| + \alpha_n \mu \|A_2(T_n x_n)\|) \|\lambda_n A_1(y) + \alpha_n \mu A_2(T_n y)\| \]

This implies that

\[
0 \leq \frac{1}{\lambda_n} \left( \|x_n - y\|^2 - \|y_n - y\|^2 + 2\beta_n \langle y - Tx_n, \lambda_n A_1(y) + \alpha_n \mu A_2(T_n y) \rangle \right) \\
+ (\lambda_n \|A_1(Tx_n)\| + \alpha_n \mu \|A_2(T_n x_n)\|) \|\lambda_n A_1(y) + \alpha_n \mu A_2(T_n y)\| \]
\[
= \left( \|x_n - y\| + \|y_n - y\| \right) \|x_n - y\| - \|y_n - y\| + \frac{2\beta_n}{\lambda_n} \langle y - Tx_n, A_1 y \rangle \]
\[
+ \mu \frac{\alpha_n}{\lambda_n} A_2(T_n y) + (\|A_1(Tx_n)\| + \mu \frac{\alpha_n}{\lambda_n} \|A_2(T_n x_n)\|) \|\lambda_n A_1(y) + \alpha_n \mu A_2(T_n y)\| \]
\[
\leq \frac{M}{\lambda_n} \|x_n - y_n\| + 2\beta_n \langle y - Tx_n, A_1(y) + \frac{\alpha_n}{\lambda_n} A_2(T_n y) \rangle \\
+ (\|A_1(Tx_n)\| + \mu \frac{\alpha_n}{\lambda_n} \|A_2(T_n x_n)\|) \|\lambda_n A_1(y) + \alpha_n \mu A_2(T_n y)\|, \quad (6.2.11)
\]

that is,

\[
0 \leq \frac{M}{2\beta_n} \|x_n - y_n\| + \left( \|y - T x_n, A_1(y) + \frac{\alpha_n}{\lambda_n} A_2(T_n y) \| \right) \\
+ \left( \|A_1(Tx_n)\| + \mu \frac{\alpha_n}{\lambda_n} \|A_2(T_n x_n)\| \right) \|\lambda_n A_1(y) + \alpha_n \mu A_2(T_n y)\|. \quad (6.2.11)
\]

Since \( T \) is nonexpansive, it is known that \( L = 1 \) and \( \{\beta_n\} \subset [a, b] \) for some \( a, b \in (0, \frac{1}{2}) \).

In terms of the conditions that \( \alpha_n \to 0, \lambda_n \to 0 \) and \( (\|x_n - y_n\| + \alpha_n) / \lambda_n \to 0 \), we deduce from (6.2.11) and \( x_n \to \hat{x} \) (\( = P_{\text{Fix}(T)}(x_0) \)) that

\[ \langle y - \hat{x}, A_1 y \rangle \geq 0, \quad \forall y \in \text{Fix}(T). \]

The condition (B1) ensures

\[ \langle y - \hat{x}, A_1 \hat{x} \rangle \geq 0, \quad \forall y \in \text{Fix}(T), \]

that is, \( \hat{x} \in VIP(\text{Fix}(T), A_1) \). Furthermore, from the conditions (A2) and (A4), we conclude that Problem 6.1.1 has a unique solution. Hence, \( VIP \ (VIP(\text{Fix}(T), A_1), A_2) \) is a
singleton. Thus we may assume that \( VIP(VIP(Fix(T), A_1), A_2) = \{ x^* \} \). This implies that \( x^* \in VIP(Fix(T), A_1) \).

Now we show that \( \hat{x} = x^* \). Indeed, since \( \hat{x}, x^* \in VIP(Fix(T), A_1) \), we have

\[
\langle A_1 \hat{x}, y - \hat{x} \rangle \geq 0, \quad \forall y \in Fix(T), \tag{6.2.12}
\]

and

\[
\langle A_1 x^*, y - x^* \rangle \geq 0, \quad \forall y \in Fix(T). \tag{6.2.13}
\]

Setting \( y = x^* \) in inequality (6.2.12) and \( y = \hat{x} \) in inequality (6.2.13), and then adding the resultant inequalities, we obtain

\[
\langle A_1 \hat{x} - A_1 x^*, \hat{x} - x^* \rangle \leq 0.
\]

Since \( A_1 \) is \( \alpha \)-inverse-strongly monotone, we have

\[
\alpha \| A_1 \hat{x} - A_1 x^* \|^2 \leq \langle A_1 \hat{x} - A_1 x^*, \hat{x} - x^* \rangle \leq 0.
\]

Consequently, \( A_1 \hat{x} = A_1 x^* \). Since \( A_1 \) is injective, we have \( \hat{x} = x^* \). \( \square \)

### 6.3 Multi-Step Explicit and Implicit Hybrid Extragradient-Like Methods

We considered the following THVIP where the HVIP is defined over the intersection of fixed point set of a nonexpansive mapping and fixed point set of strictly pseudo-contractive mapping.

**Problem 6.3.1.** Let \( C \) a nonempty closed convex subset of a real Hilbert space \( H \) and \( F : C \to H \) be \( \kappa \)-Lipschitz continuous and \( \eta \)-strongly monotone, where \( \kappa > 0 \) and \( \eta > 0 \) are constants. Let \( V : C \to H \) be \( \rho \)-contraction with coefficient \( \rho \in [0, 1) \), \( S, T_1 : C \to C \) be nonexpansive mappings, and \( T_2 : C \to C \) be \( \zeta \)-strictly pseudo-contractive
mapping with $\text{Fix}(T_1) \cap \text{Fix}(T_2) \neq \emptyset$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \leq \tau$, where \( \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \). The problem is to find $x^* \in \Xi$ such that
\[
\langle (\mu F - \gamma V)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Xi, \tag{6.3.1}
\]
where $\Xi$ denotes the solution set of the following hierarchical variational inequality problem (HVIP) of finding $x^* \in \text{Fix}(T_1) \cap \text{Fix}(T_2)$ such that
\[
\langle (\mu F - \gamma S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(T_1) \cap \text{Fix}(T_2). \tag{6.3.2}
\]

Whenever $T_1 \equiv T$ is a nonexpansive mapping and $T_2 \equiv I$ is an identity mapping, then Problem 6.3.1 reduce to Problem 6.1.1.

We also consider the following triple hierarchical variational inequality problem:

**Problem 6.3.2.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $F : C \to H$ be $\kappa$-Lipschitz continuous and $\eta$-strongly monotone, where $\kappa > 0$ and $\eta > 0$ are constants. Let $A : C \to H$ be a monotone and $L$-Lipschitz continuous mapping, $V : C \to H$ be $\rho$-contraction with coefficient $\rho \in [0, 1)$, $S, T : C \to C$ be nonexpansive mappings with $\text{Fix}(T) \cap \Theta \neq \emptyset$, where $\Theta = \text{VIP}(C, A)$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. The problem is to find $x^* \in \Xi$ such that
\[
\langle (\mu F - \gamma V)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Xi, \tag{6.3.3}
\]
where $\Xi$ denotes the solution set of the following hierarchical variational inequality problem (HVIP) of finding $x^* \in \text{Fix}(T) \cap \Theta$ such that
\[
\langle (\mu F - \gamma S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(T) \cap \Theta, \tag{6.3.4}
\]

We remark that Problem 6.3.2 is a generalization of Problem 6.3.1. Indeed, in Problem 6.3.2, if we put $T = T_1$ and $A = I - T_2$, where $T_1 : C \to C$ is a nonexpansive mapping and $T_2 : C \to C$ is a $\zeta$-strictly pseudo-contractive mapping. Then, from the definition of strictly pseudo-contractive mapping, we have
\[
\langle T_2x - T_2y, x - y \rangle \leq \|x - y\|^2 - \frac{1 - \zeta}{2} \|T_2x - (I - T_2)y\|^2, \quad \forall x, y \in C.
\]
It is clear that the mapping \( A = I - T_2 \) is \( \frac{1 - \lambda_2}{2} \)-inverse strongly monotone. Letting \( L = \frac{2}{1 - \lambda_2} \), then \( A : C \to H \) is monotone and \( L \)-Lipschitz continuous. In this case, \( \Theta = \text{Fix}(T_2) \). Therefore, Problem 6.3.2 reduces to Problem 6.3.1.

Motivated and inspired by Korpelevich’s extragradient method [217], the iterative method proposed in [30] and multi-step hybrid extragradient method by Kong et al. [114], we propose the following multi-step explicit and implicit hybrid extragradient-like methods for solving Problem 6.3.2.

**Algorithm 6.3.1.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and \( F : C \to H \) be \( \kappa \)-Lipschitz continuous and \( \eta \)-strongly monotone \( A : C \to H \) be monotone and \( L \)-Lipschitz continuous, \( V : C \to H \) be \( \rho \)-contraction with coefficient \( \rho \in [0, 1) \) and \( S, T : C \to C \) be nonexpansive mappings. Suppose that \( \{\alpha_n\} \subset [0, \infty) \), \( \{\beta_n\} \subset [0, 1] \) and \( \{\lambda_n\}, \{\delta_n\} \subset (0, 1) \). Let \( 0 < \mu < 2\eta/\kappa^2 \), \( 0 < \gamma \leq \tau \) and \( A_n = \alpha_n I + A \) for all \( n \geq 0 \), where \( \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \). The sequence \( \{x_n\} \) is generated by the following iterative scheme:

\[
\begin{align*}
x_0 & \in C \quad \text{chosen arbitrarily}, \\
y_n &= (1 - \beta_n)x_n + \beta_n P_C(x_n - \lambda A_n(x_n)), \\
z_n &= \gamma_n x_n + (1 - \gamma_n) T_P(y_n - \lambda A_n(y_n)), \\
x_{n+1} &= P_C[\lambda_n \gamma (\delta_n V(x_n) + (1 - \delta_n) S(x_n)) + (I - \lambda_n \mu F)(T(z_n))], \quad \forall n \geq 0.
\end{align*}
\]

(6.3.5)

In particular, if \( V \equiv 0 \), then (6.3.5) reduces to the following iterative scheme:

\[
\begin{align*}
x_0 & \in C \quad \text{chosen arbitrarily}, \\
y_n &= (1 - \beta_n)x_n + \beta_n P_C(x_n - \lambda A_n(x_n)), \\
z_n &= \gamma_n x_n + (1 - \gamma_n) T_P(y_n - \lambda A_n(y_n)), \\
x_{n+1} &= P_C[\lambda_n (1 - \delta_n) S(x_n) + (I - \lambda_n \mu F)(T(z_n))], \quad \forall n \geq 0.
\end{align*}
\]

(6.3.6)

If \( S \equiv V \), then (6.3.5) reduces to the following iterative scheme:
\[
\begin{align*}
  x_0 &= x \in C \quad \text{chosen arbitrarily,} \\
  y_n &= (1 - \beta_n)x_n + \beta_n P_C(x_n - \lambda A_n(x_n)), \\
  z_n &= \gamma_n x_n + (1 - \gamma_n)T(P_C(y_n - \lambda A_n(y_n))), \\
  x_{n+1} &= P_C[\lambda_n \gamma V(x_n) + (I - \lambda_n \mu F)(T(z_n))], \quad \forall n \geq 0.
\end{align*}
\] (6.3.7)

Moreover, if \( S \equiv V \equiv 0 \), then (6.3.5) reduces to the following iterative scheme:

\[
\begin{align*}
  x_0 &= x \in C \quad \text{chosen arbitrarily,} \\
  y_n &= (1 - \beta_n)x_n + \beta_n P_C(x_n - \lambda A_n(x_n)), \\
  z_n &= \gamma_n x_n + (1 - \gamma_n)T(P_C(y_n - \lambda A_n(y_n))), \\
  x_{n+1} &= P_C[(I - \lambda_n \mu F)(T(z_n))], \quad \forall n \geq 0.
\end{align*}
\] (6.3.8)

**Algorithm 6.3.2.** Let \( F : C \to H \) be \( \kappa \)-Lipschitzian and \( \eta \)-strongly monotone on the nonempty, closed and convex subset \( C \) of \( H \), \( A : C \to H \) be a monotone and \( L \)-Lipschitzian mapping, \( V : C \to H \) be a \( \rho \)-contraction with coefficient \( \rho \in [0, 1) \) and \( S, T : C \to C \) be nonexpansive mappings. Suppose that \( \{\alpha_n\} \subset [0, \infty) \), \( \{\beta_n\}, \{\gamma_n\} \subset [0, 1] \) and \( \{\lambda_n\}, \{\delta_n\} \subset (0, 1) \). Let \( 0 < \mu < 2\eta^2/\kappa^2 \), \( 0 < \gamma \leq \tau \) and \( A_n = \alpha_n I + A \) for all \( n \geq 0 \), where \( \tau = 1 - \sqrt{1 - \mu(2\eta - \mu \kappa^2)} \). Then, the sequence \( \{x_n\} \) is generated by the following iterative scheme:

\[
\begin{align*}
  x_0 &= x \in C \quad \text{chosen arbitrarily,} \\
  y_n &= (1 - \beta_n)x_n + \beta_n P_C(y_n - \lambda A_n y_n), \\
  z_n &= \gamma_n y_n + (1 - \gamma_n)TP_C(z_n - \lambda A_n z_n), \\
  x_{n+1} &= P_C[\lambda_n \gamma (\delta_n V x_n + (1 - \delta_n) S x_n) + (I - \lambda_n \mu F)T z_n], \quad \forall n \geq 0.
\end{align*}
\] (6.3.9)

In particular, if \( V \equiv 0 \), then (6.3.9) reduces to the following iterative scheme:

\[
\begin{align*}
  x_0 &= x \in C \quad \text{chosen arbitrarily,} \\
  y_n &= (1 - \beta_n)x_n + \beta_n P_C(y_n - \lambda A_n y_n), \\
  z_n &= \gamma_n y_n + (1 - \gamma_n)TP_C(z_n - \lambda A_n z_n), \\
  x_{n+1} &= P_C[\lambda_n (1 - \delta_n) S x_n + (I - \lambda_n \mu F)T z_n], \quad \forall n \geq 0.
\end{align*}
\] (6.3.10)
Further, if $S \equiv V$, then (6.3.9) reduces to the following iterative scheme:

$$
\begin{cases}
x_0 = x \in C \quad \text{chosen arbitrarily,} \\
y_n = (1 - \beta_n)x_n + \beta_n P_C(y_n - \lambda A_n y_n), \\
z_n = \gamma_n y_n + (1 - \gamma_n)TP_C(z_n - \lambda A_n z_n), \\
x_{n+1} = P_C[\lambda_n \gamma V x_n + (I - \lambda_n \mu F)Tz_n], \\
\forall n \geq 0.
\end{cases}
$$

Moreover, if $S \equiv V \equiv 0$, then equation (6.3.9) reduces to the following iterative scheme:

$$
\begin{cases}
x_0 = x \in C \quad \text{chosen arbitrarily,} \\
y_n = (1 - \beta_n)x_n + \beta_n P_C(y_n - \lambda A_n y_n), \\
z_n = \gamma_n y_n + (1 - \gamma_n)TP_C(z_n - \lambda A_n z_n), \\
x_{n+1} = P_C[(I - \lambda_n \mu F)Tz_n], \\
\forall n \geq 0.
\end{cases}
$$

Our problems and algorithms improve and extend those given in [30] in the following aspects:

(a) Problem 6.3.2 generalizes Problem 6.3.1 from the fixed point set $\text{Fix}(T)$ of a nonexpansive mapping $T$ to the intersection $\text{Fix}(T) \cap \Theta$ of the fixed point set of a nonexpansive mapping $T$ and the solution set $\Theta$ of classical VIP.

(b) The Korpelevich extragradient algorithm is extended to develop the multi-step explicit and implicit hybrid extragradient-like algorithms (that is, Algorithms 6.3.1 and 6.3.2) for solving Problem 6.3.2 by virtue of the iterative schemes in Theorem 4.1 in [30].

(c) The strong convergence of the sequences generated by Algorithms 6.3.1 and 6.3.2 hold under the lack of the same restrictions as those in Theorem 4.1 of [30].

(d) The boundedness requirement of the sequence $\{x_n\}$ in Theorem 4.1 in [30] is replaced by the boundedness requirement of the sequence $\{Sx_n\}$.

We present the convergence analysis of Algorithm 6.3.1 for solving Problem 6.3.2.
Theorem 6.3.1. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, $F : C \to H$ be $k$-Lipschitz and $\eta$-strongly monotone with constants $k$, $\eta > 0$, $A : C \to H$ be $1/L$-inverse strongly monotone, $V : G \to H$ be $\rho$-contraction with coefficient $\rho \in [0, 1)$ and $S, T : C \to C$ be nonexpansive mappings. Let $0 < \lambda < 2/L$, $0 < \mu < 2\eta/Lk^2$ and $0 < \gamma < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Assume that the solution set $\Xi$ of HVIP (6.3.4) is nonempty and the sequences $\{\alpha_n\} \subset [0, \infty)$, $\{\gamma_n\}$, $\{\gamma_n\} \subset [0, 1)$ and $\{\lambda_n\}$, $\{\delta_n\} \subset (0, 1)$ satisfy the following condition.

(C1) $\sum_{n=0}^{\infty} \alpha_n < \infty; \quad (C2) 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \quad (C3) 0 < \liminf_{n \to \infty} \delta_n \leq \limsup_{n \to \infty} \delta_n < 1; \quad (C4) \lim_{n \to \infty} \lambda_n = 0$, $\lim_{n \to \infty} \delta_n = 0$ and $\sum_{n=0}^{\infty} \delta_n \lambda_n = \infty; \quad (C5) \text{there are constants } \overline{k}, \theta > 0 \text{ such that}$

$$\|x - Tx\| \geq \overline{k}[d(x, Fix(T) \cap \Theta)]^\theta, \quad \forall x \in C;$$

(C6) $\lim_{n \to \infty} \frac{\lambda_n^{1/\theta}}{\delta_n} = 0.$

Then, the following assertions hold.

(a) If $\{x_n\}$ is the sequence generated by the scheme (6.3.5) and $\{S x_n\}$ is bounded, then $\{x_n\}$ converges strongly to a point $x^* \in Fix(T) \cap \Theta$ which is a unique solution of Problem 6.3.2 provided $\|x_{n+1} - x_n\| + \|x_n - z_n\| = o(\lambda_n)$.

(b) If $\{x_n\}$ is a sequence generated by the scheme (6.3.6) and $\{S(x_n)\}$ is bounded, then $\{x_n\}$ converges strongly to a unique solution $x^*$ of the following VIP provided $\|x_{n+1} - x_n\| + \|x_n - z_n\| = o(\lambda_n)$:

$$\text{find } x^* \in \Xi \text{ such that } \langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in \Xi. \quad (6.3.13)$$
Proof. We treat only case (a), that is, the convergence of the sequence \( \{x_n\} \) is generated by the scheme (6.3.5). Obviously, from the condition \( \Xi \neq \emptyset \), it follows that \( \text{Fix}(T) \cap \Theta \neq \emptyset \). In addition, in terms of conditions (C2) and (C3), without loss of generality, we may assume that \( \{\beta_n\} \subset [a, b] \) for some \( a, b \in (0, 1) \) and \( \{\gamma_n\} \subset [c, d] \) for some \( c, d \in (0, 1) \).

First of all, let us show that \( P_C(I - \lambda(\alpha I + A)) \) is \( \xi \)-averaged for each \( \lambda \in \left(0, \frac{2}{\alpha + L}\right) \), where \( \xi = \frac{2 + \lambda(\alpha + L)}{4} \). Indeed, since \( A \) is \( \frac{1}{L} \)-ism, that is, \( \langle Ax - Ay, x - y \rangle \geq \frac{1}{L} \|Ax - Ay\|^2 \).

Observe that

\[
(\alpha + L)\langle (\alpha I + A)x - (\alpha I + A)y, x - y \rangle \\
= (\alpha + L)[\alpha \|x - y\|^2 + \langle Ax - Ay, x - y \rangle] \\
= \alpha^2\|x - y\|^2 + \alpha\langle Ax - Ay, x - y \rangle + \alpha L\|x - y\|^2 + L\langle Ax - Ay, x - y \rangle \\
\geq \alpha^2\|x - y\|^2 + 2\alpha\langle Ax - Ay, x - y \rangle + \|Ax - Ay\|^2 \\
= \|\alpha(x - y) + Ax - Ay\|^2 \\
= \|(\alpha I + A)x - (\alpha I + A)y\|^2.
\]

Hence, it follows that \( \alpha I + A \) is \( \frac{1}{\alpha + L} \)-ism. Thus, \( \alpha I + A \) is \( \frac{1}{\lambda(\alpha + L)} \)-ism according to Proposition 2.1.2 (ii). By Proposition 2.1.2 (iii), the complement \( I - (\alpha I + A) \) is \( \frac{\lambda(\alpha + L)}{2} \)-averaged.

Therefore, noting that \( P_C \) is \( \frac{1}{2} \)-averaged and utilizing Proposition 2.1.3 (d), we know that for each \( \lambda \in \left(0, \frac{2}{\alpha + L}\right) \), \( P_C(I - \lambda(\alpha I + A)) \) is \( \xi \)-averaged with

\[
\xi = \frac{1}{2} + \frac{\lambda(\alpha + L)}{2} - \frac{1}{2} \cdot \frac{\lambda(\alpha + L)}{2} = \frac{2 + \lambda(\alpha + L)}{4} \in (0, 1).
\]

This shows that \( P_C(I - \lambda(\alpha I + A)) \) is nonexpansive. Furthermore, for \( \lambda \in \left(0, \frac{2}{\alpha + L}\right) \), utilizing the fact that
\[
\lim_{\alpha \to \infty} \frac{\lambda(\alpha + L)}{\alpha + L} = \frac{2}{L}
\]
we may assume that

\[
0 < \lambda < \frac{2}{\alpha_n + L}, \quad \forall n \geq 0.
\]

Consequently, it follows that for each integer \( n \geq 0 \), \( P_C(I - \lambda A_n) \) is \( \xi_n \)-averaged with

\[
\xi_n = \frac{1}{2} + \frac{\lambda(\alpha_n + L)}{2} - \frac{1}{2} \cdot \frac{\lambda(\alpha_n + L)}{2} = \frac{2 + \lambda(\alpha_n + L)}{4} \in (0, 1).
\]
\[ \|d\|^u + \|d - u\| \leq \\
\|d\|^u + \|d - u\| (\|u\| - 1) + \|d - u\| \leq \\
[\|d\|^u + \|d - u\|] (\|u\| - 1) + \|d - u\| \leq \\
[\|d\|^u + \|d - u\|] (\|u\| - 1) + \|d - u\| \leq \\
[\|d(\forall\forall - I) - d(\forall\forall - I)] (\|u\| - 1) + \|d - u\| \leq \\
\|d(\forall\forall - I)\|_2 - d(\forall\forall - I)\|_2 + \\
\|d(\forall\forall - I)\|_2 - u_d(\forall\forall - I)\|_2 (\|u\| - 1) + \|d - u\| \leq \\
\|d(\forall\forall - I)\|_2 - u_d(\forall\forall - I)\|_2 (\|u\| - 1) + \|d - u\| \leq \\
\|d - u_d(\forall\forall - I)\|_2 (\|u\| - 1) + \|d - u\| \leq \|d - u\| \]

and

\[ \|d\|^u + \|d - u\| = \\
\|d\|^u + \|d - u\| (\|u\| - 1) = \\
\|d(\forall\forall - I) - d(\forall\forall - I)\|_2 \|y\| + \|d - u\| (\|u\| - 1) \leq \\
\|d(\forall\forall - I)\|_2 - d(\forall\forall - I)\|_2 + \\
\|d(\forall\forall - I)\|_2 - u_d(\forall\forall - I)\|_2 (\|u\| - 1) \leq \\
\|d(\forall\forall - I)\|_2 - u_d(\forall\forall - I)\|_2 (\|u\| - 1) \leq \|d - u\| \]

Hence, we have

\[ \left( \begin{array}{c} 7 \\ 0 \end{array} \right) \in \mathcal{E} \quad \text{for } x = d = d(\forall\forall - I)\|_2 \quad \text{and} \quad d = d(\forall\forall - I)\|_2 \]

Indeed, take a fixed \( P \in \mathcal{E}(\forall\forall - I) \). Then, we get

Step 1. \( \mathcal{E} \) is bounded

Now we divide the remainder of the proof into several steps.

This immediately implies that \( P \|_{\forall\forall - I} \) is nonexpansive for all \( n \geq 0 \).
Indeed, observe that

\[ u_{\partial}(u_{\partial} \gamma - I) = u_{\partial} \gamma - I \text{ where } 0 = \| u_{\partial} \gamma - u_{\partial} \|_{0} = \| u_{\partial} - u_{\partial} \|_{0} = \| u_{\partial} - u_{\partial} \|_{0} \]

Step 2: \{u_{n}\}

and \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n}\} \{u_{n\})
\[ \|y_n - p\|^2 \]
\[ = \| (1 - \beta_n) (x_n - p) + \beta_n (u_n - p) \|^2 \]
\[ = (1 - \beta_n) \| x_n - p \|^2 + \beta_n \| u_n - p \|^2 - \beta_n (1 - \beta_n) \| x_n - u_n \|^2 \]
\[ \leq (1 - \beta_n) \| x_n - p \|^2 + \beta_n \| x_n - p \|^2 + \lambda \alpha_n \| p \|^2 - \beta_n (1 - \beta_n) \| x_n - u_n \|^2 \]
\[ = (1 - \beta_n) \| x_n - p \|^2 + \beta_n \| x_n - p \|^2 + \alpha_n \beta_n (2\lambda \| p \| \| x_n - p \| + \alpha_n \lambda^2 \| p \|^2) \]
\[ - \beta_n (1 - \beta_n) \| x_n - u_n \|^2 \]
\[ \leq (1 - \beta_n) \| x_n - p \|^2 + \beta_n \| x_n - p \|^2 + \alpha_n M_1 - \beta_n (1 - \beta_n) \| x_n - u_n \|^2 \]
\[ = \| x_n - p \|^2 + \alpha_n M_1 - \beta_n (1 - \beta_n) \| x_n - u_n \|^2, \tag{6.3.17} \]

where \( M_1 = \sup_{n \geq 0} \{ \beta_n (2\lambda \| p \| \| x_n - p \| + \alpha_n \lambda^2 \| p \|^2) \} < \infty \). Also, observe that

\[ \| x_n - p \|^2 \]
\[ = \| \gamma_n (x_n - p) + (1 - \gamma_n) (Tv_n - p) \|^2 \]
\[ = \gamma_n \| x_n - p \|^2 + (1 - \gamma_n) \| Tv_n - p \|^2 - \gamma_n (1 - \gamma_n) \| x_n - T v_n \|^2 \]
\[ \leq \gamma_n \| x_n - p \|^2 + (1 - \gamma_n) \| u_n - p \|^2 - \gamma_n (1 - \gamma_n) \| x_n - T v_n \|^2 \]
\[ \leq \gamma_n \| x_n - p \|^2 + (1 - \gamma_n) \| y_n - p \|^2 + \lambda \alpha_n \| p \|^2 - \gamma_n (1 - \gamma_n) \| x_n - T v_n \|^2 \]
\[ = \gamma_n \| x_n - p \|^2 + (1 - \gamma_n) \| y_n - p \|^2 + \alpha_n (2\lambda \| p \| \| y_n - p \| + \alpha_n \lambda^2 \| p \|^2) \]
\[ - \gamma_n (1 - \gamma_n) \| x_n - T v_n \|^2 \]
\[ \leq \gamma_n \| x_n - p \|^2 + (1 - \gamma_n) \| y_n - p \|^2 + \alpha_n M_2 - \gamma_n (1 - \gamma_n) \| x_n - T v_n \|^2 \]
\[ \leq \gamma_n \| x_n - p \|^2 + (1 - \gamma_n) \| x_n - p \|^2 + \alpha_n M_1 - \beta_n (1 - \beta_n) \| x_n - u_n \|^2 \]
\[ + \alpha_n M_2 - \gamma_n (1 - \gamma_n) \| x_n - T v_n \|^2 \]
\[ \leq \| x_n - p \|^2 + \alpha_n M_1 - (1 - \gamma_n) \beta_n (1 - \beta_n) \| x_n - u_n \|^2 \]
\[ + \alpha_n M_2 - \gamma_n (1 - \gamma_n) \| x_n - T v_n \|^2 \]
\[ = \| x_n - p \|^2 + \alpha_n (M_1 + M_2) - (1 - \gamma_n) \beta_n (1 - \beta_n) \| x_n - u_n \|^2 \]
\[ - \gamma_n (1 - \gamma_n) \| x_n - T v_n \|^2, \tag{6.3.18} \]
where $M_2 = \sup_{n \geq 0} \{(1 - \gamma_n)(2\lambda\|p\|\|y_n - p\| + \alpha_n\lambda^2\|p\|^2)\} < \infty$. Hence, it follows from (6.3.18) that

$$
\|x_{n+1} - p\|^2 
\leq \|\lambda_n \gamma_n(V x_n + (1 - \delta_n)S x_n) + (I - \lambda_n \mu F)T z_n - p\|^2
= \|\lambda_n \gamma_n(V x_n + (1 - \delta_n)S x_n) - \lambda_n \mu F T p + (I - \lambda_n \mu F)T z_n - (I - \lambda_n \mu F)T p\|^2
\leq \{\|\lambda_n \gamma_n(V x_n + (1 - \delta_n)S x_n) - \lambda_n \mu F T p\| + \|T z_n - (I - \lambda_n \mu F)T p\|^2\}^2
\leq \{\lambda_n \gamma_n(\gamma V x_n - \mu F p) + (1 - \delta_n)(\gamma S x_n - \mu F p)\| + (1 - \lambda_n \gamma)\|z_n - p\|^2\}^2
\leq \lambda_n \gamma_n \|\|\|\gamma V x_n - \mu F p\| + \|\gamma S x_n - \mu F p\|^2 + (1 - \lambda_n \gamma)\|z_n - p\|^2
\leq \lambda_n \gamma_n \|\gamma V x_n - \mu F p\| + \|\gamma S x_n - \mu F p\|^2 + \|z_n - p\|^2
\leq \lambda_n \gamma_n \|\gamma V x_n - \mu F p\| + \|\gamma S x_n - \mu F p\|^2 + \|z_n - p\|^2 + \alpha_n(M_1 + M_2)
- (1 - \gamma_n)\beta_n(1 - \beta_n)\|x_n - u_n\|^2 - \gamma_n(1 - \gamma_n)\|x_n - T v_n\|^2
\leq \|x_n - p\|^2 + \alpha_n + \lambda_n)M_3 - (1 - \gamma_n)\beta_n(1 - \beta_n)\|x_n - u_n\|^2
- \gamma_n(1 - \gamma_n)\|x_n - T v_n\|^2;
$$

(6.3.19)

where $M_3 = \sup_{n \geq 0} \{\frac{1}{\gamma_n} \|\gamma V x_n - \mu F p\| + \|\gamma S x_n - \mu F p\|^2, M_1 + M_2\}$. This implies immediately that

$$
(1 - \gamma_n)\beta_n(1 - \beta_n)\|x_n - u_n\|^2 + \gamma_n(1 - \gamma_n)\|x_n - T v_n\|^2
\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (\alpha_n + \lambda_n)M_3
= \{\|x_n - p\|^2 - \|x_{n+1} - p\|^2\}(\|x_n - p\| + \|x_{n+1} - p\|) + (\alpha_n + \lambda_n)M_3
\leq \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + (\alpha_n + \lambda_n)M_3.
$$

(6.3.20)

Note that $\lim_{n \to \infty} \alpha_n = 0, 0 < \lim_{n \to \infty} \beta_n \leq \lim\sup_{n \to \infty} \beta_n < 1$ and $0 < \lim\inf_{n \to \infty} \gamma_n \leq \lim\sup_{n \to \infty} \gamma_n < 1$. Hence, taking into account the boundedness of $\{x_n\}$ and $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$, we deduce from (6.3.20) that
\[
\lim_{n \to \infty} \|x_n - u_n\| = \lim_{n \to \infty} \|x_n - T u_n\| = 0. 
\] (6.3.21)

Thus, utilizing (6.3.5) we get
\[
\lim_{n \to \infty} \|y_n - x_n\| = \lim_{n \to \infty} \beta_n \|u_n - x_n\| = 0,
\]
and
\[
\lim_{n \to \infty} \|x_n - x_n\| = \lim_{n \to \infty} (1 - \gamma_n)\|T u_n - x_n\| = 0.
\]

Furthermore, note that
\[
\|T u_n - v_n\| \leq \|T u_n - x_n\| + \|x_n - u_n\| + \|u_n - v_n\|
= \|T u_n - x_n\| + \|x_n - u_n\| + \|P_C(I - \lambda A) x_n - P_C(I - \lambda A) y_n\|
\leq \|T u_n - x_n\| + \|x_n - u_n\| + \|x_n - y_n\|.
\]

This implies that
\[
\lim_{n \to \infty} \|T u_n - v_n\| = \lim_{n \to \infty} \|u_n - v_n\| = 0. 
\] (6.3.22)

Step 3. \(\omega_w(x_n) \subseteq \text{Fix}(T) \cap \Theta\).

Indeed, suppose that \(\xi \in \omega_w(x_n)\) and \(\{x_{n_j}\}\) is a subsequence of \(\{x_n\}\) such that \(x_{n_j} \to \xi\). Set \(T = P_C(I - \lambda A)\). Then for each \(\lambda \in (0, \frac{\lambda}{L})\), \(T\) is nonexpansive. As a matter of fact, since \(A\) is \(\frac{1}{L}\)-ism and \(\lambda A\) is \(\frac{1}{\lambda L}\)-ism. Hence, by Proposition 2.1.2 (iii) the complement \(I - \lambda A\) is \(\frac{L}{2}\)-averaged. Therefore, noting that \(P_C\) is \(\frac{1}{2}\)-averaged and applying Proposition 2.1.3 (iv), we know that for each \(\lambda \in (0, \frac{\lambda}{L})\), \(\overline{T} = P_C(I - \lambda A)\) is \(\alpha\)-averaged, with
\[
\alpha = \frac{1}{2} + \frac{\lambda L}{2} - \frac{1}{2} \cdot \frac{\lambda L}{2} = \frac{2 + \lambda L}{4} \in (0, 1).
\]

Consequently, it is clear that \(\overline{T}\) is nonexpansive.

Now observe that
\[
\|x_n - \overline{T} x_n\| \leq \|x_n - u_n\| + \|u_n - \overline{T} x_n\|
= \|x_n - u_n\| + \|P_C(I - \lambda A) x_n - P_C(I - \lambda A) x_n\|
\leq \|x_n - u_n\| + \|(I - \lambda A) x_n - (I - \lambda A) x_n\|
= \|x_n - u_n\| + \lambda \alpha_n \|x_n\|.
\]
So, from \(\|x_n - u_n\| \to 0\), \(\alpha_n \to 0\) and the boundedness of \(\{x_n\}\) it follows that

\[
\lim_{n \to \infty} \|x_n - \tilde{T}x_n\| = 0. \tag{6.3.23}
\]

Taking into account \(x_{n_j} \to \hat{x}\) and utilizing Lemma 2.1.13, we obtain \(\hat{x} \in \text{Fix}(\tilde{T})\). But \(\text{Fix}(\tilde{T}) = \Theta\); we therefore have \(\hat{x} \in \Theta\). Furthermore, since \(x_{n_j} \to \hat{x}\) and

\[
\lim_{n \to \infty} \|u_n - \hat{x}_n\| = \lim_{n \to \infty} \|u_n - v_n\| = \lim_{n \to \infty} \|Tv_n - v_n\| = 0,
\]

it is known that \(v_{n_j} \to \hat{x}\) and \(\lim_{n \to \infty} \|Tv_{n_j} - v_{n_j}\| = 0\). Thus, from Lemma 2.1.13, we get \(\hat{x} \in \text{Fix}(T)\). Therefore, we have \(\hat{x} \in \text{Fix}(T) \cap \Theta\), and thus,

\[
\omega_w(x_n) \subset \text{Fix}(T) \cap \Theta. \tag{6.3.24}
\]

**Step 4.** \(\omega_w(x_n) \subset \Xi\).

Indeed, we first note that \(0 < \gamma \leq \tau\) and

\[
\mu \eta \geq \tau
\]

\[
\iff \mu \eta \geq 1 - \sqrt{1 - \mu(2\eta - \mu \kappa^2)}
\]

\[
\iff \sqrt{1 - \mu(2\eta - \mu \kappa^2)} \geq 1 - \mu \eta
\]

\[
\iff 1 - 2\mu \eta + \mu^2 \kappa^2 \geq 1 - 2\mu \eta + \mu^2 \eta^2
\]

\[
\iff \kappa^2 \geq \eta^2
\]

\[
\iff \kappa \geq \eta. \tag{6.3.25}
\]

It is clear that

\[
(\mu F - \gamma S)x - (\mu F - \gamma S)y, x - y \geq (\mu \eta - \gamma)\|x - y\|^2, \quad \forall x, y \in C.
\]

Hence, it follows from \(0 < \gamma \leq \tau \leq \mu \eta\) that \(\mu F - \gamma S\) is monotone. Putting

\[
w_n = \lambda_n \gamma(\delta_n V x_n + (1 - \delta_n)Sx_n) + (1 - \lambda_n \mu F)Tz_n, \quad \forall n \geq 0,
\]

and noticing from equation (6.3.5)
\[ x_{n+1} = P_C w_n - w_n + \lambda_n \gamma (\delta_n V x_n + (1 - \delta_n) S x_n) + (I - \lambda_n \mu F) T z_n, \]

we obtain

\[
x_n - x_{n+1} = w_n - P_C w_n + \delta_n \lambda_n (\mu F - \gamma V) x_n + \lambda_n (1 - \delta_n) (\mu F - \gamma S) x_n + (1 - \lambda_n) (I - T) x_n + \lambda_n ((I - \mu F) x_n - (I - \mu F) T x_n) + (I - \lambda_n \mu F) T z_n - (I - \lambda_n \mu F) T z_n.
\]

(6.3.26)

Set \( e_n = \frac{x_n - x_{n+1}}{\lambda_n (1 - \delta_n)} \), \( \forall n \geq 0 \). It can be easily seen from (6.3.26) that

\[
e_n = \frac{w_n - P_C w_n}{\lambda_n (1 - \delta_n)} + (\mu F - \gamma S) x_n + \delta_n \frac{\lambda_n (\mu F - \gamma S) x_n}{1 - \delta_n} + \frac{1 - \lambda_n}{\lambda_n (1 - \delta_n)} (I - T) x_n + \frac{1}{1 - \delta_n} [(I - \mu F) x_n - (I - \mu F) T x_n] + \frac{(I - \lambda_n \mu F) T z_n - (I - \lambda_n \mu F) T z_n}{\lambda_n (1 - \delta_n)}.
\]

This yields that, for all \( w \in \text{Fix}(T) \cap \Theta \) (noticing \( x_n = P_C w_{n-1} \)),

\[
\langle e_n, x_n - w \rangle \\
= \frac{1}{\lambda_n (1 - \delta_n)} (w_n - P_C w_n, P_C w_{n-1} - w) + \langle (\mu F - \gamma S) x_n, x_n - w \rangle + \delta_n \frac{\lambda_n (\mu F - \gamma S) x_n}{1 - \delta_n} + \frac{1 - \lambda_n}{\lambda_n (1 - \delta_n)} (I - T) x_n - (I - T) w, x_n - w \rangle + \frac{1}{1 - \delta_n} [(I - \mu F) x_n - (I - \mu F) T x_n, x_n - w] + \frac{1}{\lambda_n (1 - \delta_n)} [(I - \lambda_n \mu F) T z_n - (I - \lambda_n \mu F) T z_n, x_n - w]
\]

\[
= \frac{1}{\lambda_n (1 - \delta_n)} (w_n - P_C w_n, P_C w_n - w) + \frac{1}{\lambda_n (1 - \delta_n)} (w_n - P_C w_n, P_C w_{n-1} - P_C w_n) + \langle (\mu F - \gamma S) w, x_n - w \rangle + \delta_n \frac{\lambda_n (\mu F - \gamma S) x_n}{1 - \delta_n} + \frac{1 - \lambda_n}{\lambda_n (1 - \delta_n)} (I - T) x_n - (I - T) w, x_n - w \rangle + \frac{\delta_n}{1 - \delta_n} [(\mu F - \gamma V) x_n, x_n - w].
\]
\[
+ \frac{1}{1 - \delta_n} \langle (I - \lambda_n \mu F)x_n - (I - \lambda_n \mu F)Tx_n, x_n - w \rangle \\
+ \frac{1}{\lambda_n(1 - \delta_n)} \langle (I - \lambda_n \mu F)Tx_n - (I - \lambda_n \mu F)Tz_n, x_n - w \rangle.
\]

(6.3.27)

In (6.3.27), the first term is nonnegative due to Propositions 2.2.1 and 2.2.2, and the fourth and fifth terms are also nonnegative due to the monotonicity of \(\mu F - \gamma S\) and \(I - T\). We, therefore, deduce from (6.3.27) that (noticing again \(x_{n+1} = P_G w_n\))

\[
\langle e_n, x_n - w \rangle \\
\geq \frac{1}{\lambda_n(1 - \delta_n)} \langle w_n - P_G w_n, P_G w_{n-1} - P_G w_n \rangle + \langle (\mu F - \gamma S)w, x_n - w \rangle \\
+ \frac{\delta_n}{1 - \delta_n} \langle (\mu F - \gamma V)x_n, x_n - w \rangle + \frac{1}{1 - \delta_n} \langle (I - \mu F)x_n - (I - \mu F)Tx_n, x_n - w \rangle \\
+ \frac{1}{\lambda_n(1 - \delta_n)} \langle (I - \lambda_n \mu F)Tx_n - (I - \lambda_n \mu F)Tz_n, x_n - w \rangle.
\]

(6.3.28)

Note that

\[
\| (I - \lambda_n \mu F)Tx_n - (I - \lambda_n \mu F)Tz_n \| \leq (1 - \lambda_n \tau) \| x_n - z_n \|,
\]

and

\[
\| (I - \mu F)x_n - (I - \mu F)Tx_n \| \leq (1 + \mu \kappa) \| x_n - Tx_n \|.
\]

Hence it follows from \(\| x_n - z_n \| = o(\lambda_n)\) and \(\| x_n - Tx_n \| \to 0\) that

\[
\| (I - \lambda_n \mu F)Tx_n - (I - \lambda_n \mu F)Tz_n \| / \lambda_n \to 0 \quad \text{and} \quad \| (I - \mu F)x_n - (I - \mu F)Tx_n \| \to 0,
\]

respectively. Also, since \(e_n \to 0\) (due to \(\| x_{n+1} - x_n \| = o(\lambda_n)\)), \(\delta_n \to 0\) and \(\{x_n\}\) is bounded by Step 1 which implies that \(\{w_n\}\) is bounded, we obtain from (6.3.28) that

\[
\limsup_{n \to \infty} \langle (\mu F - \gamma S)w, x_n - w \rangle \leq 0, \quad \forall w \in \text{Fix}(T) \cap \Theta.
\]

(6.3.29)
This suffices to guarantee that $\omega_w(x_n) \subset \Xi$; namely, every weak limit point of $\{x_n\}$ solves the HVIP (6.3.4). As a matter of fact, if $x_n \rightharpoonup \bar{x} \in \omega_w(x_n)$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$, then we deduce from (6.3.29) that

$$\langle (\mu F - \gamma S)w, \bar{x} - w \rangle \leq \limsup_{n \to \infty} \langle (\mu F - \gamma S)w, x_n - w \rangle \leq 0, \quad \forall w \in \Fix(T) \cap \Theta,$$

that is,

$$\langle (\mu F - \gamma S)w, w - \bar{x} \rangle \geq 0, \quad \forall w \in \Fix(T) \cap \Theta.$$

In addition, note that $\omega_w(x_n) \subset \Fix(T) \cap \Theta$ by Step 3. Since $\mu F - \gamma S$ is monotone and Lipschitz continuous, and $\Fix(T) \cap \Theta$ is nonempty, closed and convex, by the Minty lemma the last inequality is equivalent to the inequality (6.3.4). Thus, we get $\bar{x} \in \Xi$.

**Step 5.** $\{x_n\}$ converges strongly to a unique solution $x^*$ of Problem 6.3.2.

Indeed, we now take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ satisfying

$$\limsup_{n \to \infty} \langle (\mu F - \gamma V)x^*, x_n - x^* \rangle = \lim_{i \to \infty} \langle (\mu F - \gamma V)x^*, x_{n_i} - x^* \rangle.$$

Without loss of generality, we may further assume that $x_{n_i} \rightharpoonup \bar{x}$, then $\bar{x} \in \Xi$ as we just proved. Since $x^*$ is a solution of the THVIP (6.3.3), we get

$$\limsup_{n \to \infty} \langle (\mu F - \gamma V)x^*, x_n - x^* \rangle = \langle (\mu F - \gamma V)x^*, \bar{x} - x^* \rangle \geq 0. \quad (6.3.30)$$

From (6.3.5) and (6.3.18), it follows that (noticing that $x_{n+1} = P_G w_n$ and $0 < \gamma \leq \tau$)

$$\|x_{n+1} - x^*\|^2 \leq \langle w_n - x^*, x_{n+1} - x^* \rangle + \langle P_G w_n - w_n, P_G w_n - x^* \rangle$$

$$\leq \langle w_n - x^*, x_{n+1} - x^* \rangle + \delta_n \lambda_n \gamma \langle V x_n - V x^*, x_{n+1} - x^* \rangle + \lambda_n (1 - \delta_n) \gamma \langle S x_n - S x^*, x_{n+1} - x^* \rangle$$

$$\leq \langle (I - \lambda_n \mu F)T x_n - (I - \lambda_n \mu F) x^*, x_{n+1} - x^* \rangle + \delta_n \lambda_n \gamma \langle V x_n - V x^*, x_{n+1} - x^* \rangle + \lambda_n (1 - \delta_n) \gamma \langle S x_n - S x^*, x_{n+1} - x^* \rangle$$

$$+ \delta_n \lambda_n \gamma \langle (\gamma V - \mu F)x^*, x_{n+1} - x^* \rangle + \lambda_n (1 - \delta_n) \gamma \langle (\gamma S - \mu F)x^*, x_{n+1} - x^* \rangle$$
\[- \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|y_n - y^*\|^2 + \|y_{n+1} - y^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|z_n - z^*\|^2 + \|z_{n+1} - z^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|w_n - w^*\|^2 + \|w_{n+1} - w^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|u_n - u^*\|^2 + \|u_{n+1} - u^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|v_n - v^*\|^2 + \|v_{n+1} - v^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|y_n - y^*\|^2 + \|y_{n+1} - y^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|z_n - z^*\|^2 + \|z_{n+1} - z^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|w_n - w^*\|^2 + \|w_{n+1} - w^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|u_n - u^*\|^2 + \|u_{n+1} - u^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|v_n - v^*\|^2 + \|v_{n+1} - v^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|y_n - y^*\|^2 + \|y_{n+1} - y^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|z_n - z^*\|^2 + \|z_{n+1} - z^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|w_n - w^*\|^2 + \|w_{n+1} - w^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|u_n - u^*\|^2 + \|u_{n+1} - u^*\|^2) \]
\[+ \frac{1}{2} \frac{1}{\delta_n \lambda_n (1 - \delta_n)} \frac{1}{2} (\|v_n - v^*\|^2 + \|v_{n+1} - v^*\|^2) \]

It turns out that
\[\|x_{n+1} - x^*\|^2 \leq \frac{1}{1 + \delta_n \lambda_n (1 - \rho)} \|x_n - x^*\|^2 + \frac{2}{1 + \delta_n \lambda_n (1 - \rho)} \]
[\[\delta_n \lambda_n ((\gamma S - \mu F)x^*, y_{n+1} - x^*) \]
\[+ \lambda_n (1 - \delta_n)((\gamma S - \mu F)x^*, y_{n+1} - x^*) + \alpha_n (M_1 + M_2) \]
\[\leq \frac{1}{1 + \delta_n \lambda_n (1 - \rho)} \|x_n - x^*\|^2 + \frac{2}{1 + \delta_n \lambda_n (1 - \rho)} \]
[\[\delta_n \lambda_n ((\gamma S - \mu F)x^*, y_{n+1} - x^*) + \lambda_n (1 - \delta_n) \]
\[+ \alpha_n (M_1 + M_2) \]
\[\{((\gamma S - \mu F)x^*, x_{n+1} - x^*) + 2 \alpha_n (M_1 + M_2). \] (6.3.31)

However, from \(x^* \in \Xi\) and condition (C5) we obtain that
\[\{(\gamma S - \mu F)x^*, x_{n+1} - x^*\} \]
\[= \{(\gamma S - \mu F)x^*, y_{n+1} - P_{Fiz(T) \cap \Theta} x_{n+1}\} + \{(\gamma S - \mu F)x^*, P_{Fiz(T) \cap \Theta} x_{n+1} - x^*\} \]
\[\leq \{(\gamma S - \mu F)x^*, x_{n+1} - P_{Fiz(T) \cap \Theta} x_{n+1}\} \]
\[\leq \|(\gamma S - \mu F)x^*\|d(x_{n+1}, Fiz(T) \cap \Theta) \]
\[\leq \|(\gamma S - \mu F)x^*\|d \left( \frac{1}{k} \|x_{n+1} - T x_{n+1}\| \right)^{1/\theta}. \] (6.3.32)
On the other hand, we also have

\[
\|x_{n+1} - T x_{n+1}\| \leq \|x_{n+1} - T x_n\| + \|T x_n - T x_{n+1}\|
\]

\[
\leq \|x_n - x_{n+1}\| + \|\lambda_n \gamma (\delta_n V x_n + (1 - \delta_n) S x_n) + (I - \lambda_n \mu F) T x_n - T x_n\|
\]

\[
\leq \|x_n - x_{n+1}\| + \|T x_n - T x_n\| + \lambda_n \|\gamma (\delta_n V x_n + (1 - \delta_n) S x_n) - \mu F T x_n\|
\]

\[
= \|x_n - x_{n+1}\| + \|T x_n - T x_n\| + \lambda_n \|\gamma \delta_n (V x_n - S x_n) + \gamma S x_n - \mu F T x_n\|
\]

\[
\leq \|x_n - x_{n+1}\| + \|z_n - x_n\| + \lambda_n M_0. \tag{6.3.33}
\]

Hence, for a big enough constant \(k_1 > 0\), we have

\[
(\gamma S - \mu F)x^*, x_{n+1} - x^*) \leq k_1 (\lambda_n + \|x_n - x_{n+1}\| + \|z_n - x_n\|)^{1/\theta}
\]

\[
\leq k_1 \lambda_n^{1/\theta} \left(1 + \frac{\|x_n - x_{n+1}\| + \|z_n - x_n\|}{\lambda_n}\right)^{1/\theta} \tag{6.3.34}
\]

Combining (6.3.31)-(6.3.34), we get

\[
\|x_{n+1} - x^*\|^2 \leq \left[1 - \delta_n \lambda_n \gamma (1 - \rho)\right]\|x_n - x^*\|^2 + \frac{2}{1 + \delta_n \lambda_n \gamma (1 - \rho)} \delta_n \lambda_n \{(\gamma V - \mu F)x^*,
\]

\[
x_{n+1} - x^*\} + \lambda_n \{1 - \delta_n\} \{(\gamma S - \mu F)x^*, x_{n+1} - x^*\} + 2\alpha_n (M_1 + M_2)
\]

\[
\leq \left[1 - \delta_n \lambda_n \gamma (1 - \rho)\right]\|x_n - x^*\|^2 + \frac{2\delta_n \lambda_n}{1 + \delta_n \lambda_n \gamma (1 - \rho)} \{(\gamma V - \mu F)x^*, x_{n+1} - x^*\}
\]

\[
+ \frac{k_1 \lambda_n^{1/\theta}}{\delta_n} \left(1 + \frac{\|x_n - x_{n+1}\| + \|z_n - x_n\|}{\lambda_n}\right)^{1/\theta} + 2\alpha_n (M_1 + M_2)
\]

\[
= (1 - \mu_n)\|x_n - x^*\|^2 + \nu_n + 2\alpha_n (M_1 + M_2), \tag{6.3.35}
\]

where \(\mu_n = \delta_n \lambda_n \gamma (1 - \rho)\) and

\[
\nu_n = \frac{2\delta_n \lambda_n}{1 + \delta_n \lambda_n \gamma (1 - \rho)} \|(\gamma V - \mu F)x^*, x_{n+1} - x^*\|
\]

\[
+ \frac{k_1 \lambda_n^{1/\theta}}{\delta_n} \left(1 + \frac{\|x_n - x_{n+1}\| + \|z_n - x_n\|}{\lambda_n}\right)^{1/\theta}.
\]
Now condition (C4) implies that \( \sum_{n=1}^{\infty} \mu_n = \infty \). Moreover, since \( \|x_{n+1} - x_n\| + \|z_n - x_n\| = o(\lambda_n) \), condition (C6) and (6.3.30) imply that
\[
\limsup_{n \to \infty} \frac{\nu_n}{\mu_n} \leq 0.
\]
Therefore, we can apply Lemma 2.1.2 to (6.3.35) to conclude that \( x_n \to x^* \). The proof of part (a) is complete.

It is easy to see that part (b) now becomes a straightforward consequence of part (a) since, if \( V = 0 \), THVIP (6.3.3) reduces to the VIP in part (b). This completes the proof. \( \square \)

Next we consider a special case of Problem 6.3.2. In Problem 6.3.2, put \( \mu = 2 \), \( F = \frac{1}{2} I \) and \( \gamma = \tau = 1 \). In this case, the objective is to find \( z^* \in \Xi \) such that
\[
\langle (I - V)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Xi,
\]
where \( \Xi \) denotes the solution set of the following hierarchical variational inequality problem (HVIP) of finding \( z^* \in \text{Fix}(T) \cap \Theta \) such that
\[
\langle (I - S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(T) \cap \Theta.
\]

**Corollary 6.3.1.** Let \( A : C \to H \) be a \( 1/L \)-inverse strongly monotone mapping with \( 0 < \lambda < 2/L \), \( V : C \to H \) be a \( p \)-contraction with coefficient \( p \in [0, 1) \) and \( S, T : C \to C \) be nonexpansive mappings. Assume that the solution set \( \Xi \) of the HVIP (6.3.37) is nonempty and that the following conditions hold for five sequences \( \{\alpha_n\} \subset [0, \infty) \), \( \{\beta_n\} \), \( \{\gamma_n\} \subset [0, 1] \) and \( \{\lambda_n\} \), \( \{\delta_n\} \subset (0, 1) \):

(C1) \( \sum_{n=0}^{\infty} \alpha_n < \infty \);

(C2) \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \);

(C3) \( 0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1 \);

(C4) \( \lim_{n \to \infty} \lambda_n = 0 \), \( \lim_{n \to \infty} \delta_n = 0 \) and \( \sum_{n=0}^{\infty} \delta_n \lambda_n = \infty \).
(C5) there are constants $\bar{k}, \theta > 0$ such that $\|x - Tx\| \geq \bar{k}[d(x, \text{Fix}(T) \cap \Theta)]^\theta, \forall x \in C$;

(C6) $\lim_{n \to \infty} \frac{x_n}{\lambda_n} = 0$.

Then, the following assertions hold:

(a) If $\{x_n\}$ is the sequence generated by the iterative scheme

$$
\begin{align*}
    x_0 &= x \in C \quad \text{chosen arbitrarily}, \\
    y_n &= \left(1 - \beta_n\right)x_n + \beta_n P_C(x_n - \lambda A_n x_n), \\
    z_n &= \gamma_n x_n + (1 - \gamma_n) P_C(y_n - \lambda A_n y_n), \\
    x_{n+1} &= P_C[\lambda_n (\delta_n x_n + (1 - \delta_n) S x_n) + (1 - \lambda_n) T z_n], \quad \forall n \geq 0
\end{align*}
$$

and $\{S x_n\}$ is bounded, then $\{x_n\}$ converges strongly to the point $x^* \in \text{Fix}(T) \cap \Theta$ which is a unique solution of THVIP (6.3.36) provided $\|x_{n+1} - x_n\| + \|x_n - z_n\| = o(\lambda_n)$.

(b) If $\{x_n\}$ is the sequence generated by the iterative scheme

$$
\begin{align*}
    x_0 &= x \in C \quad \text{chosen arbitrarily}, \\
    y_n &= \left(1 - \beta_n\right)x_n + \beta_n P_C(x_n - \lambda A_n x_n), \\
    z_n &= \gamma_n x_n + (1 - \gamma_n) P_C(y_n - \lambda A_n y_n), \\
    x_{n+1} &= P_C[\lambda_n (1 - \delta_n) S x_n + (1 - \lambda_n) T z_n], \quad \forall n \geq 0
\end{align*}
$$

and $\{S x_n\}$ is bounded, then $\{x_n\}$ converges strongly to a unique solution $x^*$ of the following VIP provided $\|x_{n+1} - x_n\| + \|x_n - z_n\| = o(\lambda_n)$:

$$
\text{find } x^* \in \Xi \text{ such that } (x^*, x - x^*) \geq 0, \quad \forall x \in \Xi,
$$

that is, $x^*$ is the minimum-norm solution of HVIP (6.3.37).

To illustrate Theorem 6.3.1 and Algorithm 6.3.1, we present the following example.
Example 6.3.1. Let $H = \mathbb{R}$ with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$ which are defined by $(x, y) = ac + bd$ and $\|x\| = \sqrt{a^2 + b^2}$, for all $x, y \in \mathbb{R}^2$ with $x = (a, b)$ and $y = (c, d)$. Let $C = \{(a, a) : |a| \leq 1\} \neq \emptyset$. Clearly, $C$ is a nonempty, bounded, closed and convex subset of $\mathbb{R}^2$. Let $V$ be a $2 \times 2$ positive semidefinite matrix such that $0 < \|V\| < 1$. For instance, putting $V = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & \frac{1}{3} \end{bmatrix}$, we know that $\|V\| = \frac{5}{6}$ and $V : C \to C$ is a $\rho$-contraction with contractivity constant $\rho = \frac{2}{3}$. Let $F = \frac{1}{2}I$. Then, $F : C \to C$ is a $\kappa$-Lipschitzian and $\eta$-strongly monotone operator with constants $\kappa = \frac{1}{2}$ and $\eta = \frac{1}{2}$, respectively. Take $\mu = 2$ and $\gamma = 1$ such that $0 < \mu < \frac{3}{2}$ and $0 < \gamma \leq \tau$ where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} = 1$. Let $S$ and $T$ be two $2 \times 2$ positive definite matrices such that $\|S\| = \|T\| = 1$, for instance, putting $S = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$ and $T = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix}$, we know that $\|T\| = \|S\| = 1$ and that $S, T : C \to C$ are two nonexpansive mappings with $Fix(T) = C \neq \emptyset$. Put $A$ be a $2 \times 2$ zero matrix. Then $Fix(T) \cap \Theta = Fix(T) = C$ and $0 = \|x - Tx\| \geq k[d(x, Fix(T) \cap \Theta)]^\theta = 0, \forall x \in C$ for all positive constants $k$ and $\theta$.

Further, we observe that the solution set $\Xi$ of the HVIP is the following

$$
\Xi = \{x^* \in Fix(T) \cap \Theta : ((\mu F - \gamma S)x^*, z - x^*) \geq 0, \forall z \in Fix(T) \cap \Theta\}
$$

$$
= \{z^* \in Fix(T) : ((I - S)z^*, z - z^*) \geq 0, \forall z \in Fix(T)\}
$$

$$
= \{z^* \in Fix(T) : (0, z - z^*) \geq 0, \forall z \in Fix(T)\}
$$

$$
= Fix(T) = C \neq \emptyset.
$$

In the meantime, it is easy to see that there exists a unique solution $x^* = (0, 0)$ to the following THVIP: find $x^* \in \Xi = Fix(T)$ such that

$$
((\mu F - \gamma V)x^*, x - x^*) \geq 0, \forall x \in \Xi.
$$

(Namely, $((I - V)x^*, x - x^*) \geq 0, \forall x \in \Xi$.)

Also, set $\alpha_n = 0$, $\forall n \geq 0$, and choose the sequences $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\lambda_n\}$ satisfying conditions (C2)-(C4), (C6) in Theorem 6.3.1. Then, the iterative scheme (6.3.5) reduces to
\[
\begin{align*}
\begin{cases}
x_0 &= (a, a) \in C \quad \text{chosen arbitrarily}, \\
y_n &= (1 - \beta_n)x_n + \beta_n P_C(x_n - \lambda A_n x_n), \\
z_n &= \gamma_n x_n + (1 - \gamma_n) T P_C(y_n - \lambda A_n y_n), \\
x_{n+1} &= P_C\left[\lambda_n \gamma_n (\delta_n V x_n + (1 - \delta_n) S x_n) + (1 - \lambda_n \mu_F) T x_n\right] \\
&= P_C\left[\lambda_n (\delta_n V x_n + (1 - \delta_n) S x_n) + (1 - \lambda_n) T x_n\right] \\
&= \lambda_n (\delta_n V x_n + (1 - \delta_n) S x_n) + (1 - \lambda_n) T x_n, \quad \forall n \geq 0.
\end{cases}
\end{align*}
\]

We assert that
\[
x_{n+1} = \prod_{i=0}^{n} (1 - \delta_i \lambda_i) x_0, \quad \forall n \geq 0.
\]

For \( n = 0 \) we have
\[
x_1 = \lambda_0 (\delta_0 V x_0 + (1 - \delta_0) S x_0) + (1 - \lambda_0) T x_0 \\
= \lambda_0 (1 - \delta_0) x_0 + (1 - \lambda_0) x_0 \\
= (1 - \delta_0 \lambda_0) x_0.
\]

Assume that for some \( n \geq 0 \)
\[
x_n = \prod_{i=0}^{n-1} (1 - \delta_i \lambda_i) x_0.
\]

Then,
\[
x_{n+1} = \lambda_n (\delta_n V x_n + (1 - \delta_n) S x_n) + (1 - \lambda_n) T x_n \\
= \lambda_n [\delta_n V (\prod_{i=0}^{n-1} (1 - \delta_i \lambda_i) x_0) \\
+ (1 - \delta_n) S (\prod_{i=0}^{n-1} (1 - \delta_i \lambda_i) x_0)] + (1 - \lambda_n) T (\prod_{i=0}^{n-1} (1 - \delta_i \lambda_i) x_0) \\
= \lambda_n (1 - \delta_n) \prod_{i=0}^{n-1} (1 - \delta_i \lambda_i) x_0 + (1 - \lambda_n) \prod_{i=0}^{n-1} (1 - \delta_i \lambda_i) x_0 \\
= [\lambda_n (1 - \delta_n) + (1 - \lambda_n)] \prod_{i=0}^{n-1} (1 - \delta_i \lambda_i) x_0 \\
= \prod_{i=0}^{n} (1 - \delta_i \lambda_i) x_0.
\]
So, by induction the assertion is valid. Therefore, it follows that

\[ x_{n+1} = (1 - \delta_n \lambda_n) x_n, \quad \forall n \geq 0, \]

which immediately yields

\[ \|x_{n+1} - x_n\| = \delta_n \lambda_n \|x_n\|, \quad \forall n \geq 0. \]

This shows that \( \|x_{n+1} - x_n\| = o(\lambda_n) \), and hence, \( \|x_{n+1} - x_n\| + \|x_n - x_n\| = o(\lambda_n) \). Hence, by theorem 6.3.1, the sequence \( \{x_n\} \) converges to a unique solution \( x^* = (0,0) \) to the above THVIP.

By using Matlab programming for the solution of \( x_n \) in each iteration, we get following table by considering initial value \( x_1 = (-1,-1) \)

<table>
<thead>
<tr>
<th>No. of iterations</th>
<th>( y_n )</th>
<th>( z_n )</th>
<th>( x_{n+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-1,-1)</td>
<td>(-1,-1)</td>
<td>(-1,-1)</td>
</tr>
<tr>
<td>2</td>
<td>(-0.9975,-0.9975)</td>
<td>(-0.9502,-0.9502)</td>
<td>(-0.9975,-0.9975)</td>
</tr>
<tr>
<td>3</td>
<td>(-0.9875,-0.9875)</td>
<td>(-0.8996,-0.8996)</td>
<td>(-0.9875,-0.9875)</td>
</tr>
<tr>
<td>4</td>
<td>(-0.9653,-0.9653)</td>
<td>(-0.8456,-0.8456)</td>
<td>(-0.9653,-0.9653)</td>
</tr>
<tr>
<td>5</td>
<td>(-0.9267,-0.9267)</td>
<td>(-0.7861,-0.7861)</td>
<td>(-0.9267,-0.9267)</td>
</tr>
<tr>
<td>6</td>
<td>(-0.8688,-0.8688)</td>
<td>(-0.7204,-0.7204)</td>
<td>(-0.8688,-0.8688)</td>
</tr>
<tr>
<td>7</td>
<td>(-0.7906,-0.7906)</td>
<td>(-0.6480,-0.6480)</td>
<td>(-0.7906,-0.7906)</td>
</tr>
<tr>
<td>8</td>
<td>(-0.6937,-0.6937)</td>
<td>(-0.5698,-0.5698)</td>
<td>(-0.6937,-0.6937)</td>
</tr>
<tr>
<td>9</td>
<td>(-0.5827,-0.5827)</td>
<td>(-0.4873,-0.4873)</td>
<td>(-0.5827,-0.5827)</td>
</tr>
<tr>
<td>10</td>
<td>(-0.4647,-0.4647)</td>
<td>(-0.4028,-0.4028)</td>
<td>(-0.4647,-0.4647)</td>
</tr>
<tr>
<td>11</td>
<td>(-0.3486,-0.3486)</td>
<td>(-0.3195,-0.3195)</td>
<td>(-0.3486,-0.3486)</td>
</tr>
<tr>
<td>12</td>
<td>(-0.2431,-0.2431)</td>
<td>(-0.2409,-0.2409)</td>
<td>(-0.2431,-0.2431)</td>
</tr>
<tr>
<td>No. of iterations</td>
<td>$y_n$</td>
<td>$z_n$</td>
<td>$x_{n+1}$</td>
</tr>
<tr>
<td>-------------------</td>
<td>-------</td>
<td>-------</td>
<td>-----------</td>
</tr>
<tr>
<td>13</td>
<td>(-0.1556, -0.1556)</td>
<td>(-0.1708, -0.1708)</td>
<td>(-0.1556, -0.1556)</td>
</tr>
<tr>
<td>14</td>
<td>(-0.0899, -0.0899)</td>
<td>(-0.1121, -0.1121)</td>
<td>(-0.0899, -0.0899)</td>
</tr>
<tr>
<td>15</td>
<td>(-0.0458, -0.0458)</td>
<td>(-0.0670, -0.0670)</td>
<td>(-0.0458, -0.0458)</td>
</tr>
<tr>
<td>16</td>
<td>(-0.0200, -0.0200)</td>
<td>(-0.0356, -0.0356)</td>
<td>(-0.0200, -0.0200)</td>
</tr>
<tr>
<td>17</td>
<td>(-0.0072, -0.0072)</td>
<td>(-0.0163, -0.0163)</td>
<td>(-0.0072, -0.0072)</td>
</tr>
<tr>
<td>18</td>
<td>(-0.0020, -0.0020)</td>
<td>(-0.0062, -0.0062)</td>
<td>(-0.0020, -0.0020)</td>
</tr>
<tr>
<td>19</td>
<td>(1.0e-003, 1.0e-003)</td>
<td>(-0.0018, -0.0018)</td>
<td>(1.0e-003, 1.0e-003)</td>
</tr>
<tr>
<td></td>
<td>(-0.3805, -0.3805)</td>
<td>(-0.3805, -0.3805)</td>
<td>(-0.3805, -0.3805)</td>
</tr>
<tr>
<td>20</td>
<td>(1.0e-004, 1.0e-004)</td>
<td>(1.0e-003, 1.0e-003)</td>
<td>(1.0e-004, 1.0e-004)</td>
</tr>
<tr>
<td></td>
<td>(-0.3710, -0.3710)</td>
<td>(-0.3616, -0.3616)</td>
<td>(-0.3710, -0.3710)</td>
</tr>
<tr>
<td>21</td>
<td>(0, 0)</td>
<td>(1.0e-004, 1.0e-004)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td></td>
<td>(-0.3710, -0.3710)</td>
<td>(-0.3710, -0.3710)</td>
<td>(-0.3710, -0.3710)</td>
</tr>
</tbody>
</table>

where, $1.0e-003 = 10^{-3}$ and $1.0e-004 = 10^{-4}$

We now present the convergence analysis of Algorithm 6.3.2 for solving Problem 6.3.2.

**Theorem 6.3.2.** Let $F : C \to H$ be a $\kappa$-Lipschitzian and $\eta$-strongly monotone operator with constants $\kappa, \eta > 0$, respectively, $A : C \to H$ be a $1/L$-inverse strongly monotone mapping, $V : C \to H$ be a $\rho$-contraction with coefficient $\rho \in [0, 1)$ and $S, T : C \to C$ be nonexpansive mappings. Let $0 < \lambda < 2/L$, $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that the solution set $\Xi$ of the HVIP (6.3.4) is nonempty and that the following conditions hold for five sequences $\{\alpha_n\} \subset [0, \infty)$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ and $\{\lambda_n\}$, $\{\delta_n\} \subset (0, 1)$:

(C1) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(C2) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$;
(C3) \(0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1\);

(C4) \(\lim_{n \to \infty} \lambda_n = 0, \lim_{n \to \infty} \delta_n = 0\) and \(\sum_{n=0}^{\infty} \delta_n \lambda_n = \infty\);

(C5) there are constants \(\bar{k}, \theta > 0\) such that \(\|x - Tx\| \geq \bar{k}[d(x, \text{Fix}(T) \cap \Theta)]^{\theta}, \forall x \in C\);

(C6) \(\lim_{n \to \infty} \frac{\lambda_n}{\delta_n} = 0\).

Then, the following assertions hold:

(a) If \(\{x_n\}\) is the sequence generated by the scheme (6.3.9) and \(\{Sx_n\}\) is bounded, then \(\{x_n\}\) converges strongly to the point \(x^* \in \text{Fix}(T) \cap \Theta\) which is a unique solution of Problem 6.3.2 provided \(\|x_{n+1} - x_n\| + \|x_n - z_n\| = o(\lambda_n)\).

(b) If \(\{x_n\}\) is the sequence generated by the scheme (6.3.10) and \(\{Sx_n\}\) is bounded, then \(\{x_n\}\) converges strongly to a unique solution \(x^*\) of the following VIP provided \(\|x_{n+1} - x_n\| + \|x_n - z_n\| = o(\lambda_n)\):

\[
\text{Find } x^* \in \Xi \text{ such that } \langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in \Xi. \tag{6.3.38}
\]

**Proof.** We only prove case (a), that is, the sequence \(\{x_n\}\) is generated by the scheme (6.3.9). Obviously, from the condition \(\Xi \neq \emptyset\) it follows that \(\text{Fix}(T) \cap \Theta \neq \emptyset\). In addition, in terms of conditions (C2) and (C3), without loss of generality we may assume that \(\{\beta_n\} \subset [a, b]\) for some \(a, b \in (0, 1)\) and \(\{\gamma_n\} \subset [c, d]\) for some \(c, d \in (0, 1)\).

Repeating the same argument as that in Theorem 6.3.1, we can obtain the following statement: \(P_C(I - \lambda(\alpha I + A))\) is \(\xi\)-averaged for each \(\lambda \in (0, \frac{2}{\alpha + L})\), where \(\xi = \frac{2 + \lambda(\alpha + L)}{4} \in (0, 1)\), and hence \(P_C(I - \lambda(\alpha I + A))\) is nonexpansive.

Furthermore, for \(\lambda \in (0, \frac{2}{L})\), utilizing the fact that \(\lim_{n \to \infty} \frac{2}{\alpha_n + L} = \frac{2}{L}\) we may assume that

\(0 < \lambda < \frac{2}{\alpha_n + L}, \quad \forall n \geq 0\).
Consequently, it follows that for each integer $n \geq 0$, $P_C(I - \lambda(\alpha_n I + A))$ is $\xi_n$-averaged with $
olimits \xi_n = \frac{2 + \lambda(\alpha_n + \lambda)}{4} \in (0, 1)$. This immediately implies that $P_C(I - \lambda(\alpha_n I + A))$ is nonexpansive for all $n \geq 0$.

Next we divide the remainder of the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, take a fixed $p \in \text{Fix}(T) \cap \Theta$ arbitrarily. Then, we get $Tp = p$ and $P_C(I - \lambda A)p = p$ for $\lambda \in (0, \frac{3}{2})$. Hence, we have

$$
\|y_n - p\| \leq (1 - \beta_n)\|x_n - p\| + \beta_n\|P_C(I - \lambda A_n)y_n - p\| \\
= (1 - \beta_n)\|x_n - p\| + \beta_n\|P_C(I - \lambda A_n)y_n - P_C(I - \lambda A)p\| \\
\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|P_C(I - \lambda A_n)y_n - P_C(I - \lambda A_n)p\| \\
+ \|P_C(I - \lambda A_n)p - P_C(I - \lambda A)p\| \\
\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|y_n - p\| + \|(I - \lambda A_n)p - (I - \lambda A)p\| \\
= (1 - \beta_n)\|x_n - p\| + \beta_n\|y_n - p\| + \lambda\alpha_n\|p\|,
$$

which implies that

$$
\|y_n - p\| \leq \|x_n - p\| + \frac{\beta_n}{1 - \beta_n} \lambda\alpha_n\|p\| \leq \|x_n - p\| + \frac{b}{1 - b} \lambda\alpha_n\|p\|. \quad (6.3.39)
$$

Thus, we obtain that

$$
\|x_n - p\| \leq \gamma_n\|y_n - p\| + (1 - \gamma_n)\|TP_C(I - \lambda A_n)x_n - p\| \\
\leq \gamma_n\|y_n - p\| + (1 - \gamma_n)\|P_C(I - \lambda A_n)x_n - p\| \\
= \gamma_n\|y_n - p\| + (1 - \gamma_n)\|P_C(I - \lambda A_n)x_n - P_C(I - \lambda A)p\| \\
\leq \gamma_n\|y_n - p\| + (1 - \gamma_n)\|P_C(I - \lambda A_n)x_n - P_C(I - \lambda A_n)p\| \\
+ \|P_C(I - \lambda A_n)p - P_C(I - \lambda A)p\| \\
\leq \gamma_n\|y_n - p\| + (1 - \gamma_n)\|z_n - p\| + \|(I - \lambda A_n)p - (I - \lambda A)p\| \\
= \gamma_n\|y_n - p\| + (1 - \gamma_n)\|z_n - p\| + \lambda\alpha_n\|p\|,
$$
which together with (6.3.39) implies that

\[ \| z_n - p \| \leq \| y_n - p \| + \frac{1 - \gamma_n}{\gamma_n} \lambda \alpha_n \| p \| \]

\[ \leq \| x_n - p \| + \frac{b}{1 - b} \lambda \alpha_n \| p \| + \frac{1 - c}{c} \lambda \alpha_n \| p \| \]

\[ \leq \| x_n - p \| + 2 \max \left\{ \frac{b}{1 - b}, \frac{1 - c}{c} \right\} \lambda \alpha_n \| p \|. \]

Note that boundedness of \( \{Sx_n\} \), we get \( \sup_{n \geq 0} \| \gamma Sx_n - \mu Fp \| \leq M \), for some \( M \geq 0 \).

Moreover, utilizing Lemma 2.1.12 we have from (6.3.9)

\[ \| x_{n+1} - p \| \]

\[ = \| P_C [\lambda_n \gamma (\delta_n V x_n + (1 - \delta_n) S x_n) + (I - \lambda_n \mu F) T z_n] - P_C p \| \]

\[ \leq \| \lambda_n \gamma (\delta_n V x_n + (1 - \delta_n) S x_n) + (I - \lambda_n \mu F) T z_n - p \| \]

\[ = \| \lambda_n \gamma (\delta_n V x_n + (1 - \delta_n) S x_n) - \lambda_n \mu F T p + (I - \lambda_n \mu F) T z_n - (I - \lambda_n \mu F) T p \| \]

\[ \leq \| \lambda_n \gamma (\delta_n V x_n + (1 - \delta_n) S x_n) - \lambda_n \mu F T p \| + \| (I - \lambda_n \mu F) T z_n - (I - \lambda_n \mu F) T p \| \]

\[ = \lambda_n \| \delta_n (\gamma V x_n - \mu F p) + (1 - \delta_n) (\gamma S x_n - \mu F p) \| \]

\[ + \| (I - \lambda_n \mu F) T z_n - (I - \lambda_n \mu F) T p \| \]

\[ \leq \lambda_n [\delta_n \| \gamma V x_n - \mu F p \| + (1 - \delta_n) \| \gamma S x_n - \mu F p \| + (1 - \lambda_n \gamma) \| z_n - p \| \]

\[ \leq \lambda_n [\delta_n (\| \gamma V x_n - \gamma V p \| + \| \gamma V p - \mu F p \|) + (1 - \delta_n) M + (1 - \lambda_n \gamma) \| z_n - p \| \]

\[ \leq \lambda_n [\delta_n \gamma \| x_n - p \| + \delta_n \| \gamma V p - \mu F p \| + (1 - \delta_n) M] \]

\[ + (1 - \lambda_n \gamma) (\| x_n - p \| + 2 \max \left\{ \frac{b}{1 - b}, \frac{1 - c}{c} \right\} \lambda \alpha_n \| p \|) \]

\[ \leq \lambda_n [\delta_n \gamma \| x_n - p \| + \gamma \| \gamma V p - \mu F p \|] \]

\[ + (1 - \lambda_n \gamma) (\| x_n - p \| + 2 \max \left\{ \frac{b}{1 - b}, \frac{1 - c}{c} \right\} \lambda \alpha_n \| p \|) \]

\[ \leq \lambda_n [\gamma \| x_n - p \| + \gamma \max \{ M, \| \gamma V p - \mu F p \| \}] \]

\[ + (1 - \lambda_n \gamma) (\| x_n - p \| + 2 \max \left\{ \frac{b}{1 - b}, \frac{1 - c}{c} \right\} \lambda \alpha_n \| p \|) \]

\[ = [1 - (\tau - \gamma p) \lambda_n] \| x_n - p \| + \gamma \max \{ M, \| \gamma V p - \mu F p \| \} \]

\[ + 2 \max \left\{ \frac{b}{1 - b}, \frac{1 - c}{c} \right\} \lambda \alpha_n \| p \|. \]

(6.3.40)
Setting

\[ \widetilde{M} = \max \left\{ \|x_0 - p\|, \frac{M}{\tau - \gamma \rho}, \frac{\|y_p - \mu F p\|}{\tau - \gamma \rho} \right\}, \]

by induction we can derive

\[ \|x_{n+1} - p\| \leq \widetilde{M} + \sum_{j=0}^{n} 2 \max \left\{ \frac{b}{1 - b}, \frac{1 - c}{e} \right\} \lambda \alpha_j \|p\|, \quad \forall n \geq 0. \tag{6.3.41} \]

Consequently, \( \{x_n\} \) is bounded (due to \( \sum_{n=0}^{\infty} \alpha_n < \infty \)) and so \( \{y_n\}, \{x_n\}, \{Ax_n\} \) and \( \{Ay_n\} \).

**Step 2.** \( \lim_{n \to \infty} \|u_n - x_n\| = \lim_{n \to \infty} \|x_n - y_n\| = \lim_{n \to \infty} \|u_n - T u_n\| = 0 \), where \( u_n = P_C(I - \lambda A_n) x_n \) and \( u_n = P_C(I - \lambda A_n) y_n \).

Observe that

\[
\|y_n - p\|^2 \\
= \| (1 - \beta_n)(x_n - p) + \beta_n(u_n - p) \|^2 \\
= (1 - \beta_n)\|x_n - p\|^2 + \beta_n\|u_n - p\|^2 - \beta_n(1 - \beta_n)\|x_n - u_n\|^2 \\
\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n\|y_n - p\|^2 + \lambda \alpha_n\|p\|^2 - \beta_n(1 - \beta_n)\|x_n - u_n\|^2 \\
= (1 - \beta_n)\|x_n - p\|^2 + \beta_n\|y_n - p\|^2 + \alpha_n \beta_n (2\lambda\|p\|\|y_n - p\| + \alpha_n \lambda^2\|p\|^2) \\
- \beta_n(1 - \beta_n)\|x_n - u_n\|^2, \tag{6.3.42}
\]

and hence

\[
\|y_n - p\|^2 \\
\leq \|x_n - p\|^2 + \alpha_n \beta_n \left( (2\lambda\|p\|\|y_n - p\| + \alpha_n \lambda^2\|p\|^2) - \beta_n\|x_n - u_n\|^2 \right) \\
\leq \|x_n - p\|^2 + \alpha_n M_1 \beta_n \|x_n - u_n\|^2, \tag{6.3.43}
\]

where \( M_1 = \sup_{n \geq 0} \left\{ \frac{\beta_n}{1 - \beta_n} (2\lambda\|p\|\|y_n - p\| + \alpha_n \lambda^2\|p\|^2) \right\} < \infty \). We, also observe that

\[
\|x_n - p\|^2 = \| \gamma_n(y_n - p) + (1 - \gamma_n)(Tu_n - p) \|^2 \\
= \gamma_n\|y_n - p\|^2 + (1 - \gamma_n)(Tu_n - p)^2 - \gamma_n(1 - \gamma_n)\|y_n - Tu_n\|^2 \\
\leq \gamma_n\|y_n - p\|^2 + (1 - \gamma_n)(u_n - p)^2 - \gamma_n(1 - \gamma_n)\|y_n - Tu_n\|^2
\]
\[
\leq \gamma_n \|y_n - p\|^2 + (1 - \gamma_n)\|x_n - p\|^2 + \lambda \alpha_n \|p\|^2 - \gamma_n (1 - \gamma_n) \|y_n - Tu_n\|^2 \\
= \gamma_n \|y_n - p\|^2 + (1 - \gamma_n)\|x_n - p\|^2 + \alpha_n (2\lambda \|p\|\|x_n - p\| + \alpha_n \lambda^2 \|p\|^2) \\
- \gamma_n (1 - \gamma_n) \|y_n - Tu_n\|^2,
\]
which together with (6.3.43) yields that
\[
\|x_n - p\|^2 \leq \|y_n - p\|^2 + \alpha_n \frac{1 - \gamma_n}{\gamma_n} (2\lambda \|p\|\|x_n - p\| + \alpha_n \lambda^2 \|p\|^2) - (1 - \gamma_n) \|y_n - Tu_n\|^2 \\
\leq \|y_n - p\|^2 + \alpha_n M_2 - (1 - \gamma_n) \|y_n - Tu_n\|^2 \\
\leq \|x_n - p\|^2 + \alpha_n M_1 - \beta_n \|x_n - v_n\|^2 + \alpha_n M_2 - (1 - \gamma_n) \|y_n - Tu_n\|^2 \\
= \|x_n - p\|^2 + \alpha_n (M_1 + M_2) - \beta_n \|x_n - v_n\|^2 - (1 - \gamma_n) \|y_n - Tu_n\|^2, \tag{6.3.45}
\]
where \(M_2 = \sup_{n \geq 0} \left\{ \frac{1 - \gamma_n}{\gamma_n} (2\lambda \|p\|\|x_n - p\| + \alpha_n \lambda^2 \|p\|^2) \right\} < \infty.\)

Hence, it follows from (6.3.44) that
\[
\|x_{n+1} - p\|^2 \leq \|\lambda_n \gamma (\delta_n V x_n + (1 - \delta_n) S x_n) + (I - \lambda_n \mu F) T z_n - p\|^2 \\
= \|\lambda_n \gamma (\delta_n V x_n + (1 - \delta_n) S x_n) - \lambda_n \mu F T p + (I - \lambda_n \mu F) T z_n - (I - \lambda_n \mu F) T p\|^2 \\
\leq \{\|\lambda_n \gamma (\delta_n V x_n + (1 - \delta_n) S x_n) - \lambda_n \mu F T p\|^2 \\
+ \|\lambda_n \gamma (\delta_n V x_n + (1 - \delta_n) S x_n) - \lambda_n \mu F T p\|^2\} + \|x_n - p\|^2 \\
\leq \{\lambda_n (\|\delta_n \gamma (V x_n - \mu F p) + (1 - \delta_n) (\gamma S x_n - \mu F p)\| + (1 - \lambda_n \tau) \|x_n - p\|\}^2 \\
\leq \lambda_n \frac{1}{\tau^2} (\|\gamma V x_n - \mu F p\|^2 + (1 - \delta_n)\|\gamma S x_n - \mu F p\|^2 + (1 - \lambda_n \tau) \|x_n - p\|^2 \\
\leq \lambda_n \frac{1}{\tau^2} (\|\gamma V x_n - \mu F p\|^2 + (1 - \delta_n)\|\gamma S x_n - \mu F p\|^2 + (1 - \lambda_n \tau) \|x_n - p\|^2 \\
\leq \lambda_n \frac{1}{\tau}\|\gamma V x_n - \mu F p\| + \|\gamma S x_n - \mu F p\| + \|x_n - p\| + \alpha_n (M_1 + M_2) \\
- \beta_n \|x_n - v_n\|^2 - (1 - \gamma_n) \|y_n - Tu_n\|^2 \\
\leq \|x_n - p\|^2 + (\alpha_n + \lambda_n) M_3 - \beta_n \|x_n - v_n\|^2 - (1 - \gamma_n) \|y_n - Tu_n\|^2, \tag{6.3.46}
\]
where $M_3 = \sup_{n \geq 0} \left\{ \frac{1}{\gamma} \left( \| \gamma V x_n - \mu F p \| + \| \gamma S x_n - \mu F p \| \right)^2, M_1 + M_2 \right\}$. This implies immediately that

\[
\beta_n \| x_n - v_n \|^2 + (1 - \gamma_n) \| y_n - Tu_n \|^2 \\
\leq \| x_n - p \|^2 - \| x_{n+1} - p \|^2 + (\alpha_n + \lambda_n) M_3 \\
= (\| x_n - p \| - \| x_{n+1} - p \|)(\| x_n - p \| + \| x_{n+1} - p \|) + (\alpha_n + \lambda_n) M_3 \\
\leq \| x_n - x_{n+1} \|(\| x_n - p \| + \| x_{n+1} - p \|) + (\alpha_n + \lambda_n) M_3.
\]

(6.3.47)

Note that $\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \lambda_n = 0$, $0 < \lim_{n \to \infty} \beta_n \leq \lim_{n \to \infty} \beta_n < 1$ and $0 < \lim_{n \to \infty} \gamma_n \leq \lim_{n \to \infty} \gamma_n < 1$. Hence, taking into account the boundedness of $\{x_n\}$ and $\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0$, we deduce from (6.3.47) that

\[
\lim_{n \to \infty} \| x_n - v_n \| = \lim_{n \to \infty} \| y_n - Tu_n \| = 0.
\]

(6.3.48)

Thus, utilizing (6.3.9) we get

\[
\lim_{n \to \infty} \| y_n - x_n \| = \lim_{n \to \infty} \beta_n \| v_n - x_n \| = 0,
\]

\[
\lim_{n \to \infty} \| x_n - y_n \| = \lim_{n \to \infty} (1 - \gamma_n) \| Tu_n - y_n \| = 0,
\]

and

\[
\| x_n - x_k \| \leq \| x_n - y_n \| + \| y_n - x_k \| \to 0 \quad \text{as } n \to \infty.
\]

Furthermore, note that

\[
\| Tu_n - v_n \| \leq \| Tu_n - Tu_n \| + \| Tu_n - y_n \| + \| y_n - v_n \| \\
\leq \| u_n - u_n \| + \| Tu_n - y_n \| + \| y_n - v_n \| \\
= \| P_C(I - \lambda A_n) y_n - P_C(I - \lambda A_n) x_n \| + \| Tu_n - y_n \| + (1 - \beta_n) \| x_k - v_n \| \\
\leq \| y_n - x_n \| + \| Tu_n - y_n \| + (1 - \beta_n) \| x_k - v_n \|.
\]

This implies that

\[
\lim_{n \to \infty} \| Tu_n - v_n \| = \lim_{n \to \infty} \| u_n - v_n \| = 0,
\]

(6.3.49)
and so
\[ \|u_n - x_n\| \leq \|u_n - v_n\| + \|v_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \]

**Step 3.** \( \omega_w(x_n) \subset \text{Fix}(T) \cap \Theta. \)

Indeed, suppose that \( \hat{x} \in \omega_w(x_n) \) and \( \{x_{n_j}\} \) is a subsequence of \( \{x_n\} \) such that \( x_{n_j} \to \hat{x} \). Repeating the same arguments as in Step 3 of the proof of Theorem 6.3.1, we know that
\[ \tilde{T} = P_{C}(I - \lambda A) \] is nonexpansive. Now observe that
\[
\begin{align*}
\|x_n - \tilde{T}x_n\| &\leq \|x_n - u_n\| + \|u_n - \tilde{T}z_n\| + \|\tilde{T}z_n - \tilde{T}x_n\| \\
&\leq \|x_n - u_n\| + \|z_n - x_n\| + \|u_n - \tilde{T}z_n\| \\
&= \|x_n - u_n\| + \|z_n - x_n\| + \|P_C(I - \lambda A_n)z_n - P_C(I - \lambda A)z_n\| \\
&\leq \|x_n - u_n\| + \|z_n - x_n\| + \|P_C(I - \lambda A_n)z_n - (I - \lambda A)z_n\| \\
&= \|x_n - u_n\| + \|z_n - x_n\| + \|\lambda z_n\|.
\end{align*}
\]
So, from \( \|x_n - u_n\| \to 0, \|z_n - x_n\| \to 0, \alpha_n \to 0 \) and the boundedness of \( \{x_n\} \) it follows that
\[ \lim_{n \to \infty} \|x_n - \tilde{T}x_n\| = 0. \quad (6.3.50) \]

Taking into account \( x_{n_j} \to \hat{x} \) and utilizing Lemma 2.1.13, we obtain \( \hat{x} \in \text{Fix}(\tilde{T}) \). But \( \text{Fix}(\tilde{T}) = \Theta \); we therefore have \( \hat{x} \in \Theta \). Furthermore, since \( x_{n_j} \to \hat{x} \) and
\[ \lim_{n \to \infty} \|u_n - x_n\| = \lim_{n \to \infty} \|u_n - v_n\| = \lim_{n \to \infty} \|Tv_n - v_n\| = 0, \]
it is known that \( u_{n_j} \to \hat{x} \) and \( \lim_{n \to \infty} \|Tv_{n_j} - v_{n_j}\| = 0 \). Thus, from Lemma 2.1.13 we get \( \hat{x} \in \text{Fix}(T) \). Therefore, we have \( \hat{x} \in \text{Fix}(T) \cap \Theta \). This shows that there holds
\[ \omega_w(x_n) \subset \text{Fix}(T) \cap \Theta. \quad (6.3.51) \]

**Step 4.** \( \omega_w(x_n) \subset \Xi. \)

Indeed, we first note that \( 0 < \gamma \leq \tau \) and from equation (6.3.25). It is clear that
\[
\langle (\mu F - \gamma S)x - (\mu F - \gamma S)y, x - y \rangle \geq (\mu \gamma - \gamma)\|x - y\|^2, \quad \forall x, y \in C.
\]
Hence, it follows from $0 < \gamma \leq \tau \leq \mu n$ that $\mu F - \gamma S$ is monotone. Putting

$$w_n = \lambda_n \gamma(\delta_n V x_n + (1 - \delta_n)S x_n) + (I - \lambda_n \mu F)T z_n, \quad \forall n \geq 0.$$ 

From (6.3.9)

$$x_{n+1} = P_C w_n - w_n + \lambda_n \gamma(\delta_n V x_n + (1 - \delta_n)S x_n) + (I - \lambda_n \mu F)T z_n,$$

we obtain

$$x_n - x_{n+1}$$

$$= w_n - P_C w_n + \delta_n \lambda_n(\mu F - \gamma V)x_n + \lambda_n(1 - \delta_n)(\mu F - \gamma S)x_n$$

$$+ (1 - \lambda_n)(I - T)x_n + \lambda_n[(I - \mu F)x_n - (I - \mu F)T x_n]$$

$$+ (I - \lambda_n \mu F)T z_n - (I - \lambda_n \mu F)T z_n.$$  \hspace{1cm} (6.3.52)

Set $e_n = \frac{x_n - x_{n+1}}{\lambda_n(1 - \delta_n)}$, $\forall n \geq 0$. It follows from (6.3.52) that

$$e_n = \frac{w_n - P_C w_n}{\lambda_n(1 - \delta_n)} + (\mu F - \gamma S)x_n + \frac{\delta_n}{1 - \delta_n}(\mu F - \gamma V)x_n + \frac{1 - \lambda_n}{\lambda_n(1 - \delta_n)}(I - T)x_n$$

$$+ \frac{1}{1 - \delta_n}[(I - \mu F)x_n - (I - \mu F)T x_n] + \frac{(I - \lambda_n \mu F)T z_n - (I - \lambda_n \mu F)T z_n}{\lambda_n(1 - \delta_n)}.$$

This yields that, for all $w \in Fix(T) \cap \Theta$ (noticing $x_n = P_C w_{n-1}$),

$$(e_n, x_n - w)$$

$$= \frac{1}{\lambda_n(1 - \delta_n)}(w_n - P_C w_n, P_C w_{n-1} - w) + \langle (\mu F - \gamma S)x_n, x_n - w \rangle$$

$$+ \frac{\delta_n}{1 - \delta_n}(\mu F - \gamma V)x_n, x_n - w \rangle + \frac{1 - \lambda_n}{\lambda_n(1 - \delta_n)}(I - T)x_n - (I - T)w, x_n - w \rangle$$

$$+ \frac{1}{1 - \delta_n}[(I - \mu F)x_n - (I - \mu F)T x_n, x_n - w \rangle$$

$$+ \frac{1}{\lambda_n(1 - \delta_n)}[(I - \lambda_n \mu F)T z_n - (I - \lambda_n \mu F)T z_n, x_n - w \rangle$$

$$= \frac{1}{\lambda_n(1 - \delta_n)}(w_n - P_C w_n, P_C w_{n-1} - P_C w_n) + \frac{1}{\lambda_n(1 - \delta_n)}(w_n - P_C w_n, P_C w_{n-1} - P_C w_n)$$

$$+ \langle (\mu F - \gamma S)w, x_n - w \rangle + \langle (\mu F - \gamma S)x_n - (\mu F - \gamma S)w, x_n - w \rangle$$
\begin{align*}
&\frac{1 - \lambda_n}{\lambda_n(1 - \delta_n)} ((I - T)x_n - (I - T)w, x_n - w) + \frac{\delta_n}{1 - \delta_n} ((\mu F - \gamma V)x_n, x_n - w) \\
&+ \frac{1}{1 - \delta_n} ((I - \mu F)x_n - (I - \mu F)Tx_n, x_n - w) \\
&+ \frac{1}{\lambda_n(1 - \delta_n)} ((I - \lambda_n \mu F)Tx_n - (I - \lambda_n \mu F)Tz_n, x_n - w). \tag{6.3.53}
\end{align*}

In (6.3.53), the first term is nonnegative due to Propositions 2.2.1 and 2.2.2, and the fourth and fifth terms are also nonnegative due to the monotonicity of $\mu F - \gamma S$ and $I - T$. We, therefore, deduce from (6.3.53) that (noticing again $x_{n+1} = P_C w_n$)

\begin{align*}
&\langle e_n, x_n - w \rangle \\
&\geq \frac{1}{\lambda_n(1 - \delta_n)} \langle w_n - P_C w_n, P_C w_{n-1} - P_C w_n \rangle + \langle (\mu F - \gamma S)w, x_n - w \rangle \\
&+ \frac{\delta_n}{1 - \delta_n} \langle (\mu F - \gamma V)x_n, x_n - w \rangle + \frac{1}{1 - \delta_n} \langle (I - \mu F)x_n - (I - \mu F)Tx_n, x_n - w \rangle \\
&+ \frac{1}{\lambda_n(1 - \delta_n)} \langle (I - \lambda_n \mu F)Tx_n - (I - \lambda_n \mu F)Tz_n, x_n - w \rangle. \tag{6.3.54}
\end{align*}

Note that

\[\|(I - \lambda_n \mu F)Tx_n - (I - \lambda_n \mu F)Tz_n\| \leq (1 - \lambda_n \tau)\|x_n - z_n\|,\]

and

\[\|(I - \mu F)x_n - (I - \mu F)Tx_n\| \leq (1 + \mu \kappa)\|x_n - Tx_n\|,\]

Hence it follows from $\|x_n - z_n\| = o(\lambda_n)$ and $\|x_n - Tx_n\| \to 0$ that

\[\|(I - \lambda_n \mu F)Tx_n - (I - \lambda_n \mu F)Tz_n\| \to 0 \text{ and } \|(I - \mu F)x_n - (I - \mu F)Tx_n\| \to 0,\]

respectively. Also, since $e_n \to 0$ (due to $\|x_{n+1} - x_n\| = o(\lambda_n)$), $\delta_n \to 0$ and $\{x_n\}$ is bounded by Step 1 which implies that $\{u_n\}$ is bounded, we obtain from (6.3.54) that

\[\limsup_{n \to \infty} \langle (\mu F - \gamma S)w, x_n - w \rangle \leq 0, \quad \forall w \in \text{Fix}(T) \cap \Theta. \tag{6.3.55}\]
This suffices to guarantee that $\omega_w(x_n) \subseteq \Xi$; namely, every weak limit point of $\{x_n\}$ solves the HVIP (6.3.4). As a matter of fact, if $x_{n_i} \rightharpoonup \tilde{x} \in \omega_w(x_n)$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$, then we deduce from (6.3.55) that

$$
\langle (\mu F - \gamma S)w, \tilde{x} - w \rangle \leq \limsup_{n \to \infty} \langle (\mu F - \gamma S)w, x_n - w \rangle \leq 0, \quad \forall w \in \text{Fix}(T) \cap \Theta,
$$

that is,

$$
\langle (\mu F - \gamma S)w, w - \tilde{x} \rangle \geq 0, \quad \forall w \in \text{Fix}(T) \cap \Theta.
$$

In addition, note that $\omega_w(x_n) \subseteq \text{Fix}(T) \cap \Theta$ by Step 3. Since $\mu F - \gamma S$ is monotone and Lipschitz continuous, and $\text{Fix}(T) \cap \Theta$ is nonempty, closed and convex, by the Minty lemma the last inequality is equivalent to the inequality (6.3.4). Thus, we get $\tilde{x} \in \Xi$.

Step 5. $\{x_n\}$ converges strongly to a unique solution $x^*$ of Problem 6.3.2.

Indeed, we now take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ satisfying

$$
\limsup_{n \to \infty} \langle (\mu F - \gamma V)x^*, x_n - x^* \rangle = \lim_{i \to \infty} \langle (\mu F - \gamma V)x^*, x_{n_i} - x^* \rangle.
$$

Without loss of generality, we may further assume that $x_{n_i} \rightharpoonup \tilde{x}$; then $\tilde{x} \in \Xi$ as we just proved. Since $x^*$ is a solution of the THVIP (6.3.3), we get

$$
\limsup_{n \to \infty} \langle (\mu F - \gamma V)x^*, x_n - x^* \rangle = \langle (\mu F - \gamma V)x^*, \tilde{x} - x^* \rangle \geq 0. \quad (6.3.56)
$$

From (6.3.9), it follows from (6.3.44) that (noticing that $x_{n+1} = P_C w_n$ and $0 < \gamma \leq \tau$)

$$
\|x_{n+1} - x^*\|^2
$$

$$
= \langle w_n - x^*, x_{n+1} - x^* \rangle + \langle P_C w_n - w_n, P_C w_n - x^* \rangle
$$

$$
\leq \langle w_n - x^*, x_{n+1} - x^* \rangle
$$

$$
= ((I - \lambda_n \mu F)Tx_n - (I - \lambda_n \mu F)x^*, x_{n+1} - x^*)
$$

$$
+ \delta_n \lambda_n \gamma \langle Vx_n - Vx^*, x_{n+1} - x^* \rangle + \lambda_n (1 - \delta_n) \gamma \langle Sx_n - Sx^*, \tilde{x}_{n+1} - x^* \rangle
$$

$$
+ \delta_n \lambda_n \langle (\gamma V - \mu F)x^*, x_{n+1} - x^* \rangle + \lambda_n (1 - \delta_n) \langle (\gamma S - \mu F)x^*, x_{n+1} - x^* \rangle
$$
\[ (1 - \lambda_n \tau) \|x_n - x^*\| \|x_{n+1} - x^*\| + [\delta_n \lambda_n \gamma \rho + \lambda_n (1 - \delta_n) \gamma] \|x_n - x^*\| \|x_{n+1} - x^*\| \\
+ \delta_n \lambda_n ((\gamma V - \mu F)x^*, x_{n+1} - x^*) + \lambda_n (1 - \delta_n) ((\gamma S - \mu F)x^*, x_{n+1} - x^*) \\
\leq (1 - \lambda_n \tau)^{1/2} (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
+ [\delta_n \lambda_n \gamma \rho + \lambda_n (1 - \delta_n) \gamma^{1/2}] (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
+ \delta_n \lambda_n ((\gamma V - \mu F)x^*, x_{n+1} - x^*) + \lambda_n (1 - \delta_n) ((\gamma S - \mu F)x^*, x_{n+1} - x^*) \\
\leq (1 - \lambda_n \tau)^{1/2} (\|x_n - x^*\|^2 + \alpha_n (M_1 + M_2) + \|x_{n+1} - x^*\|^2) \\
+ [\delta_n \lambda_n \gamma \rho + \lambda_n (1 - \delta_n) \gamma^{1/2}] (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
+ \delta_n \lambda_n ((\gamma V - \mu F)x^*, x_{n+1} - x^*) + \lambda_n (1 - \delta_n) ((\gamma S - \mu F)x^*, x_{n+1} - x^*) \\
\leq [1 - \delta_n \lambda_n \gamma (1 - \rho)]^{1/2} (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \delta_n \lambda_n ((\gamma V - \mu F)x^*, x_{n+1} - x^*) \\
+ \lambda_n (1 - \delta_n) ((\gamma S - \mu F)x^*, x_{n+1} - x^*) + \alpha_n (M_1 + M_2). \tag{6.3.57} \]

It turns out that
\[ \|x_{n+1} - x^*\|^2 \]
\[ \leq \frac{1 - \delta_n \lambda_n \gamma (1 - \rho)}{1 + \delta_n \lambda_n \gamma (1 - \rho)} \|x_n - x^*\|^2 + \frac{2}{1 + \delta_n \lambda_n \gamma (1 - \rho)} [\delta_n \lambda_n ((\gamma V - \mu F)x^*, x_{n+1} - x^*) \\
+ \lambda_n (1 - \delta_n) ((\gamma S - \mu F)x^*, x_{n+1} - x^*) + \alpha_n (M_1 + M_2)] \\
\leq [1 - \delta_n \lambda_n \gamma (1 - \rho)] \|x_n - x^*\|^2 + \frac{2}{1 + \delta_n \lambda_n \gamma (1 - \rho)} [\delta_n \lambda_n ((\gamma V - \mu F)x^*, x_{n+1} - x^*) \\
+ \lambda_n (1 - \delta_n) ((\gamma S - \mu F)x^*, x_{n+1} - x^*) + 2 \alpha_n (M_1 + M_2). \tag{6.3.57} \]

However, from \( x^* \in \Xi \) and condition (C5) we obtain that
\[ ((\gamma S - \mu F)x^*, x_{n+1} - x^*) \]
\[ = ((\gamma S - \mu F)x^*, x_{n+1} - P_{Fix(T) \cap \Theta} x_{n+1}) + ((\gamma S - \mu F)x^*, P_{Fix(T) \cap \Theta} x_{n+1} - x^*) \]
\[ \leq ((\gamma S - \mu F)x^*, x_{n+1} - P_{Fix(T) \cap \Theta} x_{n+1}) \]
\[ \leq \| (\gamma S - \mu F)x^* \| d(x_{n+1}, Fix(T) \cap \Theta) \]
\[ \leq \| (\gamma S - \mu F)x^* \| (\frac{1}{k} \| x_{n+1} - Tx_{n+1} \|)^{1/\theta}. \tag{6.3.58} \]

On the other hand, we also have
\[ \|x_{n+1} - T x_{n+1}\| \]
\[ \leq \|x_{n+1} - T x_n\| + \|T x_n - T x_{n+1}\| \]
\[ \leq \|x_n - x_{n+1}\| + \|\lambda_n (\delta_n V x_n + (1 - \delta_n)S x_n) + (1 - \lambda_n)T x_n - T x_n\| \]
\[ \leq \|x_n - x_{n+1}\| + \|T x_n - T x_n\| + \lambda_n \|\gamma (\delta_n V x_n + (1 - \delta_n)S x_n) - \mu F T x_n\| \]
\[ = \|x_n - x_{n+1}\| + \|T x_n - T x_n\| + \lambda_n \|\gamma \delta_n (V x_n - S x_n) + \gamma S x_n - \mu F T x_n\| \]
\[ \leq \|x_n - x_{n+1}\| + \|z_n - x_n\| + \lambda_n \|M_0\|. \quad (6.3.59) \]

Hence, for a big enough constant \( \bar{k}_1 > 0 \), we have

\[ \langle (\gamma S - \mu F) x^*, x_{n+1} - x^* \rangle \]
\[ \leq \bar{k}_1 (\lambda_n + \|x_n - x_{n+1}\| + \|z_n - x_n\|)^1/\theta \]
\[ \leq \bar{k}_1 \lambda_n^{1/\theta} (1 + \frac{\|x_n - x_{n+1}\| + \|z_n - x_n\|}{\lambda_n})^{1/\theta}. \quad (6.3.60) \]

Combining (6.3.57) - (6.3.60), we get

\[ \|x_{n+1} - x^*\|^2 \]
\[ \leq \left[ 1 - \delta_n \lambda_n \gamma (1 - \rho) \right] \|x_n - x^*\|^2 + \frac{2}{1 + \delta_n \lambda_n \gamma (1 - \rho)} \delta_n \lambda_n \langle (\gamma V - \mu F) x^*, x_{n+1} - x^* \rangle + \lambda_n \langle (\gamma S - \mu F) x^*, x_{n+1} - x^* \rangle + 2\alpha_n (M_1 + M_2) \]
\[ \leq \left[ 1 - \delta_n \lambda_n \gamma (1 - \rho) \right] \|x_n - x^*\|^2 + \frac{2\delta_n \lambda_n}{1 + \delta_n \lambda_n \gamma (1 - \rho)} \langle (\gamma V - \mu F) x^*, x_{n+1} - x^* \rangle \]
\[ + \frac{\bar{k}_1 \lambda_n^{1/\theta}}{\delta_n} (1 + \frac{\|x_n - x_{n+1}\| + \|z_n - x_n\|}{\lambda_n})^{1/\theta} + 2\alpha_n (M_1 + M_2) \]
\[ = (1 - \mu_n) \|x_n - x^*\|^2 + \nu_n + 2\alpha_n (M_1 + M_2), \quad (6.3.61) \]

where \( \mu_n = \delta_n \lambda_n \gamma (1 - \rho) \) and

\[ \nu_n = \frac{2\delta_n \lambda_n}{1 + \delta_n \lambda_n \gamma (1 - \rho)} \left[ \langle (\gamma V - \mu F) x^*, x_{n+1} - x^* \rangle + \frac{\bar{k}_1 \lambda_n^{1/\theta}}{\delta_n} (1 + \frac{\|x_n - x_{n+1}\| + \|z_n - x_n\|}{\lambda_n})^{1/\theta} \right]. \]
Now condition (C4) implies that $\sum_{n=1}^{\infty} \mu_n = \infty$. Moreover, since $\|x_{n+1} - x_n\| + \|x_n - x_n\| = o(\lambda_n)$, condition (C6) and (6.3.56) imply that

$$\limsup_{n \to \infty} \frac{\nu_n}{\mu_n} \leq 0.$$ 

Therefore, we can apply Lemma 2.1.2 to (6.3.61) to conclude that $x_n \to x^*$. The proof of part (a) is complete.

It is easy to see that part (b) now becomes a straightforward consequence of part (a) since, if $V = 0$, THVIP (6.3.3) reduces to the VIP in part (b). This completes the proof. $\square$

Next we consider a special case of Problem 6.3.2. In Problem 6.3.2, put $\mu = 2$, $F = \frac{1}{2}I$ and $\gamma = \tau = 1$. In this case, the objective is to find $x^* \in \Xi$ such that

$$((I - V)x^*, x - x^*) \geq 0, \quad \forall x \in \Xi,$$

where $\Xi$ denotes the solution set of the following hierarchical variational inequality problem (HVIP) of finding $z^* \in \text{Fix}(T) \cap \Theta$ such that

$$((I - S)z^*, z - z^*) \geq 0, \quad \forall z \in \text{Fix}(T) \cap \Theta.$$

**Corollary 6.3.2.** Let $A : C \to H$ be a $1/L$-inverse strongly monotone mapping with $0 < \lambda < 2/L$, $V : C \to H$ be a $p$-contraction with coefficient $p \in [0, 1)$ and $S, T : C \to C$ be nonexpansive mappings. Assume that the solution set $\Xi$ of the HVIP (6.3.37) is nonempty and that the following conditions hold for five sequences $\{\alpha_n\} \subset [0, \infty)$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ and $\{\lambda_n\}$, $\{\delta_n\} \subset (0, 1)$:

(C1) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(C2) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$;

(C3) $0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1$;

(C4) $\lim_{n \to \infty} \lambda_n = 0$, $\lim_{n \to \infty} \delta_n = 0$ and $\sum_{n=0}^{\infty} \delta_n \lambda_n = \infty$.
(C5) there are constants $\bar{K}, \theta > 0$ such that $\|x - Tz\| \geq \bar{K}[d(x, Fix(T) \cap \Theta)]^\theta, \forall x \in C$;

(C6) $\lim_{n \to \infty} {\delta_n \over \lambda_n} = 0$.

Then, the following assertions hold.

(a) If $\{x_n\}$ is the sequence generated by the iterative scheme

$$
\begin{align*}
x_0 &= x \in C \text{ chosen arbitrarily}, \\
y_n &= (1 - \beta_n)x_n + \beta_n P_C(y_n - \lambda A_n y_n), \\
z_n &= \gamma_n y_n + (1 - \gamma_n)TP_C(z_n - \lambda A_n z_n), \\
x_{n+1} &= P_C[\lambda_n(\beta_n V x_n + (1 - \delta_n)S x_n) + (1 - \lambda_n)Tz_n], \quad \forall n \geq 0
\end{align*}
$$

and $\{S x_n\}$ is bounded, then $\{x_n\}$ converges strongly to the point $x^* \in Fix(T) \cap \Theta$ which is a unique solution of THVIP (6.3.36) provided $\|x_{n+1} - x_n\| + \|x_n - z_n\| = o(\lambda_n)$.

(b) If $\{x_n\}$ is the sequence generated by the iterative scheme

$$
\begin{align*}
x_0 &= x \in C \text{ chosen arbitrarily}, \\
y_n &= (1 - \beta_n)x_n + \beta_n P_C(y_n - \lambda A_n y_n), \\
z_n &= \gamma_n y_n + (1 - \gamma_n)TP_C(z_n - \lambda A_n z_n), \\
x_{n+1} &= P_C[\lambda_n(1 - \delta_n)S x_n + (1 - \lambda_n)Tz_n], \quad \forall n \geq 0
\end{align*}
$$

and $\{S x_n\}$ is bounded, then $\{x_n\}$ converges strongly to a unique solution $x^*$ of the following VIP provided $\|x_{n+1} - x_n\| + \|x_n - z_n\| = o(\lambda_n)$:

find $x^* \in \Xi$ such that $\langle x^*, x - x^* \rangle \geq 0, \forall x \in \Xi$,

that is, $x^*$ is the minimum-norm solution of HVIP (6.3.37).

6.4 System of Hierarchical Variational Inequalities

We consider the following problems of system of hierarchical variational inequalities (in short, SHVI).
Problem 6.4.1. Let $F : C \to H$ be $\kappa$-Lipschitzian and $\eta$-strongly monotone on the nonempty, closed and convex subset $C$ of $H$, where $\kappa$ and $\eta$ are positive constants. Let $A : C \to H$ be a monotone and $L$-Lipschitzian mapping, $V : C \to H$ be a $\rho$-contraction with coefficient $\rho \in [0,1)$ and $S,T : C \to C$ be nonexpansive mappings such that $\text{Fix}(T) \cap \Theta \neq \emptyset$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Then the objective is to find $x^* \in \text{Fix}(T) \cap \Theta$ such that

\[
\begin{cases}
\langle (\mu F - \gamma V)x^*, x - x^* \rangle \geq 0, & \forall x \in \text{Fix}(T) \cap \Theta, \\
\langle (\mu F - \gamma S)x^*, y - x^* \rangle \geq 0, & \forall y \in \text{Fix}(T) \cap \Theta.
\end{cases}
\tag{6.4.1}
\]

In particular, if $T \equiv T_1$ and $A \equiv I - T_2$, where $T_2 : C \to C$ is $\zeta_2$-strictly pseudo-contractive, Problem 6.4.1 reduces to the following:

Problem 6.4.2. Let $F : C \to H$ be $\kappa$-Lipschitzian and $\eta$-strongly monotone on the nonempty, closed and convex subset $C$ of $H$, where $\kappa$ and $\eta$ are positive constants. Let $V : C \to H$ be a $\rho$-contraction with coefficient $\rho \in [0,1)$, $S,T_1 : C \to C$ be nonexpansive mappings and $T_2 : C \to C$ be $\zeta_2$-strictly pseudo-contractive mapping such that $\text{Fix}(T_1) \cap \text{Fix}(T_2) \neq \emptyset$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Then the objective is to find $x^* \in \text{Fix}(T_1) \cap \text{Fix}(T_2)$ such that

\[
\begin{cases}
\langle (\mu F - \gamma V)x^*, x - x^* \rangle \geq 0, & \forall x \in \text{Fix}(T_1) \cap \text{Fix}(T_2), \\
\langle (\mu F - \gamma S)x^*, y - x^* \rangle \geq 0, & \forall y \in \text{Fix}(T_1) \cap \text{Fix}(T_2).
\end{cases}
\tag{6.4.2}
\]

We derive the following strong convergence result of Algorithm 6.3.1 for finding a unique solution of Problem 6.4.1.

Theorem 6.4.1. Let $F : C \to H$ be a $\kappa$-Lipschitzian and $\eta$-strongly monotone operator with constants $\kappa, \eta > 0$, respectively, $A : C \to H$ be a $1/L$-inverse strongly monotone mapping, $V : C \to H$ be a $\rho$-contraction with coefficient $\rho \in [0,1)$ and $S,T : C \to C$ be nonexpansive mappings. Let $0 < \lambda < 2/L$, $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that Problem 6.4.1 has a solution and that the
following conditions hold for five sequences \( \{\alpha_n\} \subset [0, \infty), \{\beta_n\}, \{\gamma_n\} \subset [0, 1] \) and \( \{\lambda_n\}, \{\delta_n\} \subset (0, 1) \):

(C1) \( \sum_{n=0}^{\infty} \alpha_n < \infty \);

(C2) \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \);

(C3) \( 0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1 \);

(C4) \( 0 < \liminf_{n \to \infty} \delta_n \leq \limsup_{n \to \infty} \delta_n < 1 \);

(C5) \( \lim_{n \to \infty} \lambda_n = 0 \) and \( \sum_{n=0}^{\infty} \lambda_n = \infty \).

Then, the following assertions hold.

(a) If \( \{x_n\} \) is the sequence generated by the scheme (6.3.5) and \( \{Sx_n\} \) is bounded, then \( \{x_n\} \) converges strongly to a unique solution of Problem 6.4.1 provided \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \).

(b) If \( \{x_n\} \) is the sequence generated by the scheme (6.3.6) and \( \{Sx_n\} \) is bounded, then \( \{x_n\} \) converges strongly to a unique solution \( x^* \in \text{Fix}(T) \cap \Theta \) of the following system of variational inequalities provided \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \):

\[
\begin{align*}
(Fx^*, x - x^*) & \geq 0, \quad \forall x \in \text{Fix}(T) \cap \Theta, \\
((\mu F - \gamma S)x^*, y - x^*) & \geq 0, \quad \forall y \in \text{Fix}(T) \cap \Theta.
\end{align*}
\]

(6.4.3)

Proof. We only prove (a), that is, the sequence \( \{x_n\} \) is generated by the scheme (6.3.5). First of all, it is seen easily that \( 0 < \gamma \leq \tau \) and \( \kappa \geq \eta \Leftrightarrow \mu \eta \geq \tau \). Hence it follows from the \( \rho \)-contractiveness of \( V \) and \( \gamma \rho < \gamma \leq \tau \leq \mu \eta \) that \( \mu F - \gamma V \) is \( (\mu \eta - \gamma \rho) \)-strongly monotone and Lipschitz continuous. So, there exists a unique solution \( x^* \) of the following VIP:

find \( x^* \in \text{Fix}(T) \cap \Theta \) such that \( ((\mu F - \gamma V)x^*, x - x^*) \geq 0, \quad \forall x \in \text{Fix}(T) \cap \Theta \).
Consequently, it is easy to see that Problem 6.4.1 has a unique solution $x^* \in \text{Fix}(T) \cap \Theta$. In addition, taking into account condition (C4), without loss of generality we may assume that \( \{\delta_n\} \subset [a, b] \) for some \( a, b \in (0, 1) \).

Next we divide the rest of the proof into several steps.

Step 1. \( \{x_n\} \) is bounded.

Indeed, repeating the same argument as in Step 1 of the proof of Theorem 6.3.1 we can derive the claim.

Step 2. \( \lim_{n \to \infty} \|w_n - x_n\| = \lim_{n \to \infty} \|x_n - y_n\| = \lim_{n \to \infty} \|v_n - Tv_n\| = 0 \), where \( w_n = P_C(I - \lambda A_n)x_n \) and \( v_n = P_C(I - \lambda A_n)y_n \).

Indeed, repeating the same argument as in Step 2 of the proof of Theorem 6.3.1 we can derive the claim.

Step 3. \( \omega(w(x_n)) \subset \text{Fix}(T) \cap \Theta \).

Indeed, repeating the same argument as in Step 3 of the proof of Theorem 6.3.1 we can derive the claim.

Step 4. \( \{x_n\} \) converges strongly to a unique solution $x^*$ of Problem 6.4.1.

Indeed, according to \( \|x_{n+1} - x_n\| \to 0 \) we can take a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) satisfying

\[
\limsup_{n \to \infty} ((\gamma V - \mu F)x^*, x_{n+1} - x^*) = \limsup_{n \to \infty} ((\gamma V - \mu F)x^*, x_n - x^*) = \lim_{i \to \infty} ((\gamma V - \mu F)x^*, x_{n_i} - x^*). \tag{6.4.4}
\]

Without loss of generality, we may further assume that \( x_{n_i} \to \bar{x} \), then \( \bar{x} \in \text{Fix}(T) \cap \Theta \) due to Step 3. Since $x^*$ is a solution of Problem 6.4.1, we get

\[
\limsup_{n \to \infty} ((\gamma V - \mu F)x^*, x_{n+1} - x^*) = ((\gamma V - \mu F)x^*, \bar{x} - x^*) \leq 0. \tag{6.4.5}
\]

Repeating the same argument as that of (6.4.5), we have

\[
\limsup_{n \to \infty} ((\gamma S - \mu F)x^*, x_{n+1} - x^*) \leq 0. \tag{6.4.6}
\]
Repeating the same argument as (6.3.31) in the proof of Theorem 6.3.1, we obtain

\[
\|x_{n+1} - x^*\|^2 \\
\leq [1 - \delta_n \lambda_n \gamma(1 - \rho)] \|x_n - x^*\|^2 + \frac{2}{1 + \delta_n \lambda_n \gamma(1 - \rho)} [\delta_n \lambda_n ((\gamma V - \mu F)x^*, x_{n+1} - x^*) \\
+ \lambda_n (1 - \delta_n)((\gamma S - \mu F)x^*, x_{n+1} - x^*)] + 2\alpha_n (M_1 + M_2).
\]  

(6.4.7)

Put \( \sigma_n = 2\alpha_n (M_1 + M_2) \), \( s_n = \delta_n \lambda_n \gamma(1 - \rho) \) and

\[
t_n = \frac{2}{\gamma(1 - \rho)[1 + \delta_n \lambda_n \gamma(1 - \rho)]} ((\gamma V - \mu F)x^*, x_{n+1} - x^*) \\
+ \frac{2(1 - \delta_n)}{\delta_n \gamma(1 - \rho)[1 + \delta_n \lambda_n \gamma(1 - \rho)]} ((\gamma S - \mu F)x^*, x_{n+1} - x^*).
\]

Then, the inequality (6.4.6) can be rewritten as

\[
\|x_{n+1} - x^*\|^2 \leq (1 - s_n)\|x_n - x^*\|^2 + s_n t_n + \sigma_n.
\]

In terms of conditions (C4) and (C5), we conclude from \( 0 < 1 - \rho \leq 1 \) that

\[
\{s_n\} \subset (0,1) \quad \text{and} \quad \sum_{n=0}^{\infty} s_n = \infty.
\]

Note that

\[
\frac{2}{\gamma(1 - \rho)[1 + \delta_n \lambda_n \gamma(1 - \rho)]} \leq \frac{2}{\gamma(1 - \rho)}
\]

and

\[
\frac{2(1 - \delta_n)}{\delta_n \gamma(1 - \rho)[1 + \delta_n \lambda_n \gamma(1 - \rho)]} \leq \frac{2}{\alpha \gamma(1 - \rho)}.
\]

Consequently, utilizing Lemma 2.1.4 we obtain that

\[
\limsup_{n \to \infty} t_n \leq \limsup_{n \to \infty} \frac{2}{\gamma(1 - \rho)[1 + \delta_n \lambda_n \gamma(1 - \rho)]} ((\gamma V - \mu F)x^*, x_{n+1} - x^*) \\
+ \limsup_{n \to \infty} \frac{2(1 - \delta_n)}{\delta_n \gamma(1 - \rho)[1 + \delta_n \lambda_n \gamma(1 - \rho)]} ((\gamma S - \mu F)x^*, x_{n+1} - x^*) \\
\leq 0.
\]

(6.4.8)

So, this together with Lemma 2.1.2 leads to \( \lim_{n \to \infty} \|x_n - x^*\| = 0 \). The proof is complete. \( \square \)
Utilizing Theorem 6.4.1 we immediately derive the following result.

**Corollary 6.4.1.** Let \( F : C \rightarrow H \) be a \( \kappa \)-Lipschitzian and \( \eta \)-strongly monotone operator with constants \( \kappa, \eta > 0 \), respectively, \( A : C \rightarrow H \) be a \( 1/L \)-inverse strongly monotone mapping, \( V : C \rightarrow H \) be a \( \mu \)-contraction with coefficient \( \mu \in [0, 1) \) and \( T : C \rightarrow C \) be a nonexpansive mapping such that \( \text{Fix}(T) \cap \Theta \neq \emptyset \). Let \( 0 < \lambda < 2/L \), \( 0 < \mu < 2\eta/\kappa^2 \) and \( 0 < \gamma \leq \tau \), where \( \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \). Assume that the following conditions hold for four sequences \( \{\alpha_n\} \subset [0, \infty), \{\beta_n\}, \{\gamma_n\} \subset [0, 1] \) and \( \{\lambda_n\} \subset (0, 1) \):

1. \( \sum_{n=0}^{\infty} \alpha_n < \infty \);
2. \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \);
3. \( 0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1 \);
4. \( \lim_{n \to \infty} \lambda_n = 0 \) and \( \sum_{n=0}^{\infty} \lambda_n = \infty \).

Then, the following assertions hold.

(a) If \( \{x_n\} \) is the sequence generated by the scheme (6.3.7) and \( \{Vx_n\} \) is bounded, then \( \{x_n\} \) converges strongly to a unique solution of the following VIP provided

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\]

\( \text{find } x^* \in \text{Fix}(T) \cap \Theta \) such that

\[
((\mu F - \gamma V)x^*, x - x^*) \geq 0, \quad \forall x \in \text{Fix}(T) \cap \Theta. \tag{6.4.9}
\]

(b) If \( \{x_n\} \) is the sequence generated by the scheme (6.3.8), then \( \{x_n\} \) converges strongly to a unique solution of the following VIP provided

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\]

\( \text{find } x^* \in \text{Fix}(T) \cap \Theta \) such that

\[
\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(T) \cap \Theta. \tag{6.4.10}
\]
Proof. In Theorem 6.4.1, putting $S = V$, we know that the iterative scheme (6.3.5) reduces to (6.3.7) since there holds for any $\{\delta_n\} \subset (0, 1)$

$$x_{n+1} = P_C[\lambda_n \gamma(\delta_n Vx_n + (1 - \delta_n)Sx_n) + (I - \lambda_n \mu F)Tz_n]$$

$$= P_C[\lambda_n \gamma(\delta_n Vx_n + (1 - \delta_n)Vx_n) + (I - \lambda_n \mu F)Tz_n]$$

$$= P_C[\lambda_n \gamma Vx_n + (I - \lambda_n \mu F)Tz_n].$$

In this case, the SHVI (6.4.1) is equivalent to the VIP (6.4.9). Thus, utilizing Theorem 6.4.1 (a) we obtain the desired conclusion (a). As for the conclusion (b), we immediately derive it from $S = V \equiv 0$ and Theorem 6.4.1 (b).

By repeating the argument similar to that in the proof of Theorem 6.3.2 we can derive the following strong convergence result of Algorithm 6.3.1 for finding a unique solution of Problem 6.4.1.

**Theorem 6.4.2.** Let $F : C \rightarrow H$ be a $\kappa$-Lipschitzian and $\eta$-strongly monotone operator with constants $\kappa, \eta > 0$, respectively, $A : C \rightarrow H$ be a $1/L$-inverse strongly monotone mapping, $V : C \rightarrow H$ be a $\rho$-contraction with coefficient $\rho \in [0, 1)$ and $S, T : C \rightarrow C$ be nonexpansive mappings. Let $0 < \lambda < 2/L$, $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu \kappa^2)}$. Assume that Problem 6.4.1 has a solution and that the following conditions hold for five sequences $\{\alpha_n\} \subset [0, \infty)$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1)$ and $\{\lambda_n\}$, $\{\delta_n\} \subset (0, 1)$:

(C1) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(C2) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;

(C3) $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$;

(C4) $0 < \liminf_{n \rightarrow \infty} \delta_n \leq \limsup_{n \rightarrow \infty} \delta_n < 1$;

(C5) $\lim_{n \rightarrow \infty} \lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$. 

Then, the following assertions hold.

(a) If \( \{x_n\} \) is the sequence generated by the scheme (6.3.9) and \( \{Sx_n\} \) is bounded, then \( \{x_n\} \) converges strongly to a unique solution of Problem 6.4.1 provided \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \).

(b) If \( \{x_n\} \) is the sequence generated by the scheme (6.3.10) and \( \{Sx_n\} \) is bounded, then \( \{x_n\} \) converges strongly to a unique solution \( x^* \in \text{Fix}(T) \cap \Theta \) of the following SHVI provided \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \):

\[
\begin{align*}
&\quad (Fx^*, x - x^*) \geq 0, \quad \forall x \in \text{Fix}(T) \cap \Theta, \\
&\quad ((\mu F - \gamma S)x^*, y - x^*) \geq 0, \quad \forall y \in \text{Fix}(T) \cap \Theta.
\end{align*}
\tag{6.4.11}
\]

Utilizing Theorem 6.4.2 we immediately derive the following result.

**Corollary 6.4.2.** Let \( F : C \to H \) be a \( \kappa \)-Lipschitzian and \( \eta \)-strongly monotone operator with constants \( \kappa, \eta > 0 \), respectively, \( A : C \to H \) be a \( 1/L \)-inverse strongly monotone mapping, \( V : C \to H \) be a \( \rho \)-contraction with coefficient \( \rho \in [0, 1) \) and \( T : C \to C \) be a nonexpansive mapping such that \( \text{Fix}(T) \cap \Theta \neq \emptyset \). Let \( 0 < \lambda < 2/L, \ 0 < \mu < 2\eta/\kappa^2 \) and \( 0 < \gamma \leq \tau, \) where \( \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \). Assume that the following conditions hold for four sequences \( \{\alpha_n\} \subset (0, \infty), \{\beta_n\}, \{\gamma_n\} \subset [0, 1) \) and \( \{\lambda_n\} \subset (0, 1) \):

1. \( \sum_{n=0}^{\infty} \alpha_n < \infty; \)

2. \( 0 < \lim \inf_{n \to \infty} \beta_n \leq \lim \sup_{n \to \infty} \beta_n < 1; \)

3. \( 0 < \lim \inf_{n \to \infty} \gamma_n \leq \lim \sup_{n \to \infty} \gamma_n < 1; \)

4. \( \lim_{n \to \infty} \lambda_n = 0 \) and \( \sum_{n=0}^{\infty} \lambda_n = \infty. \)

Then, the following assertions hold.
(a) If \( \{x_n\} \) is the sequence generated by the scheme (6.3.11) and \( \{Vz_n\} \) is bounded, then \( \{x_n\} \) converges strongly to a unique solution of the following VIP provided

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0:
\]

\[
\text{find } x^* \in \text{Fix}(T) \cap \Theta \text{ such that } \\
\langle (\mu F - \gamma V)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(T) \cap \Theta. \quad (6.4.12)
\]

(b) If \( \{x_n\} \) is the sequence generated by the scheme (6.3.12), then \( \{x_n\} \) converges strongly to a unique solution of the following VIP provided \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0: \)

\[
\text{find } x^* \in \text{Fix}(T) \cap \Theta \text{ such that } \\
\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(T) \cap \Theta. \quad (6.4.13)
\]

**Proof.** In Theorem 6.4.2, putting \( S = V \), we know that the iterative scheme (6.3.9) reduces to (6.3.11) since there holds for any \( \{\delta_n\} \subset (0,1) \)

\[
x_{n+1} = P_0[\lambda_n \gamma (\delta_n Vx_n + (1 - \delta_n)Sx_n) + (I - \lambda_n \mu F)Tz_n] \\
= P_0[\lambda_n \gamma (\delta_n Vx_n + (1 - \delta_n)Vx_n) + (I - \lambda_n \mu F)Tz_n] \\
= P_0[\lambda_n \gamma Vx_n + (I - \lambda_n \mu F)Tz_n].
\]

In this case, the SHVI (6.4.1) is equivalent to the VIP (6.4.12). Thus, utilizing Theorem 6.4.2 (a) we obtain the desired conclusion (a). As for the conclusion (b), we immediately derive it from \( S = V \equiv 0 \) and Theorem 6.4.2 (b). \( \square \)